The Logic of Compound Statements cont.
Logical Arguments

CSE 215, Foundations of Computer Science

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Logical Arguments

- An *argument* (*form*) is a (finite) sequence of statements (forms), usually written as follows:

$$p_1$$
$$\dots$$
$$p_n$$
$$\therefore q$$

- We call $p_1, \dots, p_n$ the *premises* (or *assumptions* or *hypotheses*) and q the *conclusion*, of the argument.
- We read: "p_1, p_2, …, p_n, therefore q" OR

  "From premises p_1, p_2, …, p_n infer conclusion q"

Logical Arguments

- Argument forms are also called *inference rules.*
- An argument form consisting of two premises and a conclusion is called a *syllogism*.

Logical Arguments

- An inference rule is said to be *valid*, or (*logically*) *sound*, if it is the case that, for each truth valuation, if all the premises true, then the conclusion is also true!

Theorem: An inference rule is valid if, and only if, the conditional $p_1 \land p_2 \land \ldots \land p_n \rightarrow q$ is a tautology.

Determining Validity or Invalidity

- **Testing an Argument Form for Validity**
  1. Identify the premises and conclusion of the argument form.
  2. Construct a truth table showing the truth values of all the premises and the conclusion.
  3. A row of the truth table in which **all the premises are true** is called a **critical row.**
     a) If there is a critical row in which the conclusion is false, then the argument form is invalid.
     b) If the conclusion in every critical row is true, then the argument form is valid.

Determining Validity or Invalidity

$$p \rightarrow q \vee \sim r$$
$$q \rightarrow p \wedge r$$
$$\therefore p \rightarrow r$$

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \rightarrow q \vee \sim r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	T	F	T	F
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows it is possible for an argument of this form to have true premises and a false conclusion. Hence this form of argument is invalid.

Modus Ponens

- Modus Ponens:

"method of affirming" in Latin

$$p \rightarrow q$$
$$p$$
$$\therefore q$$

		premises		conclusion	
p	q	$p \rightarrow q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T	F	
F	T	T	F	T	
F	F	T	F	F	

Modus Ponens

- The following argument is valid:

  If Socrates is a man, then Socrates is mortal.

  Socrates is a man.

  ∴ Socrates is mortal.

Modus Ponens

- **Example:**

  If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

  The sum of the digits of 371,487 is divisible by 3.

  ∴ 371,487 is divisible by 3.

Modus Tollens

- Modus Tonens: $p \rightarrow q$

"method of denying" $\sim q$

in Latin $\therefore \sim p$

- Modus Tollens is valid because :
  - modus ponens is valid and the fact that a conditional statement is logically equivalent to its contrapositive, OR
  - it can be established formally by using a truth table.

Modus Tollens

- Example:

  (1) If Zeus is human, then Zeus is mortal.

  (2) Zeus is not mortal.

  ∴ Zeus is not human.

- An intuitive proof is proof by contradiction
  - if Zeus were human, then by (1) he would be mortal.
  - But by (2) he is not mortal (which would be a contradiction).
  - Hence, Zeus cannot be human.

Recognizing Modus Ponens and Modus Tollens

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.

There are more pigeons than there are pigeonholes.

∴ ?

Recognizing Modus Ponens and Modus Tollens

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.

There are more pigeons than there are pigeonholes.

∴ At least two pigeons roost in the same hole.

by modus ponens

Recognizing Modus Ponens and Modus Tollens

If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is **not** divisible by 3.

$\therefore$?

Recognizing Modus Ponens and Modus Tollens

If 870,232 is divisible by 6, then it is divisible by 3.

870,232 is **not** divisible by 3.

∴ 870,232 is not divisible by 6. by modus tollens

Other Rules of Inference

- **Generalization:**

$$p \qquad \textbf{and} \qquad q$$

$$\therefore p \vee q \qquad\qquad \therefore p \vee q$$

- Example:

Anton is a junior.

∴ (more generally) Anton is a junior or Anton is a senior.

Other Rules of Inference

- **Specialization:**

$p \land q$ **and** $p \land q$

$\therefore p$ $\therefore q$

- Example:

Ana knows numerical analysis and

Ana knows graph algorithms.

∴ (in particular) Ana knows graph algorithms.

Other Rules of Inference

- **Elimination :**

$p \vee q$ **and** $p \vee q$

$\sim q$ $\sim p$

$\therefore p$ $\therefore q$

  - If we have only two possibilities and we can rule one out, the other one must be the case.
- Example:

$x - 3 = 0$ or $x + 2 = 0$

$x + 2 \neq 0.$

$\therefore x - 3 = 0.$

Other Rules of Inference

- **Transitivity :**

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\therefore p \rightarrow r$$

- Example:

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

∴ If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Proof Techniques

- **Proof by Contradiction:**

$$\sim p \rightarrow c, \text{ where } c \text{ is a contradiction}$$
$$\therefore p$$

- The usual way to derive a conditional $\sim p \rightarrow c$ is to assume $\sim p$ and then derive c (i.e., a contradiction).
- Thus, if one can derive a contradiction from $\sim p$, then one may conclude that p is true.

Knights and Knaves: knights always tell the truth and knaves always lie

A says: B is a knight.

B says: A and I are of opposite type.

Suppose A is a knight.

∴ What A says is true.	by definition of knight
∴ B is also a knight.	That's what A said.
∴ What B says is true.	by definition of knight
∴ A and B are of opposite types.	That's what B said.

∴ We have arrived at the following contradiction: A and B are both knights and A and B are of opposite type.

∴ The supposition is false.	by the contradiction rule

Therefore:

∴ A is not a knight.	negation of supposition
∴ A is a knave.	since A is not a knight, A is a knave.
∴ What A says is false.	by definition of knave
∴ B is not a knight.	~(what A said) by definition of knave
∴ B is also a knave.	by elimination

Proof Techniques

- **Proof by Division into Cases:**

$$p \lor q$$
$$p \rightarrow r$$
$$q \rightarrow r$$
$$\therefore r$$

  - If a disjunction $p \lor q$ has been derived and the goal is to prove r, then according to this inference rule it would be sufficient to derive $p \rightarrow r$ and $q \rightarrow r$.
- Example: x is positive or x is negative.

  If x is positive, then $x^2 > 0$.

  If x is negative, then $x^2 > 0$.

  $\therefore x^2 > 0$.

Quine's Method

- The following method can be used to determine whether a given propositional formula is a tautology, a contradiction, or a contingency.

Let p be a propositional formula.

- If p contains no variables, it can be simplified to T or F, and hence is either a tautology or a contradiction.
- If p contains a variable, then
  1. select a variable, say q,
  2. simplify both $p[q := \mathrm{T}]$ and $p[q := \mathrm{F}]$, denoting the simplified formulas by p_1 and p_2, respectively, and
  3. apply the method recursively to p_1 and p_2.
- If p_1 and p_2 are both tautologies, so is p.
- If p_1 and p_2 are both contradictions, so is p.
- In all other cases, p is a contingency.

Quine's Method Example

$(p \wedge \sim q \to r) \wedge (r \to p \vee q) \wedge (p \to \sim r) \wedge (p \vee q \vee r) \to q$

We first select a variable, say q, and then consider the two cases, $q := \mathrm{T}$ and $q := \mathrm{F}$.

1. For $q := \mathrm{T}$, the formula $\ldots \to \mathrm{T}$ can be simplified to T.
2. For $q := \mathrm{F}$,

$(p \wedge \sim \mathrm{F} \to r) \wedge (r \to p \vee \mathrm{F}) \wedge (p \to \sim r) \wedge (p \vee \mathrm{F} \vee r) \to \mathrm{F}$

$\equiv (p \wedge \mathrm{T} \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r) \to \mathrm{F}$

$\equiv (p \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r) \to \mathrm{F}$

$\equiv \sim[(p \to r) \wedge (r \to p) \wedge (p \to \sim r) \wedge (p \vee r)]$

Quine's Method Example cont.

$\sim[(p \rightarrow r) \wedge (r \rightarrow p) \wedge (p \rightarrow \sim r) \wedge (p \vee r)]$

We select the variable p

1. For $p := \mathrm{T}$

$\sim[(\mathrm{T} \rightarrow r) \wedge (r \rightarrow \mathrm{T}) \wedge (\mathrm{T} \rightarrow \sim r) \wedge (\mathrm{T} \vee r)]$

$\equiv \sim[r \wedge \mathrm{T} \wedge \sim r \wedge \mathrm{T}] \equiv \sim[r \wedge \sim r] \equiv \sim\mathrm{F} \equiv \mathrm{T}$

2. For $p := \mathrm{F}$

$\sim[(\mathrm{F} \rightarrow r) \wedge (r \rightarrow \mathrm{F}) \wedge (\mathrm{F} \rightarrow \sim r) \wedge (\mathrm{F} \vee r)]$

$\equiv \sim[\mathrm{T} \wedge \sim r \wedge \mathrm{T} \wedge r] \equiv \sim[\sim r \wedge r] \equiv \sim\mathrm{F} \equiv \mathrm{T}$

- This completes the process. All formulas considered, including the original formula, are tautologies.