CSE 215: Foundations of Computer Science
(Predicate Logic)

Pramod Ganapathli
Department of Computer Science
State University of New York at Stony Brook

September 3, 2019
Predicate Logic
(First-Order Logic)
What is a propositional function or predicate?

Definition

- A **propositional function** or **predicate** is a sentence that contains one or more variables.
- A predicate is neither true nor false.
- A predicate becomes a proposition when the variables are substituted with specific values.
- The **domain** of a predicate variable is the set of all values that may be substituted for the variable.

Examples

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Predicate</th>
<th>Domain</th>
<th>Propositions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>( x &gt; 5 )</td>
<td>( x \in \mathbb{R} )</td>
<td>( p(6), p(-3.6), p(0), \ldots )</td>
</tr>
<tr>
<td>( p(x, y) )</td>
<td>( x + y \text{ is odd} )</td>
<td>( x \in \mathbb{Z}, y \in \mathbb{Z} )</td>
<td>( p(4, 5), p(-4, -4), \ldots )</td>
</tr>
<tr>
<td>( p(x, y) )</td>
<td>( x^2 + y^2 = 4 )</td>
<td>( x \in \mathbb{R}, y \in \mathbb{R} )</td>
<td>( p(-1.7, 8.9), p(-\sqrt{3}, -1), \ldots )</td>
</tr>
</tbody>
</table>
What is a truth set?

**Definition**

- A **truth set** of a predicate is the set of all values of the predicate that makes the predicate true.
- If \( p(x) \) is a predicate and \( x \) has domain \( D \), then the truth set of \( p(x) \) is the set of all elements of \( D \) that makes \( p(x) \) true when the values are substituted for \( x \). That is,

\[
\text{Truth set of } p(x) = \{ x \in D \mid p(x) \}
\]

**Examples**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Predicate</th>
<th>Domain</th>
<th>Truth set</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>( x &gt; 5 )</td>
<td>( x \in \mathbb{R} )</td>
<td>{ ( p(6), p(13.6), p(5.001), \ldots ) }</td>
</tr>
<tr>
<td>( p(x, y) )</td>
<td>( x + y \text{ is odd} )</td>
<td>( x \in \mathbb{Z}, y \in \mathbb{Z} )</td>
<td>{ ( p(4, 5), p(-4, -3), \ldots ) }</td>
</tr>
<tr>
<td>( p(x, y) )</td>
<td>( x^2 + y^2 = 4 )</td>
<td>( x \in \mathbb{R}, y \in \mathbb{R} )</td>
<td>{ ( p(-2, 2), p(-\sqrt{3}, -1), \ldots ) }</td>
</tr>
</tbody>
</table>
Predicates to propositions

There are two methods to obtain propositions from predicates
1. Assign specific values to variables
2. Add quantifiers

1. Assign specific values to variables

Predicates

Propositions

2. Add quantifiers
What are quantifiers?

**Definition**

- **Quantifiers** are words that refer to quantities such as “all” or “some” and they tell for how many elements a given predicate is true.

- Introduced into logic by logicians Charles Sanders Pierce and Gottlob Frege during late 19th century.

- Two types of quantifiers:
  1. Universal quantifier ($\forall$)
  2. Existential quantifier ($\exists$)
Universal quantifier (∀)

**Definition**

- Let \( p(x) \) be a predicate and \( D \) be the domain of \( x \)
- A **universal statement** is a statement of the form

\[
\forall x \in D, p(x)
\]

- Forms:
  - “\( p(x) \) is true for all values of \( x \)”
  - “For all \( x \), \( p(x) \)”
  - “For each \( x \), \( p(x) \)”
  - “For every \( x \), \( p(x) \)”
  - “Given any \( x \), \( p(x) \)”
- It is true if \( p(x) \) is true for each \( x \) in \( D \); It is false if \( p(x) \) is false for at least one \( x \) in \( D \)
- A **counterexample** to the universal statement is the value of \( x \) for which \( p(x) \) is false
Universal quantifier (∀)

Examples

<table>
<thead>
<tr>
<th>Universal st.s</th>
<th>Domain</th>
<th>Truth value</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀x ∈ D, x^2 ≥ x</td>
<td>D = {1, 2, 3}</td>
<td>True</td>
<td>Method of exhaustion</td>
</tr>
<tr>
<td>∀x ∈ R, x^2 ≥ x</td>
<td>R</td>
<td>False</td>
<td>Counterexample</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>x = 0.1</td>
</tr>
</tbody>
</table>

Caution

- Method of exhaustion cannot be used to prove universal statements for infinite sets
Existential quantifier (\(\exists\))

**Definition**

- Let \(p(x)\) be a predicate and \(D\) be the domain of \(x\)
- An **existential statement** is a statement of the form

\[
\exists x \in D, p(x)
\]

- **Forms:**
  - “There exists an \(x\) such that \(p(x)\)”
  - “For some \(x\), \(p(x)\)”
  - “We can find an \(x\), such that \(p(x)\)”
  - “There is some \(x\) such that \(p(x)\)”
  - “There is at least one \(x\) such that \(p(x)\)”
- It is true if \(p(x)\) is true for at least one \(x\) in \(D\); It is false if \(p(x)\) is false for all \(x\) in \(D\)
- A **counterproof** to the existential statement is the proof to show that \(p(x)\) is true is for no \(x\)
### Existential quantifier (\(\exists\))

#### Examples

<table>
<thead>
<tr>
<th>Universal st.s</th>
<th>Domain</th>
<th>Truth value</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\exists x \in D, x^2 \geq x)</td>
<td>(D = {1, 2, 3})</td>
<td>True</td>
<td>Method of exhaust.</td>
</tr>
<tr>
<td>(\exists x \in \mathbb{R}, x^2 \geq x)</td>
<td>(\mathbb{R})</td>
<td>True</td>
<td>Example</td>
</tr>
<tr>
<td>(\exists x \in \mathbb{Z}, x + 1 \leq x)</td>
<td>(\mathbb{Z})</td>
<td>False</td>
<td>How?</td>
</tr>
</tbody>
</table>
Formal and informal languages

Example

- $\forall x \in \mathbb{R}, x^2 \geq 0$
  - Every real number has a nonnegative square
  - All real numbers have nonnegative squares
  - Any real number has a nonnegative square
  - The square of each real number is nonnegative
  - No real numbers have negative squares
  - $x^2$ is nonnegative for every real $x$
  - $x^2$ is not less than zero for each real number $x$
### Universal conditional statement ($\forall, \rightarrow$)

#### Definition

- A **universal conditional statement** is of the form
  \[ \forall x, \text{ if } p(x) \text{ then } q(x) \]

#### Examples

- $\forall x \in \mathbb{R}$, if $x > 2$ then $x^2 > 4$
- $\forall$ real number $x$, if $x$ is an integer then $x$ is rational
  \[ \forall \text{ integer } x, x \text{ is rational} \]
  ▶ Logically equivalent
- $\forall x$, if $x$ is a square then $x$ is a rectangle
  \[ \forall \text{ square } x, x \text{ is a rectangle} \]
  ▶ Logically equivalent
- $\forall x \in U$, if $p(x)$ then $q(x)$
  \[ \forall x \in D, q(x) \]
  (where, $D = \{x \in U \mid p(x) \text{ is true}\}$)
  ▶ Logically equivalent

- Can be extended to **existential conditional statement** ($\exists, \rightarrow$)
## Implicit quantification (⇒, ⇔)

### Examples

- **If a number** is an integer, then it is a rational number
  Implicit meaning: ∀ number $x$, if $x$ is an integer, $x$ is rational
- **The number** 10 can be written as a sum of two prime numbers
  Implicit meaning: ∃ prime numbers $p$ and $q$ such that $10 = p + q$
- If $x > 2$, then $x^2 > 4$
  Implicit meaning: ∀ real $x$, if $x > 2$, then $x^2 > 4$

### Definition

- Let $p(x)$ and $q(x)$ be predicates and $D$ be the common domain of $x$. Then implicit quant. symbols $⇒$, $⇔$ are defined as:

  $$p(x) ⇒ q(x) ≡ ∀ x, p(x) → q(x)$$

  $$p(x) ⇔ q(x) ≡ ∀ x, p(x) ↔ q(x)$$
### Implicit quantification ($\Rightarrow$, $\iff$)

#### Problem

- $q(n)$: $n$ is a factor of 8; $r(n)$: $n$ is a factor of 4
- $s(n)$: $n < 5$ and $n \neq 3$
  
  Domain of $n$ is $\mathbb{Z}^+$ (i.e., positive integers)

- **What are the relationships between $q(n)$, $r(n)$, and $s(n)$ using symbols $\Rightarrow$ and $\iff$?**
Implicit quantification \( (⇒, ⇔) \)

**Problem**

- \( q(n) \): \( n \) is a factor of 8; \( r(n) \): \( n \) is a factor of 4
- \( s(n) \): \( n < 5 \) and \( n \neq 3 \)

Domain of \( n \) is \( \mathbb{Z}^+ \) (i.e., positive integers)
- What are the relationships between \( q(n) \), \( r(n) \), and \( s(n) \) using symbols \( ⇒ \) and \( ⇔ \)?

**Solution**

- Truth set of \( q(n) = \{1, 2, 4, 8\} \); Truth set of \( r(n) = \{1, 2, 4\} \); Truth set of \( s(n) = \{1, 2, 4\} \)
- \( \forall n \text{ in } \mathbb{Z}^+, r(n) \rightarrow q(n) \) i.e., \( r(n) ⇒ q(n) \)
  - i.e., “\( n \) is a factor of 4” ⇒ “\( n \) is a factor of 8”
- \( \forall n \text{ in } \mathbb{Z}^+, r(n) ⇔ s(n) \) i.e., \( r(n) ⇔ s(n) \)
  - i.e., “\( n \) is a factor of 4” ⇔ “\( n < 5 \) and \( n \neq 3 \)”
- \( \forall n \text{ in } \mathbb{Z}^+, s(n) \rightarrow q(n) \) i.e., \( s(n) ⇒ q(n) \)
  - i.e., “\( n < 5 \) and \( n \neq 3 \)” ⇒ “\( n \) is a factor of 8”
Negation of quantified statements (∼)

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Formally,</td>
</tr>
<tr>
<td>[ \sim (\forall x \in D, p(x)) \equiv \exists x \in D, \sim p(x) ]</td>
</tr>
<tr>
<td>[ \sim (\exists x \in D, p(x)) \equiv \forall x \in D, \sim p(x) ]</td>
</tr>
<tr>
<td>• Negation of a <strong>universal</strong> statement (“all are”) is logically equivalent to an <strong>existential</strong> statement (“there is at least one that is not”)</td>
</tr>
<tr>
<td>Negation of an <strong>existential</strong> statement (“some are”) is logically equivalent to a <strong>universal</strong> statement (“all are not”)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two methods to avoid errors while finding negations:</td>
</tr>
<tr>
<td>1. Write the statements formally and then take negations</td>
</tr>
<tr>
<td>2. Ask “What exactly would it mean for the given statement to be false?”</td>
</tr>
</tbody>
</table>
Examples

- All mathematicians wear glasses
  Negation (incorrect): No mathematician wears glasses
  Negation (incorrect + ambiguous): All mathematicians do not wear glasses
  Negation (correct): There is at least one mathematician who does not wear glasses

- Some snowflakes are the same
  Negation (incorrect): Some snowflakes are different
  Negation (correct): All snowflakes are different
Negation of quantified statements (\(\sim\))

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>• (\forall) primes (p), (p) is odd</td>
</tr>
<tr>
<td>\hspace{1cm} Negation: (\exists) primes (p), (p) is even</td>
</tr>
<tr>
<td>• (\exists) triangle (T), sum of angles of (T) equals 200°</td>
</tr>
<tr>
<td>\hspace{1cm} (\forall) triangles (T), sum of angles of (T) does not equal 200°</td>
</tr>
<tr>
<td>• No politicians are honest</td>
</tr>
<tr>
<td>\hspace{1cm} Formal statement: (\forall) politicians (x), (x) is not honest</td>
</tr>
<tr>
<td>\hspace{1cm} Formal negation: (\exists) politician (x), (x) is honest</td>
</tr>
<tr>
<td>\hspace{1cm} Informal negation: Some politicians are honest</td>
</tr>
<tr>
<td>• 1357 is not divisible by any integer between 1 and 37</td>
</tr>
<tr>
<td>\hspace{1cm} Formal statement: (\forall n \in [1, 37]), 1357 is not divisible by (n)</td>
</tr>
<tr>
<td>\hspace{1cm} Formal negation: (\exists n \in [1, 37]), 1357 is divisible by (n)</td>
</tr>
<tr>
<td>\hspace{1cm} Informal negation: 1357 is divisible by some integer between 1 and 37</td>
</tr>
</tbody>
</table>
**Negation of universal conditional statements**

<table>
<thead>
<tr>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Formally,</td>
</tr>
<tr>
<td>[ \sim (\forall x, p(x) \rightarrow q(x)) \equiv \exists x, \sim (p(x) \rightarrow q(x)) \equiv \exists x, (p(x) \land \sim q(x)) ]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
</table>
| • \( \forall \) real \( x \), if \( x > 10 \), then \( x^2 > 100 \)  
  Negation: \( \exists \) real \( x \), if \( x > 10 \), then \( x^2 \leq 100 \) |
| • If a computer program has more than 100,000 lines, then it contains a bug  
  Negation: There is at least one computer program that has more than 100,000 lines and does not contain a bug |
## Relation between quantifiers \((\forall, \exists)\) and \((\land, \lor)\)

<table>
<thead>
<tr>
<th>Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Universal statements are generalizations of <em>and</em> statements</td>
</tr>
<tr>
<td>Existential statements are generalizations of <em>or</em> statements</td>
</tr>
<tr>
<td>If (p(x)) is a predicate and (D = {x_1, x_2, \ldots, x_n}) is the domain of (x), then</td>
</tr>
</tbody>
</table>

\[
\forall x \in D, p(x) \equiv p(x_1) \land p(x_2) \land \cdots \land p(x_n) \\
\exists x \in D, p(x) \equiv p(x_1) \lor p(x_2) \lor \cdots \lor p(x_n)
\]
### Problem

- Consider the bowl and the balls
- Consider the statement: All the balls in the bowl are blue
- Is the statement true?

### Solution

- The statement is false iff its negation is true
- Negation: There exists a ball in the bowl that is not blue
- The negation is false; So, the statement is true, by default

### Definition

- A statement of the form

\[
\forall x \text{ in } D, \text{ if } p(x), \text{ then } q(x)
\]

is **vacuously true** or **true by default**, if and only if \( p(x) \) is false for all \( x \) in \( D \)
Universal conditional statements \((\forall x, p(x) \rightarrow q(x))\)

**Definitions**

- **Statement**: \(\forall x, \text{if } p(x) \text{ then } q(x)\)
- **Contrapositive** of the statement is \(\forall x, \text{if } \sim q(x) \text{ then } \sim p(x)\)
- **Converse** of the statement is \(\forall x, \text{if } q(x) \text{ then } p(x)\)
- **Inverse** of the statement is \(\forall x, \text{if } \sim p(x) \text{ then } \sim q(x)\)

**Identities**

- Conditional \(\equiv\) Contrapositive \(\triangleright\) Useful for proofs
- Conditional \(\not\equiv\) Converse
- Conditional \(\not\equiv\) Inverse
- Converse \(\equiv\) Inverse

**Formulas**

- \(\forall x, p(x) \rightarrow q(x) \equiv \forall x, \sim q(x) \rightarrow \sim p(x)\) \(\triangleright\) Useful for proofs
- \(\forall x, p(x) \rightarrow q(x) \not\equiv \forall x, q(x) \rightarrow p(x)\)
- \(\forall x, p(x) \rightarrow q(x) \not\equiv \forall x, \sim p(x) \rightarrow \sim q(x)\)
- \(\forall x, q(x) \rightarrow p(x) \equiv \forall x, \sim p(x) \rightarrow \sim q(x)\)
### Universal conditional statement \( \forall x, p(x) \rightarrow q(x) \)

#### Definitions

- \( \forall x, p(x) \) is a **sufficient condition** for \( q(x) \) means
  \[
  \forall x, \text{ if } p(x) \text{ then } q(x)
  \]
- \( \forall x, p(x) \) is a **necessary condition** for \( q(x) \) means
  \[
  \forall x, \text{ if } \neg p(x) \text{ then } \neg q(x) \equiv \forall x, \text{ if } q(x) \text{ then } p(x)
  \]
- \( \forall x, p(x) \) only if \( q(x) \) means
  \[
  \forall x, \text{ if } q(x) \text{ then } p(x) \equiv \forall x, \text{ if } \neg q(x) \text{ then } \neg p(x)
  \]

#### Example

- For real \( x \), \( x = 1 \) is a sufficient condition for \( x^2 = 1 \)
  i.e., \( \forall x, \text{ if } x = 1 \text{ then } x^2 = 1 \) \( \triangleright \) True
- For real \( x \), \( x^2 = 1 \) is a necessary condition for \( x = 1 \)
  i.e., \( \forall x, \text{ if } x^2 \neq 1 \text{ then } x \neq 1 \) \( \triangleright \) True
- For real \( x \), \( x = 1 \) only if \( x^2 = 1 \)
  i.e., \( \forall x, \text{ if } x \neq 1 \text{ then } x^2 \neq 1 \) \( \triangleright \) False
  Counterexample: When \( x = -1 \), then \( x^2 = 1 \)