Predicate Logic
(First-Order Logic)
What is a propositional function or predicate?

Definition

- A propositional function or predicate is a sentence that contains one or more variables.
- A predicate is neither true nor false.
- A predicate becomes a proposition when the variables are substituted with specific values.
- The domain of a predicate variable is the set of all values that may be substituted for the variable.

Examples

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Predicate</th>
<th>Domain</th>
<th>Propositions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$x &gt; 5$</td>
<td>$x \in \mathbb{R}$</td>
<td>$p(6), p(-3.6), p(0), \ldots$</td>
</tr>
<tr>
<td>$p(x, y)$</td>
<td>$x + y$ is odd</td>
<td>$x \in \mathbb{Z}, y \in \mathbb{Z}$</td>
<td>$p(4, 5), p(-4, -4), \ldots$</td>
</tr>
<tr>
<td>$p(x, y)$</td>
<td>$x^2 + y^2 = 4$</td>
<td>$x \in \mathbb{R}, y \in \mathbb{R}$</td>
<td>$p(-1.7, 8.9), p(-\sqrt{3}, -1), \ldots$</td>
</tr>
</tbody>
</table>
What is a truth set?

**Definition**

- A **truth set** of a predicate is the set of all values of the predicate that makes the predicate **true**.
- If $p(x)$ is a predicate and $x$ has domain $D$, then the truth set of $p(x)$ is the set of all elements of $D$ that makes $p(x)$ true when the values are substituted for $x$. That is,

$$\text{Truth set of } p(x) = \{ x \in D \mid p(x) \}$$

**Examples**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Predicate</th>
<th>Domain</th>
<th>Truth set</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>$x &gt; 5$</td>
<td>$x \in \mathbb{R}$</td>
<td>${p(6), p(13.6), p(5.001), \ldots}$</td>
</tr>
<tr>
<td>$p(x, y)$</td>
<td>$x + y \text{ is odd}$</td>
<td>$x \in \mathbb{Z}, y \in \mathbb{Z}$</td>
<td>${p(4, 5), p(-4, -3), \ldots}$</td>
</tr>
<tr>
<td>$p(x, y)$</td>
<td>$x^2 + y^2 = 4$</td>
<td>$x \in \mathbb{R}, y \in \mathbb{R}$</td>
<td>${p(-2, 2), p(-\sqrt{3}, -1), \ldots}$</td>
</tr>
</tbody>
</table>
There are two methods to obtain propositions from predicates:
1. Assign specific values to variables
2. Add quantifiers
What are quantifiers?

Definition

- **Quantifiers** are words that refer to quantities such as “all” or “some” and they tell for how many elements a given predicate is true.

- Introduced into logic by logicians Charles Sanders Pierce and Gottlob Frege during late 19th century.

- Two types of quantifiers:
  1. Universal quantifier (∀)
  2. Existential quantifier (∃)
Universal quantifier \((\forall)\)

Definition

- Let \(p(x)\) be a predicate and \(D\) be the domain of \(x\)
- A universal statement is a statement of the form

\[
\forall x \in D, p(x)
\]

- Forms:
  - “\(p(x)\) is true for all values of \(x\)”
  - “For all \(x\), \(p(x)\)”
  - “For each \(x\), \(p(x)\)”
  - “For every \(x\), \(p(x)\)”
  - “Given any \(x\), \(p(x)\)”
- It is true if \(p(x)\) is true for each \(x\) in \(D\); It is false if \(p(x)\) is false for at least one \(x\) in \(D\)
- A counterexample to the universal statement is the value of \(x\) for which \(p(x)\) is false
### Universal quantifier (∀)

#### Examples

<table>
<thead>
<tr>
<th>Universal st.s</th>
<th>Domain</th>
<th>Truth value</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀x ∈ D, x^2 ≥ x</td>
<td>D = {1, 2, 3}</td>
<td>True</td>
<td>Method of exhaustion</td>
</tr>
<tr>
<td>∀x ∈ (\mathbb{R}), x^2 ≥ x</td>
<td>(\mathbb{R})</td>
<td>False</td>
<td>Counterexample</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(x = 0.1)</td>
</tr>
</tbody>
</table>

#### Caution

- Method of exhaustion cannot be used to prove universal statements for infinite sets
Existential quantifier (∃)

Definition

- Let $p(x)$ be a predicate and $D$ be the domain of $x$
- An existential statement is a statement of the form

$$\exists x \in D, p(x)$$

- Forms:
  - “There exists an $x$ such that $p(x)$”
  - “For some $x$, $p(x)$”
  - “We can find an $x$, such that $p(x)$”
  - “There is some $x$ such that $p(x)$”
  - “There is at least one $x$ such that $p(x)$”
- It is true if $p(x)$ is true for at least one $x$ in $D$; It is false if $p(x)$ is false for all $x$ in $D$
- A counterproof to the existential statement is the proof to show that $p(x)$ is true is for no $x$
**Existential quantifier** $(\exists)$

### Examples

<table>
<thead>
<tr>
<th>Universal st.s</th>
<th>Domain</th>
<th>Truth value</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\exists x \in D, x^2 \geq x$</td>
<td>$D = {1, 2, 3}$</td>
<td>True</td>
<td>Method of exhaust.</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{R}, x^2 \geq x$</td>
<td>$\mathbb{R}$</td>
<td>True</td>
<td>Example</td>
</tr>
<tr>
<td>$\exists x \in \mathbb{Z}, x + 1 \leq x$</td>
<td>$\mathbb{Z}$</td>
<td>False</td>
<td>How?</td>
</tr>
</tbody>
</table>
Formal and informal languages

Example

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  - Every real number has a nonnegative square
  - All real numbers have nonnegative squares
  - Any real number has a nonnegative square
  - The square of each real number is nonnegative
  - No real numbers have negative squares
  - $x^2$ is nonnegative for every real $x$
  - $x^2$ is not less than zero for each real number $x$
Universal conditional statement \((\forall, \rightarrow)\)

**Definition**

- A universal conditional statement is of the form

\[ \forall x, \text{ if } p(x) \text{ then } q(x) \]

**Examples**

- \(\forall x \in \mathbb{R}, \text{ if } x > 2 \text{ then } x^2 > 4\)
- \(\forall \text{ real number } x, \text{ if } x \text{ is an integer then } x \text{ is rational}\)
  
  \(\forall \text{ integer } x, x \text{ is rational}\)
  
  \(\triangleright\) Logically equivalent

- \(\forall x, \text{ if } x \text{ is a square then } x \text{ is a rectangle}\)
  
  \(\forall \text{ square } x, x \text{ is a rectangle}\)
  
  \(\triangleright\) Logically equivalent

- \(\forall x \in U, \text{ if } p(x) \text{ then } q(x)\)

  \(\forall x \in D, q(x)\)

  (where, \(D = \{x \in U \mid p(x) \text{ is true}\}\))

\(\triangleright\) Logically equivalent

- Can be extended to existential conditional statement \((\exists, \rightarrow)\)
Implicit quantification \((\Rightarrow, \Leftrightarrow)\)

### Examples

- **If a number** is an integer, then it is a rational number
  - Implicit meaning: \(\forall\) number \(x\), if \(x\) is an integer, \(x\) is rational
- **The number** 10 can be written as a sum of two prime numbers
  - Implicit meaning: \(\exists\) prime numbers \(p\) and \(q\) such that \(10 = p + q\)
- **If** \(x > 2\), then \(x^2 > 4\)
  - Implicit meaning: \(\forall\) real \(x\), if \(x > 2\), then \(x^2 > 4\)

### Definition

- Let \(p(x)\) and \(q(x)\) be predicates and \(D\) be the common domain of \(x\). Then implicit quant. symbols \(\Rightarrow, \Leftrightarrow\) are defined as:

\[
\begin{align*}
p(x) \Rightarrow q(x) & \equiv \forall x, p(x) \rightarrow q(x) \\
p(x) \Leftrightarrow q(x) & \equiv \forall x, p(x) \leftrightarrow q(x)
\end{align*}
\]
Implicit quantification ($\Rightarrow, \Leftarrow$)

Problem

- $q(n)$: $n$ is a factor of 8;
- $r(n)$: $n$ is a factor of 4;
- $s(n)$: $n < 5$ and $n \neq 3$;

Domain of $n$ is $\mathbb{Z}^+$ (i.e., positive integers)

- What are the relationships between $q(n)$, $r(n)$, and $s(n)$ using symbols $\Rightarrow$ and $\Leftarrow$?
Implicit quantification \((\Rightarrow, \iff)\)

**Problem**
- \(q(n)\): \(n\) is a factor of 8; \(r(n)\): \(n\) is a factor of 4
- \(s(n)\): \(n < 5\) and \(n \neq 3\)
  - Domain of \(n\) is \(\mathbb{Z}^+\) (i.e., positive integers)
- What are the relationships between \(q(n)\), \(r(n)\), and \(s(n)\) using symbols \(\Rightarrow\) and \(\iff\)?

**Solution**
- Truth set of \(q(n) = \{1, 2, 4, 8\}\); Truth set of \(r(n) = \{1, 2, 4\}\); Truth set of \(s(n) = \{1, 2, 4\}\)
- \(\forall n\) in \(\mathbb{Z}^+, r(n) \rightarrow q(n)\) i.e., \(r(n) \Rightarrow q(n)\)
  - i.e., “\(n\) is a factor of 4” \(\Rightarrow\) “\(n\) is a factor of 8”
- \(\forall n\) in \(\mathbb{Z}^+, r(n) \leftrightarrow s(n)\) i.e., \(r(n) \iff s(n)\)
  - i.e., “\(n\) is a factor of 4” \(\iff\) “\(n < 5\) and \(n \neq 3\)”
- \(\forall n\) in \(\mathbb{Z}^+, s(n) \rightarrow q(n)\) i.e., \(s(n) \Rightarrow q(n)\)
  - i.e., “\(n < 5\) and \(n \neq 3\)” \(\Rightarrow\) “\(n\) is a factor of 8”
### Negation of quantified statements ($\sim$)

#### Definition

- Formally,

\[
\sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x)
\]

\[
\sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x)
\]

- Negation of a **universal** statement ("all are") is logically equivalent to an **existential** statement ("there is at least one that is not")

- Negation of an **existential** statement ("some are") is logically equivalent to a **universal** statement ("all are not")

#### Methods

Two methods to avoid errors while finding negations:

1. Write the statements formally and then take negations
2. Ask "What exactly would it mean for the given statement to be false?"
Negation of quantified statements ($\sim$)

**Examples**

- All mathematicians wear glasses
  - Negation (incorrect): No mathematician wears glasses
  - Negation (incorrect + ambiguous): All mathematicians do not wear glasses
  - Negation (correct): There is at least one mathematician who does not wear glasses

- Some snowflakes are the same
  - Negation (incorrect): Some snowflakes are different
  - Negation (correct): All snowflakes are different
Negation of quantified statements ($\sim$)

**Examples**

- $\forall$ primes $p$, $p$ is odd
  
  **Negation:** $\exists$ primes $p$, $p$ is even

- $\exists$ triangle $T$, sum of angles of $T$ equals $200^\circ$
  
  $\forall$ triangles $T$, sum of angles of $T$ does not equal $200^\circ$

- No politicians are honest
  
  **Formal statement:** $\forall$ politicians $x$, $x$ is not honest
  
  **Formal negation:** $\exists$ politician $x$, $x$ is honest
  
  **Informal negation:** Some politicians are honest

- 1357 is not divisible by any integer between 1 and 37
  
  **Formal statement:** $\forall n \in [1, 37]$, 1357 is not divisible by $n$
  
  **Formal negation:** $\exists n \in [1, 37]$, 1357 is divisible by $n$
  
  **Informal negation:** 1357 is divisible by some integer between 1 and 37
### Definition

- Formally,

\[ \sim (\forall x, p(x) \rightarrow q(x)) \equiv \exists x, \sim (p(x) \rightarrow q(x)) \equiv \exists x, (p(x) \land \sim q(x)) \]

### Examples

- \( \forall \) real \( x \), if \( x > 10 \), then \( x^2 > 100 \).
  Negation: \( \exists \) real \( x \) such that \( x > 10 \) and \( x^2 \leq 100 \).
- If a computer program has more than 100,000 lines, then it contains a bug.
  Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug.
Relation between quantifiers $(\forall, \exists)$ and $(\land, \lor)$

<table>
<thead>
<tr>
<th>Relation</th>
</tr>
</thead>
</table>
| • Universal statements are generalizations of and statements  
  Existential statements are generalizations of or statements |
| • If $p(x)$ is a predicate and $D = \{x_1, x_2, \ldots, x_n\}$ is the domain of $x$, then |

\[
\forall x \in D, p(x) \equiv p(x_1) \land p(x_2) \land \cdots \land p(x_n)
\]

\[
\exists x \in D, p(x) \equiv p(x_1) \lor p(x_2) \lor \cdots \lor p(x_n)
\]
# Vacuous truth of universal statements

## Problem

- Consider the bowl and the balls
- Consider the statement: All the balls in the bowl are blue
- Is the statement true?

## Solution

- The statement is false iff its negation is true
- Negation: There exists a ball in the bowl that is not blue
- The negation is false; So, the statement is true, by default

## Definition

- A statement of the form

  \[ \forall x \text{ in } D, \text{ if } p(x), \text{ then } q(x) \]

  is **vacuously true** or **true by default**, if and only if \( p(x) \) is false for all \( x \text{ in } D \)
## Universal conditional statements \((\forall x, p(x) \rightarrow q(x))\)

### Definitions

- **Statement**: \(\forall x, \text{if } p(x) \text{ then } q(x)\)
- **Contrapositive** of the statement is \(\forall x, \text{if } \sim q(x) \text{ then } \sim p(x)\)
- **Converse** of the statement is \(\forall x, \text{if } q(x) \text{ then } p(x)\)
- **Inverse** of the statement is \(\forall x, \text{if } \sim p(x) \text{ then } \sim q(x)\)

### Identities

- Conditional \(\equiv\) Contrapositive \(\triangleright\) Useful for proofs
- Conditional \(\not\equiv\) Converse
- Conditional \(\not\equiv\) Inverse
- Converse \(\equiv\) Inverse

### Formulas

- \(\forall x, p(x) \rightarrow q(x) \equiv \forall x, \sim q(x) \rightarrow \sim p(x)\) \(\triangleright\) Useful for proofs
- \(\forall x, p(x) \rightarrow q(x) \not\equiv \forall x, q(x) \rightarrow p(x)\)
- \(\forall x, p(x) \rightarrow q(x) \not\equiv \forall x, \sim p(x) \rightarrow \sim q(x)\)
- \(\forall x, q(x) \rightarrow p(x) \equiv \forall x, \sim p(x) \rightarrow \sim q(x)\)
**Universal conditional statement** $\forall x, p(x) \rightarrow q(x)$

### Definitions

- $\forall x, p(x)$ is a **sufficient condition** for $q(x)$ means
  $\forall x$, if $p(x)$ then $q(x)$
- $\forall x, p(x)$ is a **necessary condition** for $q(x)$ means
  $\forall x$, if $\sim p(x)$ then $\sim q(x) \equiv \forall x$, if $q(x)$ then $p(x)$
- $\forall x, p(x)$ only if $q(x)$ means
  $\forall x$, if $\sim q(x)$ then $\sim p(x) \equiv \forall x$, if $p(x)$ then $q(x)$

### Example

- For real $x$, $x = 1$ is a sufficient condition for $x^2 = 1$
  i.e., $\forall x$, if $x = 1$ then $x^2 = 1$  $\triangleright$ **True**
- For real $x$, $x^2 = 1$ is a necessary condition for $x = 1$
  i.e., $\forall x$, if $x^2 \neq 1$ then $x \neq 1$  $\triangleright$ **True**
- For real $x$, $x = 1$ only if $x^2 = 1$
  i.e., $\forall x$, if $x^2 \neq 1$ then $x \neq 1$  $\triangleright$ **True**
Statements with Multiple Quantifiers
Problem

- What is the interpretation for the following statement?
  “There is a person supervising every detail of the production process.”

Ambiguous interpretations

1. There is one single person who supervises all the details of the production process.
   \[ \exists \text{ person } p \text{ such that } \forall \text{ detail } d, p \text{ supervises } d \]
2. For any particular production detail, there is a person who supervises that detail, but there might be different supervisors for different details.
   \[ \forall \text{ detail } d, \exists \text{ person } p \text{ such that } p \text{ supervises } d \]
Statements with multiple quantifiers

Definitions

1. Statement form:

\[ \forall x \in D, \exists y \in E \text{ such that } P(x, y) \]

Interpretation: Allow someone else to pick whatever element \( x \) in \( D \) they wish. Then, you must find an element \( y \) in \( E \) that “works” for that particular \( x \).

2. Statement form:

\[ \exists x \in D \text{ such that } \forall y \in E, P(x, y) \]

Interpretation: Your job is to find one particular \( x \) in \( D \) that will “work” no matter what \( y \) in \( E \) anyone might choose to challenge you with.
Example: Tarski world

Problem

- For all triangles $x$, there is a square $y$ such that $x$ and $y$ have the same color. **Truth value?**

Answer

- True. **How?**
Problem

- There is a triangle $x$ such that for all circles $y$, $x$ is to the right of $y$. Truth value?

Answer

- True. How?
Example: College cafeteria

**Problem**

- $\exists$ an item $I$ such that $\forall$ students $S$, $S$ chose $I$.
- **Informal statement? Truth value?**

**Solution**

- There is an item that was chosen by every student.
- True. **How?**
Example: College cafetaria

Problem

- $\exists$ a student $S$ such that $\forall$ items $I$, $S$ chose $I$.
- Informal statement? Truth value?

Solution

- There is a student who chose every available item.
- False. How?
Example: College cafetaria

Problem

- $\exists$ a student $S$ such that $\forall$ stations $Z$, $\exists$ an item $I$ in $Z$ such that $S$ chose $I$.

- Informal statement? Truth value?

Solution

- There is a student who chose at least one item from every station.

- True. How?
Example: College cafetaria

Problem

∀ students $S$ and ∀ stations $Z$, $\exists$ an item $I$ in $Z$ such that $S$ chose $I$.

- Informal statement? Truth value?

Solution

- Every student chose at least one item from every station.
- False. How?
<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Every nonzero real number has a reciprocal.</td>
<td>• ∃ nonzero real numbers ( u ), ∃ a real number ( v ) such that ( uv = 1 ).</td>
</tr>
<tr>
<td>• There is a real number with no reciprocal.</td>
<td>• ∃ a real number ( c ) such that ( \forall ) real numbers ( d ), ( cd \neq 1 ).</td>
</tr>
<tr>
<td>Problem</td>
<td>Solution</td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>--------------------------------------------------------------------------</td>
</tr>
<tr>
<td>• There is a smallest positive integer.</td>
<td>• ∃ a positive integer $m$ such that $\forall$ positive integers $n$, $m \leq n$.</td>
</tr>
<tr>
<td>Problem</td>
<td>Solution</td>
</tr>
<tr>
<td>• There is no smallest positive real number.</td>
<td>• $\forall$ positive real numbers $x$, $\exists$ a positive real number $y$ such that $y &lt; x$.</td>
</tr>
<tr>
<td>Problem</td>
<td>Solution</td>
</tr>
<tr>
<td>• $\lim_{n \to \infty} a_n = L$</td>
<td>• $\forall \epsilon &gt; 0$, $\exists$ an integer $N$ such that $\forall$ integers $n$, if $n &gt; N$ then $L - \epsilon &lt; a_n &lt; L + \epsilon$.</td>
</tr>
</tbody>
</table>
Negations of multiply-quantified statements

<table>
<thead>
<tr>
<th>Definitions</th>
</tr>
</thead>
</table>
| • \( \sim (\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \)  
  \[ \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y) \]  
• \( \sim (\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \)  
  \[ \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y) \] |
Example: Tarski world

Problem

- For all squares $x$, there is a circle $y$ such that $x$ and $y$ have the same color. Negation?

Answer

- $\exists$ a square $x$ such that $\forall$ circles $y$, $x$ and $y$ do not have the same color. True. How?
Example: Tarski world

Problem

- There is a triangle $x$ such that for all squares $y$, $x$ is to the right of $y$. Negation?

Answer

- $\forall$ triangles $x$, $\exists$ a square $y$ such that $x$ is not to the right of $y$. True. How?
## Order of quantifiers

**Order**

- The order of quantifiers are important when multiple quantifiers are involved

**Example**

- $\exists$ a person $y$ such that $\forall$ people $x$, $x$ loves $y$.  
  *Maybe possible.*

- $\forall$ people $x$, $\exists$ a person $y$ such that $x$ loves $y$.  
  *Quite impossible.*
Order of quantifiers

Example

- For every square $x$ there is a triangle $y$ such that $x$ and $y$ have different colors
  - $\triangleright$ True
- There exists a triangle $y$ such that for every square $x$, $x$ and $y$ have different colors.
  - $\triangleright$ False
Order of quantifiers

Example

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>∀ ∈ ℤ, ∃ y ∈ ℜ (xy &lt; 1)</td>
</tr>
<tr>
<td></td>
<td>Two cases:</td>
</tr>
<tr>
<td>a</td>
<td>For x ≤ 0, let y = 1, then xy &lt; 1</td>
</tr>
<tr>
<td>b</td>
<td>For x &gt; 0, let y = 1/(x + 1), then xy &lt; 1</td>
</tr>
<tr>
<td>2</td>
<td>∃ y ∈ ℜ, ∀ ∈ ℤ (xy &lt; 1)</td>
</tr>
<tr>
<td></td>
<td>False</td>
</tr>
<tr>
<td></td>
<td>Counterexample</td>
</tr>
<tr>
<td></td>
<td>For y = 1, let x = 2, then xy &lt; 1</td>
</tr>
</tbody>
</table>

True

False