Problem 1. [5 points]
Construct a truth table for the following statement form: 
\((p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \land q) \rightarrow r)\).

Problem 2. [5 points]

\(P | Q\) denotes the NAND gate and \(P \downarrow Q\) denotes the NOR gate. Show using an input-output table that \((P | P) \mid (Q | Q) \equiv (P \downarrow Q) \downarrow (P \downarrow Q)\).

Problem 3. [5 points]

A set of premises and a conclusion are given. Use the valid argument forms to deduce the conclusion from the premises, giving a reason for each step. Assume all variables are statement variables.

\[(a) \sim p \rightarrow r \land \sim s\]
\[(b) t \rightarrow s\]
\[(c) u \rightarrow \sim p\]
\[(d) \sim w\]
\[(e) u \lor w\]
\[(f) \therefore \sim t\]

Problem 4. [5 points]

Indicate which of the following statements are true and which are false. Justification is not needed.

\[(a) \forall x \in \mathbb{R}^+, \exists y \in \mathbb{R}^+ \text{ such that } xy = 1.\]
\[(b) \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } xy = 1.\]
\[(c) \forall x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, \exists z \in \mathbb{Z}^+ \text{ such that } z = x - y.\]
\[(d) \forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \exists z \in \mathbb{Z} \text{ such that } z = x - y.\]
\[(e) \exists u \in \mathbb{R}^+ \text{ such that } \forall v \in \mathbb{R}^+, uv < v.\]

Problem 5. [5 points]

Some of the arguments given here are valid by universal modus ponens or universal modus tollens; others are invalid and exhibit the converse or the inverse error. State
which are valid and which are invalid. Justify your answers by mentioning the name of the valid argument form or error.

(a) All honest people pay their taxes.
   Darth is not honest.
   .: Darth does not pay his taxes.

(b) For all students $x$, if $x$ studies discrete mathematics, then $x$ is good at logic.
   Tarik studies discrete mathematics.
   .: Tarik is good at logic.

(c) If compilation of a computer program produces error messages, then the program is not correct.
   Compilation of this program does not produce error messages.
   .: This program is correct.

(d) Any sum of two rational numbers is rational.
   The sum $r + s$ is rational.
   .: The numbers $r$ and $s$ are both rational.

(e) If a number is even, then twice that number is even.
   The number $2n$ is even, for a particular number $n$.
   .: The particular number $n$ is even.

Problem 6. [5 points]
Prove that for all real numbers $x$ and $y$, $|x| \cdot |y| = |xy|$.

Problem 7. [5 points]
Prove the statement in two ways: (a) by contraposition and (b) by contradiction.
For all integers $m$ and $n$, if $mn$ is even then $m$ is even or $n$ is even.

Problem 8. [5 points]
Prove using mathematical induction that if $n \in \mathbb{N}$ then
\[
\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}
\]

Problem 9. [5 points]
Write a recursive algorithm (pseudocode) for solving the Towers of Hanoi problem. Let $M(k)$ be the minimum number of moves required to move $k$ disks from one peg to another peg. Write a recurrence for $M(k)$ using the algorithm. Prove using mathematical induction that $M(k) = 2^k - 1$ for all $k \geq 1$.

Problem 10. [5 points]
Use an element argument to prove the statement. Assume that all sets are subsets of a universal set $U$. For all sets $A$, $B$, and $C$, we have $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Problem 11. [5 points]
Construct an algebraic proof (use identities) for the given statement. Cite a property for every step. For all sets $A$, $B$, and $C$, we have $(A - B) - C = A - (B \cup C)$.
Problem 12. [5 points]
Prove that if $f : X \to Y$ is a one-to-one and onto function with inverse function $f^{-1} : Y \to X$, then $f \circ f^{-1} = I_Y$, where $I_Y$ is the identity function on $Y$.

Problem 13. [5 points]
Set $S$ denotes the set of real numbers strictly between 0 and 1. That is, $S = \{x \in \mathbb{R} \mid 0 < x < 1\}$. Let $a$ and $b$ be real numbers with $a < b$, and suppose that $W = \{x \in \mathbb{R} \mid a < x < b\}$. Prove that $S$ and $W$ have the same cardinality by finding a 1:1 correspondence function.

Problem 14. [5 points]
Determine whether the given relation is reflexive, symmetric, transitive, or none of these. Justify your answers.
Let $A$ be the set of all lines in the plane. A relation $R$ is defined on $A$ as follows: For all lines $\ell_1$ and $\ell_2$ in $A$, we have $\ell_1 \perp \ell_2 \iff \ell_1$ is perpendicular to $\ell_2$.

Problem 15. [5 points]
Prove that the relation is an equivalence relation, and describe the distinct equivalence classes of each relation.
Let $A$ be the set of all statement forms in three variables $p, q,$ and $r$. $R$ is the relation defined on $A$ as follows: For all $P$ and $Q$ in $A$, we have $P \equiv Q \iff P$ and $Q$ have the same truth table.