# What are sequences?

## Types of sequences

- **Finite sequence:** \( a_m, a_{m+1}, a_{m+2}, \ldots, a_n \)
  - e.g.: \( 1^1, 2^2, 3^2, \ldots, 100^2 \)
- **Infinite sequence:** \( a_m, a_{m+1}, a_{m+2}, \ldots \)
  - e.g.: \( \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \)

## Term

- **Closed-form formula:** \( a_k = f(k) \)
  - e.g.: \( a_k = \frac{k}{k+1} \)
- **Recursive formula:** \( a_k = g(k, a_{k-1}, \ldots, a_{k-c}) \)
  - e.g.: \( a_k = a_{k-1} + (k - 1)a_{k-2} \)
What are sequences?

**Growth of sequences**

- **Increasing sequence**
  e.g.: 2, 3, 5, 7, 11, 13, 17, ...
- **Decreasing sequence**
  e.g.: $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$
- **Oscillating sequence**
  e.g.: 1, $-1$, 1, $-1$, ...

**Problem-solving**

- Compute $a_k$ given $a_1, a_2, a_3, \ldots$
  e.g.: Compute $a_k$ given $\frac{1}{n}$, $\frac{2}{n+1}$, $\frac{3}{n+2}$, $\ldots$
- Compute $a_1, a_2, a_3, \ldots$ given $a_k$
  e.g.: Compute $a_1, a_2, a_3, \ldots$ given $a_k = \frac{k}{k+1}$
Sums and products of sequences

**Sum**

- **Summation form:**

  \[
  \sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n
  \]

  where, \( k = \text{index}, \ m = \text{lower limit}, \ n = \text{upper limit} \)

  e.g.: \( \sum_{k=m}^{n} (-1)^k \frac{k}{k+1} \)

**Product**

- **Product form:**

  \[
  \prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots \cdot a_n
  \]

  where, \( k = \text{index}, \ m = \text{lower limit}, \ n = \text{upper limit} \)

  e.g.: \( \sum_{k=m}^{n} \frac{k}{k+1} \)
Properties of sums and products

- Suppose $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and $c$ is any real number.

<table>
<thead>
<tr>
<th>Sum</th>
</tr>
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<tbody>
<tr>
<td>$\sum_{k=m}^{n} a_k = \sum_{k=m}^{i} a_k + \sum_{k=i+1}^{n} a_k$ for $m \leq i &lt; n$ where, $i$ is between $m$ and $n - 1$ (inclusive)</td>
</tr>
<tr>
<td>$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} (c \cdot a_k)$</td>
</tr>
<tr>
<td>$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\prod_{k=m}^{n} a_k) \cdot (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n} (a_k \cdot b_k)$</td>
</tr>
</tbody>
</table>
Change of variable

\[
\sum_{k=0}^{99} \frac{(-1)^k}{k + 1} = \sum_{j=0}^{99} \frac{(-1)^j}{j + 1} \quad \text{(Set } j = k) \tag{1}
\]

\[
= \sum_{i=1}^{100} \frac{(-1)^{i-1}}{i} \quad \text{(Set } i = j + 1) \tag{2}
\]
Definitions

- The factorial of a whole number \( n \), denoted by \( n! \), is defined as follows:

\[
n! = \begin{cases} 
1 & \text{if } n = 0, \\
 n \cdot (n - 1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 & \text{if } n > 0.
\end{cases}
\]

\( n! = \begin{cases} 
1 & \text{if } n = 0, \\
 n \cdot (n - 1)! & \text{if } n > 0.
\end{cases} \qquad \text{ Recursive definition} \)
The combination function of whole numbers $n$ and $r$ ($r \leq n$), denoted by $\binom{n}{k}$, read as $n$ choose $k$, is defined as the number of subsets of size $r$ that can be chosen from a set with $n$ elements.

The combination function is computed as:

$$
\binom{n}{k} = \begin{cases} 
1 & \text{if } r > n, \\
\frac{n!}{r!(n-r)!} & \text{if } r \leq n.
\end{cases}
$$
Proof by mathematical induction

- Mathematical induction is **aesthetically beautiful and insanely powerful** proof technique
- Mathematical induction is probably the **greatest** of all proof techniques and probably the **simplest**

**Core idea**

- A starting domino falls. From the starting domino, every successive domino falls. Then, every domino from the starting domino falls.

Source: https://i.stack.imgur.com/Z3I92.jpg
**Proposition**

For all integers \( n \geq a \), a property \( P(n) \) is true.

<table>
<thead>
<tr>
<th>Proof</th>
</tr>
</thead>
</table>
| **1. Basis step.**  
  Show that \( P(a) \) is true. |
| **2. Induction step.**  
  Show that for all integers \( k \geq a \), if \( P(k) \) is true then \( P(k+1) \) is true.  
To perform this step, suppose that \( P(k) \) is true, where \( k \) is any particular but arbitrarily chosen integer with \( k \geq a \). [This supposition is called the inductive hypothesis.]  
Now, show that \( P(k+1) \) is true. |
Proof by mathematical induction: Example 0

Proposition

- \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \) for all integers \( n \geq 1 \).

Proof

Let \( P(n) \) denote \( 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \).

1. Basis step.
   \( P(1) \) is true. \( \triangleright \) How?

2. Induction step.
   Suppose that \( P(k) \) is true for any \( k \geq 1 \).
   Now, we want to show that \( P(k+1) \) is true.

   \[
P(k + 1) = 1 + 2 + \cdots + k + (k + 1)
   = P(k) + (k + 1)
   = \frac{k(k + 1)}{2} + (k + 1) = \frac{(k + 1)(k + 2)}{2}
   
   Hence, \( P(k + 1) \) is true.