CHAPTER 10

Introduction to Intuitionistic Logic

Intuitionistic logic has developed as a result of certain philosophical views on the foundation of mathematics, known as *intuitionism.* Intuitionism was originated by L. E. J. Brouwer in 1908. The first Hilbert style formalization of the intuitionistic logic, formulated as a proof system, is due to A. Heyting (1930). In this chapter we present a Hilbert style proof system that is equivalent to the Heyting's original formalization and discuss the relationship between intuitionistic and classical logic.

There have been, of course, several successful attempts at creating models for the intuitionistic logic, and hence to define formally a notion of the intuitionistic tautology. The most known are Kripke models and topological and algebraic models. Kripke models were defined by Kripke in 1964. The topological and algebraic models were initiated by Stone and Tarski in 1937, 1938, respectively. An uniform theory and presentation of topological and algebraic models was given by Rasiowa and Sikorski in 1964. We present both approaches, Kripke and Rasiowa and Sikorski in the respective chapters on Kripke and Algebraic Models. We also give there the respective proofs of the Completeness Theorem for the proof systems presented in this section.

The goal of this chapter is to give a presentation of the intuitionistic logic formulated as a proof system and discuss the basic theorems that establish the relationship between classical and intuitionistic logics.

1 Philosophical Motivation

Intuitionists' view-point on the meaning of the basic logical and set theoretical concepts used in mathematics is different from that of most mathematicians in their research.

The basic difference lies in the interpretation of the word *exists.* For example, let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence

$$\exists x A(x) \tag{1}$$

is true if it is a theorem of arithmetic, i.e. if it can be *deduced* from the axioms of arithmetic by means of classical logic. If a mathematician proves sentence (1), this does not mean that he is able to indicate a *method of construction* of a natural number n such that $A(n)$ holds.

For the intuitionist the sentence (1) is true only he is able to provide a constructive method of finding a number n such that $A(n)$ is true.

Moreover, the mathematician often obtains the proof of the existential sentence (1), i.e. of the sentence $\exists x A(x)$ by proving first a sentence

$$\neg \forall x \neg A(x). \tag{2}$$

Next he makes use of a classical tautology

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)). \tag{3}$$

By applying Modus Ponens to (2) and (3) he obtains (1).

For the intuitionist such method is not acceptable, for it does not give any *method of constructing* a number n such that $A(n)$ holds. For this reason the intuitionist do not accept the classical tautology (3) i.e. $(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x))$ as intuitionistic tautology, or as as an intuitionistically provable sentence.

Let us denote by $\vdash_I A$ and $\models_I A$ the fact that A is intuitionistically provable and intuitionistic tautology, respectively. The proof system I for the intuitionistic logic has hence to be such that

$$\nvdash_I (\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)).$$

The intuitionistic semantics or definition of intuitionistic model I has to be such that one can prove in that also

$$\not\models_I (\neg \forall x \neg A(x)) \Rightarrow \exists x A(x)).$$

The above means also that intuitionists interpret differently the meaning of propositional connectives.

Intuitionistic implication

The intuitionistic implication $(A \Rightarrow B)$ is considered by to be true if there exists a method by which a *proof of* B can be deduced from the proof of A. In the case of the implication

$$(\neg \forall x \neg A(x)) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence $(\neg \forall x \neg A(x))$, permits is to obtain an intuitionistic proof of the sentence $\exists x A(x)$, i.e. to construct a number n such that $A(n)$ holds, hence we can't accept it as an intuitionistic theorem or tautology.

Intuitionistic negation

The negation and the disjunction are also understood differently. The sentence $\neg A$ is considered intuitionistically true if the acceptance of the sentence A leads to absurdity.

As a result of above understanding of negation and implication we have that in the intuitionistic logic I

$$\vdash_I (A \Rightarrow \neg\neg A)$$

but

$$\nvdash_I (\neg\neg A \Rightarrow A).$$

Consequently, in any intuitionistic model I,

$$\models_I (A \Rightarrow \neg\neg A)$$

and

$$\not\models_I (\neg\neg A \Rightarrow A).$$

Intuitionistic disjunction

The intuitionist regards a disjunction $(A \cup B)$ as true if one of the sentences A, B is true and there is a method by which it is possible to find out which of them is true. As a consequence a classical law of excluded middle

$$(A \cup \neg A)$$

is not acceptable by the intuitionists since there is no general method of finding out, for any given sentence A, whether A or $\neg A$ is true. This means that the intuitionistic logic must be such that

$$\nvdash_I (A \cup \neg A)$$

and

$$\models_I (A \cup \neg A).$$

Intuitionists' view of the concept of infinite set also differs from that which is generally accepted in mathematics. Intuitionists reject the idea of infinite set as a closed whole. They look upon an infinite set as something which is constantly in a state of formation. Thus, for example, the set of all natural numbers is infinite in the sense that to any given finite set of natural numbers it is always possible to add one more natural number. The notion of the set of all subsets of the set of all natural numbers is not regarded meaningful. Thus intuitionists reject the general idea of a set as defined by a modern set theory.

An exact exposition of the basic ideas of intuitionism is outside the range of our investigations. Our goal is to give a presentation of of the intuitionistic logic, which is a sort of reflection of intuitionistic ideas formulated as a proof system.

2 Hilbert System for Intuitionistic Propositional Logic

Language

We adopt a propositional language $\mathcal{L} = \mathcal{L}_{\{\cup,\cap,\Rightarrow,\neg\}}$ with the set of formulas denoted by $\mathcal{F}$.

Axioms

The set of logical axioms of the Hilbert style proof system for intuitionistic logic consists of all formulas of the forms

A1 $((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))$,

A2 $(A \Rightarrow (A \cup B))$,

A3 $(B \Rightarrow (A \cup B))$,

A4 $((A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow ((A \cup B) \Rightarrow C)))$,

A5 $((A \cap B) \Rightarrow A)$,

A6 $((A \cap B) \Rightarrow B)$,

A7 $((C \Rightarrow A) \Rightarrow ((C \Rightarrow B) \Rightarrow (C \Rightarrow (A \cap B)))$,

A8 $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \cap B) \Rightarrow C))$,

A9 $(((A \cap B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))$,

A10 $(A \cap \neg A) \Rightarrow B)$,

A11 $((A \Rightarrow (A \cap \neg A)) \Rightarrow \neg A)$,

where A, B, C are any formulas in $\mathcal{L}$.

Rules of inference

We adopt a Modus Ponens rule

$$(\mathbf{MP})\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

as the only inference rule.

Proof system I

A proof system

$$I = (\ \mathcal{L}, \mathcal{F}\ \mathbf{A1} - \mathbf{A11},\ (\mathbf{MP})\ ),$$

for $\mathcal{L}$, **A1 - A11** defined above, is called Hilbert Style Formalization for Intuitionistic Propositional Logic.

The above set of axioms is due to Rasiowa (1959). It differs from Heyting original set of axioms but they are equivalent.

We introduce, as usual, the notion of a formal proof in I and denote by

$$\vdash_I A$$

the fact that A has a formal proof in I, or that that A is *intuitionistically provable.*

2.1 Completeness Theorem for I

There are several ways one can define a semantics for the intuitionistic logic, i.e. the semantics for a Heyting proof system or for its equivalent form I. It means it is possible to define a notion of intuitionistic tautology in a way that the system I (and hence the equivalent original Heyting system) is a complete proof systems under this notion.

This notion of tautology will be defined formally and discussed later. For a moment we denote by

$$\models_I A$$

the fact that A is an intuitionistic tautology and state that the following theorem holds.

Theorem 2.1 (Completeness Theorem for I)

For any formula $A \in \mathcal{F}$,

$$\vdash_I A \text{ ı } and\ only\ if \models_I A.$$

The Completeness Theorem gives us the right to replace the notion of a theorem of a given intuitionistic proof system by a more general (independent of the proof system) and intuitive (we all have some notion of truthfulness) notion of the intuitionistic tautology.

The intuitionistic logic has been created as a rival to the classical one. So a question about the relationship between these two is a natural one. We present here some examples of tautologies and some historic results about the connection between the classical and intuitionistic logic. In this the way we can form some intuitions about what the intuitionistic tautology really is, even if we haven't defined it yet.

2.2 Examples of intuitionistic propositional tautologies

The following classical tautologies are provable in I and hence are also intuitionistic tautologies.

$$(A \Rightarrow A), \tag{4}$$

$$(A \Rightarrow (B \Rightarrow A)), \tag{5}$$

$$(A \Rightarrow (B \Rightarrow (A \cap B))), \tag{6}$$

$$((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))), \tag{7}$$

$$(A \Rightarrow \neg\neg A), \tag{8}$$

$$\neg(A \cap \neg A), \tag{9}$$

$$((\neg A \cup B) \Rightarrow (A \Rightarrow B)), \tag{10}$$

$$(\neg(A \cup B) \Rightarrow (\neg A \cap \neg B)), \tag{11}$$

$$((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B)), \tag{12}$$

$$((\neg A \cup \neg B) \Rightarrow (\neg A \cap \neg B)), \tag{13}$$

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)), \tag{14}$$

$$((A \Rightarrow \neg B) \Rightarrow (B \Rightarrow \neg A)), \tag{15}$$

$$(\neg\neg\neg A \Rightarrow \neg A), \tag{16}$$

$$(\neg A \Rightarrow \neg\neg\neg A), \tag{17}$$

$$(\neg\neg(A \Rightarrow B) \Rightarrow (A \Rightarrow \neg\neg B)), \tag{18}$$

$$((C \Rightarrow A) \Rightarrow ((C \Rightarrow (A \Rightarrow B)) \Rightarrow (C \Rightarrow B)), \tag{19}$$

2.3 Examples of classical tautologies that are not intuitionistic tautologies

The following classical tautologies **are not intuitionistic tautologies.**

$$(A \cup \neg A), \tag{20}$$

$$(\neg\neg A \Rightarrow A), \tag{21}$$

$$((A \Rightarrow B) \Rightarrow (\neg A \cup B)), \tag{22}$$

$$(\neg(A \cap B) \Rightarrow (\neg A \cup \neg B)), \tag{23}$$

$$((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow A)), \tag{24}$$

$$((\neg A \Rightarrow \neg B) \Rightarrow (B \Rightarrow A)), \tag{25}$$

$$(((A \Rightarrow B) \Rightarrow A) \Rightarrow A), \tag{26}$$

Homework Exercise

The algebraic models for the intuitionistic logic are defined in terms of *Pseudo-Boolean Algebras* in the following way. A formula A is said to be an intuitionistic tautology if and only if $v \models A$, for all v and all Pseudo-Boolean Algebras, where v maps VAR into universe of a Pseudo-Boolean Algebra. I.e. A is an intuitionistic tautology if and only if it is true in all Pseudo-Boolean Algebras under all possible variable assignments.

A 3 element Heyting algebra as defined in the section "Some three valued logics" is a 3 element Pseudo-Boolean Algebra.

1. Show that the 3 element Heyting algebra is a model for all formulas (4) - (19).

2. Find for which of the formulas (20) - (26) it acts as a counter-model.

3 Connection Between Classical and Intuitionistic Tautologies

The first connection is quite obvious. Let us observe that if we add the axiom

A12 $(A \cup \neg A)$

to the set of axioms of our system I we obtain a complete Hilbert proof system C for the classical logic. This proves the following.

Theorem 3.1 *Every formula that is derivable intuitionistically is classically derivable, i.e.*

$$if \quad \vdash_I A, \quad then \quad \vdash A,$$

where we use symbol $\vdash$ *for classical (complete classical proof system) provability.*

We write

$$\models A$$

and

$$\models_I A$$

to denote that A is a classical and intuitionistic tautology, respectively.

As both proof systems, I and C are complete under respective semantics, we can state this as the following relationship between classical and intuitionistic tautologies.

Theorem 3.2

For any formula $A \in \mathcal{F}$,

$$if \ \ \models_I A, \ \ then \ \ \models \ A.$$

The next relationship shows how to obtain intuitionistic tautologies from the classical tautologies and vice versa. It has been proved by Glivenko in 1929 and independently by Tarski in 1938.

Theorem 3.3 *(Glivenko) For any formula $A \in \mathcal{F}$, A is a classically provable if and only if $\neg\neg A$ is an intuitionistically provable, i.e.*

$$\vdash_I A \quad iff \quad \vdash \neg\neg A$$

where we use symbol $\vdash$ for classical (complete classical proof system) provability.

Theorem 3.4 *(Tarski) For any formula $A \in \mathcal{F}$, A is a classical tautology if and only if $\neg\neg A$ is an intuitionistic tautology, i.e.*

$$\models \ A \ \ if \ and \ only \ if \ \ \models_I \ \neg\neg A.$$

The following relationships were proved by Gödel in 1331.

Theorem 3.5 *(Gödel) For any $A, B \in \mathcal{F}$, a formula $(A \Rightarrow \neg B)$ is a classically provable if and only if it is an intuitionistically provable, i.e.*

$$\vdash \ (A \Rightarrow \neg B) \ \ if \ and \ only \ if \ \ \vdash_I \ (A \Rightarrow \neg B).$$

Theorem 3.6 *(Gödel) If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classically provable if and only if it is an intuitionistically provable.*

By the Completeness Theorems for classical and intuitionisctic logics we get the following equivalent semantic form of theorems 3.5and 3.6.

Theorem 3.7

A formula $(A \Rightarrow \neg B)$ is a classical tautology if and only if it is an intuitionistic tautology, i.e.

$$\models\ (A \Rightarrow \neg B)\ \ if\ and\ only\ if\ \ \models_I (A \Rightarrow \neg B).$$

Theorem 3.8

If a formula A contains no connectives except $\cap$ and $\neg$, then A is a classical tautology if and only if it is an intuitionistic tautology.

3.1 On intuitionistically derivable disjunction

In a classical logic it is possible for the disjunction $(A \cup B)$ to be a tautology when neither A nor B is a tautology. The tautology $(A \cup \neg A)$ is the simplest example. This does not hold for the intuitionistic logic.

This fact was stated without the proof by Gödel in 1931 and proved by Gentzen in 1935 via his proof system **LI** which is presented and discussed in chapter 12.

Theorem 3.9

A disjunction $(A \cup B)$ is intuitionistically provable if and only if either A or B is intuitionistically provable , i.e.

$$\vdash_I (A \cup B) \quad if\ and\ only\ if\ \ \vdash_I A\ \ or\ \ \vdash_I B.$$

We obtain, via the Completeness Theorem 2.1 the following equivalent semantic version of the above,

Theorem 3.10

A disjunction $(A \cup B)$ is intuitionistic tautology if and only if either A or B is intuitionistic tautology, i.e.

$$\models_I (A \cup B) \quad if\ and\ only\ if\ \ \models_I A\ \ or\ \ \models_I B.$$