Weighted Graphs and Applications

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Objectives

• To represent weighted edges using adjacency matrices and adjacency lists
• To model weighted graphs using the **WeightedGraph** class that extends the **AbstractGraph** class
• To design and implement the algorithm for finding a minimum spanning tree
  • To define the **MST** class that extends the **Tree** class
• To design and implement the algorithm for finding single-source shortest paths
  • To define the **ShortestPathTree** class that extends the **Tree** class
• To solve the weighted nine tail problem using the shortest-path algorithm
A graph is a **weighted graph** if each edge is assigned a weight.

For example: assume that the edges represent the driving distances among the cities.
Representing Weighted Graphs

- Representing Weighted Edges
  - Edge Array
  - Weighted Adjacency Matrices
  - Adjacency Lists
Representing Weighted Edges: Edge Array

```java
int[][] edges = {{0, 1, 2}, {0, 3, 8},
                 {1, 0, 2}, {1, 2, 7}, {1, 3, 3},
                 {2, 1, 7}, {2, 3, 4}, {2, 4, 5},
                 {3, 0, 8}, {3, 1, 3}, {3, 2, 4}, {3, 4, 6},
                 {4, 2, 5}, {4, 3, 6}};
```
Weighted Adjacency Matrices

Integer[][] adjacencyMatrix = {
    {null, 2, null, 8, null },
    {2, null, 7, 3, null },
    {null, 7, null, 4, 5},
    {8, 3, 4, null, 6},
    {null, null, 5, 6, null}
};
java.util.List<WeightedEdge>[] list =
new java.util.List[5];

<table>
<thead>
<tr>
<th>list[0]</th>
<th>WeightedEdge(0, 1, 2)</th>
<th>WeightedEdge(0, 3, 8)</th>
</tr>
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<tbody>
<tr>
<td>list[1]</td>
<td>WeightedEdge(1, 0, 2)</td>
<td>WeightedEdge(1, 3, 3)</td>
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<td></td>
<td></td>
<td>WeightedEdge(1, 2, 7)</td>
</tr>
<tr>
<td>list[2]</td>
<td>WeightedEdge(2, 3, 4)</td>
<td>WeightedEdge(2, 4, 5)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>WeightedEdge(2, 1, 7)</td>
</tr>
<tr>
<td>list[3]</td>
<td>WeightedEdge(3, 1, 3)</td>
<td>WeightedEdge(3, 2, 4)</td>
</tr>
<tr>
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<td></td>
<td>WeightedEdge(3, 4, 6)</td>
</tr>
<tr>
<td>list[4]</td>
<td>WeightedEdge(4, 2, 5)</td>
<td>WeightedEdge(4, 3, 6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>WeightedEdge(3, 0, 8)</td>
</tr>
</tbody>
</table>
Edge Adjacency Lists

For flexibility, we will use an `ArrayList` rather than a fixed-sized array to represent the list as follows:

```java
List<List<WeightedEdge>> list = new java.util.ArrayList<>();
```

<table>
<thead>
<tr>
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</thead>
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<td>WeightedEdge(3, 0, 8)</td>
<td>WeightedEdge(4, 3, 6)</td>
<td></td>
</tr>
</tbody>
</table>

![Graph Diagram]
public class WeightedEdge extends AbstractGraph.Edge implements Comparable<WeightedEdge> {
    public double weight; // The weight on edge (u, v)

    /** Create a weighted edge on (u, v) */
    public WeightedEdge(int u, int v, double weight) {
        super(u, v);
        this.weight = weight;
    }

    /** Compare two edges on weights */
    public int compareTo(WeightedEdge edge) {
        if (weight > edge.weight) {
            return 1;
        } else if (weight == edge.weight) {
            return 0;
        } else {
            return -1;
        }
    }
}
<interface>
    Graph\langle V \rangle
</interface>

AbstractGraph\langle V \rangle

WeightedGraph\langle V \rangle

+WeightedGraph()
+WeightedGraph(\text{vertices: } V[], \text{edges: int[][]})
+WeightedGraph(\text{vertices: List\langle V \rangle, edges: List\langle WeightedEdge \rangle})
+WeightedGraph(\text{edges: int[][]}, \text{numberOfVertices: int})
+WeightedGraph(\text{edges: List\langle WeightedEdge \rangle, numberOfVertices: int})
+printWeightedEdges(): void
+getWeight(\text{int } u, \text{int } v): double
+addEdges(\text{int } u, \text{int } v, \text{weight: double}): void

+getMinimumSpanningTree(): MST
+getMinimumSpanningTree(index: int): MST
+getShortestPath(index: int): ShortestPathTree

Constructs an empty graph.
Constructs a weighted graph with the specified edges and the number of vertices in arrays.
Constructs a weighted graph with the specified edges and the number of vertices.
Constructs a weighted graph with the specified edges in an array and the number of vertices.
Constructs a weighted graph with the specified edges in a list and the number of vertices.
Displays all edges and weights.
Returns the weight on the edge from \text{u} to \text{v}. Throw an exception if the edge does not exist.
Adds a weighted edge to the graph and throws an IllegalArgumentException if \text{u}, \text{v}, or \text{w} is invalid. If \text{(u, v)} is already in the graph, the new weight is set.
Returns a minimum spanning tree starting from vertex 0.
Returns a minimum spanning tree starting from vertex \text{v}.
Returns all single-source shortest paths.
import java.util.*;
public class WeightedGraph<V> extends AbstractGraph<V> {
    /** Construct an empty */
    public WeightedGraph() {
    }

    /** Construct a WeightedGraph from vertices and edges in list */
    public WeightedGraph(int[][] edges, int numberOfVertices) {
        List<V> vertices = new ArrayList<>();
        for (int i = 0; i < numberOfVertices; i++)
            vertices.add((V)(new Integer(i)));
        createWeightedGraph(vertices, edges);
    }

    /** Construct a WeightedGraph from vertices and edged in arrays */
    public WeightedGraph(V[] vertices, int[][] edges) {
        createWeightedGraph(java.util.Arrays.asList(vertices), edges);
    }

    /** Construct a WeightedGraph for vertices 0, 1, 2 and edge list */
    public WeightedGraph(List<V> vertices, List<WeightedEdge> edges) {
        createWeightedGraph(vertices, edges);
    }
}
/** Create adjacency lists from edge arrays */
private void createWeightedGraph(List<V> vertices, int[][] edges) {
    this.vertices = vertices;
    for (int i = 0; i < vertices.size(); i++)
        neighbors.add(new ArrayList<Edge>()); // Create a list for vertices
    for (int i = 0; i < edges.length; i++)
        neighbors.get(edges[i][0]).add(new WeightedEdge(edges[i][0], edges[i][1], edges[i][2]));
}

/** Create adjacency lists from edge lists */
private void createWeightedGraph(List<V> vertices, List<WeightedEdge> edges) {
    this.vertices = vertices;
    for (int i = 0; i < vertices.size(); i++)
        neighbors.add(new ArrayList<Edge>()); // Create a list for vertices
    for (WeightedEdge edge : edges)
        neighbors.get(edge.u).add(edge); // Add an edge into the list
}

/** Construct a WeightedGraph from vertices 0, 1, and edge array */
public WeightedGraph(List<WeightedEdge> edges, int numberOfVertices) {
    List<V> vertices = new ArrayList<>();
    for (int i = 0; i < numberOfVertices; i++)
        vertices.add((V)(new Integer(i)));
    createWeightedGraph(vertices, edges);
}
/** Return the weight on the edge (u, v) */
public double getWeight(int u, int v) throws Exception {
    for (Edge edge : neighbors.get(u))
        if (edge.v == v)
            return ((WeightedEdge)edge).weight;
    throw new Exception("Edge does not exit");
}

/** Add edges to the weighted graph */
public boolean addEdge(int u, int v, double weight) {
    return addEdge(new WeightedEdge(u, v, weight));
}

/** Display edges with weights */
public void printWeightedEdges() {
    for (int i = 0; i < getSize(); i++) {
        System.out.print(getVertex(i) + " (" + i + ")": ");
        for (Edge edge : neighbors.get(i))
            System.out.print("(" + edge.u +
                ", " + edge.v + ", " + ((WeightedEdge)edge).weight + ") ");
        System.out.println();
    }
}

// to continue later with minimum spanning tree and
// single-source shortest paths
public class TestWeightedGraph {
    public static void main(String[] args) {

        int[][] edges = {
            {0, 1, 807}, {0, 3, 1331}, {0, 5, 2097},
            {1, 0, 807}, {1, 2, 381}, {1, 3, 1267},
            {2, 1, 381}, {2, 3, 1015}, {2, 4, 1663}, {2, 10, 1435},
            {3, 0, 1331}, {3, 1, 1267}, {3, 2, 1015}, {3, 4, 599}, {3, 5, 1003},
            {4, 2, 1663}, {4, 3, 599}, {4, 5, 533}, {4, 7, 1260}, {4, 8, 864}, {4, 10, 496},
            {5, 0, 2097}, {5, 3, 1003}, {5, 4, 533}, {5, 6, 983}, {5, 7, 787},
            {6, 5, 983}, {6, 7, 214},
            {7, 4, 1260}, {7, 5, 787}, {7, 6, 214}, {7, 8, 888},
            {8, 4, 864}, {8, 7, 888}, {8, 9, 661}, {8, 10, 781}, {8, 11, 810},
            {9, 8, 661}, {9, 11, 1187},
            {10, 2, 1435}, {10, 4, 496}, {10, 8, 781}, {10, 11, 239},
            {11, 8, 810}, {11, 9, 1187}, {11, 10, 239}
        };

        WeightedGraph<String> graph1 = new WeightedGraph<>(vertices, edges);

        System.out.println("The number of vertices in graph1: " + graph1.getSize());
        System.out.println("The vertex with index 1 is " + graph1.getVertex(1));
        System.out.println("The index for Miami is " + graph1.getIndex("Miami"));
        System.out.println("The edges for graph1: ");
        graph1.printWeightedEdges();
    }
}
edges = new int[][] {
    {0, 1, 2}, {0, 3, 8},
    {1, 0, 2}, {1, 2, 7}, {1, 3, 3},
    {2, 1, 7}, {2, 3, 4}, {2, 4, 5},
    {3, 0, 8}, {3, 1, 3}, {3, 2, 4}, {3, 4, 6},
    {4, 2, 5}, {4, 3, 6}
};
WeightedGraph<Integer> graph2 = new WeightedGraph<>(edges, 5);
System.out.println("\nThe edges for graph2:");
graph2.printWeightedEdges();
}
The number of vertices in graph1: 12
The vertex with index 1 is San Francisco
The index for Miami is 9
The edges for graph1:
Vertex 0: (0, 1, 807) (0, 3, 1331) (0, 5, 2097)
Vertex 1: (1, 2, 381) (1, 0, 807) (1, 3, 1267)
Vertex 2: (2, 1, 381) (2, 3, 1015) (2, 4, 1663) (2, 10, 1435)
Vertex 3: (3, 4, 599) (3, 5, 1003) (3, 1, 1267)
 (3, 0, 1331) (3, 2, 1015)
Vertex 4: (4, 10, 496) (4, 8, 864) (4, 5, 533) (4, 2, 1663)
 (4, 7, 1260) (4, 3, 599)
Vertex 5: (5, 4, 533) (5, 7, 787) (5, 3, 1003)
 (5, 0, 2097) (5, 6, 983)
Vertex 6: (6, 7, 214) (6, 5, 983)
Vertex 7: (7, 6, 214) (7, 8, 888) (7, 5, 787) (7, 4, 1260)
Vertex 8: (8, 9, 661) (8, 10, 781) (8, 4, 864)
 (8, 7, 888) (8, 11, 810)
Vertex 9: (9, 8, 661) (9, 11, 1187)
Vertex 10: (10, 11, 239) (10, 4, 496) (10, 8, 781) (10, 2, 1435)
Vertex 11: (11, 10, 239) (11, 9, 1187) (11, 8, 810)

The edges for graph2:
Vertex 0: (0, 1, 2) (0, 3, 8)
Vertex 1: (1, 0, 2) (1, 2, 7) (1, 3, 3)
Vertex 2: (2, 3, 4) (2, 1, 7) (2, 4, 5)
Vertex 3: (3, 1, 3) (3, 4, 6) (3, 2, 4) (3, 0, 8)
Vertex 4: (4, 2, 5) (4, 3, 6)
Minimum Spanning Trees

• A graph may have many spanning trees
• A minimum spanning tree is a spanning tree (i.e., a subset of the edges of a connected undirected graph that connects all the vertices together without any cycles) with the minimum total weight
Minimum Spanning Trees

- Application example: a company wants to lease telephone lines to connect all the customers together
- There are many ways to connect all customers together
- Different lines have different rate (e.g., length)
- The cheapest way is to find a spanning tree with the minimum total rate

Total w: 38
Minimum Spanning Trees

Total w: 42
Not a minimum spanning tree

Total w: 38
Total w: 38
Minimum Spanning Trees

- An algorithm was developed in 1930 by Czech mathematician Vojtěch Jarník and later rediscovered and republished by computer scientists Robert C. Prim in 1957 and Edsger W. Dijkstra in 1959.
Prim’s Minimum Spanning Tree Algorithm

Input: G = (V, E, W).
Output: a MST

Tree minimumSpanningTree() {
    Let V denote the set of vertices in the graph;
    Let T be a set for the vertices in the spanning tree;
    Initially, add the starting vertex to T;
    while (size of T < n) {
        find u in T and v in V - T with the smallest weight on the edge (u, v), as shown in the figure;
        add v to T;
    }
}

Vertices already in the spanning tree
Vertices not currently in the spanning tree
Minimum Spanning Tree Algorithm Example
Refined Version of Prim’s Minimum Spanning Tree Algorithm

- To make it easy to identify the next vertex to add into the tree, we use $\text{cost}[v]$ to store the cost of adding a vertex $v$ to the spanning tree $T$
- Initially $\text{cost}[s]$ is 0 for a starting vertex and assign infinity to $\text{cost}[v]$ for all other vertices
- The algorithm repeatedly finds a vertex $u$ in $V - T$ with the smallest $\text{cost}[u]$ and moves $u$ to $T$
- For all edges $(u, v)$ where $v$ in $V - T$ update the $\text{cost}[v]$ with $\min$ between the old cost and the weight of $(u, v)$
Refined Version of Prim’s Minimum Spanning Tree Algorithm

Input: a graph $G = (V, E)$ with non-negative weights
Output: a minimum spanning tree with the starting vertex $s$ as the root

MST getMinimumSpanningTree(s) {

Let $T$ be a set that contains the vertices in the spanning tree;
Initially $T$ is empty;
Set $\text{cost}[s] = 0$; and
$\text{cost}[v] = \text{infinity}$ for all other vertices in $V$;
while (size of $T < n$) {
Find $u$ not in $T$ with the smallest $\text{cost}[u]$;
Add $u$ to $T$;
for (each $(u, v)$ in $E$)
    if ($v$ not in $T$ && $\text{cost}[v] > w(u, v)$) {
        $\text{cost}[v] = w(u, v)$;
        $\text{parent}[v] = u$;
    }
}
}

- Testing whether a vertex $v$ is in $T$ by invoking $T.$contains$(v)$ takes $O(|V|)$ time since $T$ is a list
- Therefore, the overall time complexity for this implementation is $O(|V|^2|E|)$, but it can be reduced to with binary heaps and hashing.
Implementing MST Algorithm

AbstractGraph.Tree

WeightedGraph.MST

- totalWeight: int

+ MST(root: int, parent: int[], searchOrder: List<Integer> totalWeight: int)
+ getTotalWeight(): int

Total weight of the tree.
Constructs an MST with the specified root, parent array, searchOrder, and total weight for the tree.
Returns the totalWeight of the tree.
/** MST is an inner class in WeightedGraph */
public class MST extends Tree {
    private double totalWeight; // Total weight of all edges in the tree
    public MST(int root, int[] parent, List<Integer> searchOrder,
                double totalWeight) {
        super(root, parent, searchOrder);
        this.totalWeight = totalWeight;
    }
    public double getTotalWeight() {
        return totalWeight;
    }
}

/** Get a minimum spanning tree rooted at vertex 0 */
public MST getMinimumSpanningTree() {
    return getMinimumSpanningTree(0);
}

/** Get a minimum spanning tree rooted at a specified vertex */
public MST getMinimumSpanningTree(int startingVertex) {
    // cost[v] stores the cost by adding v to the tree
    double[] cost = new double[getSize()];
    for (int i = 0; i < cost.length; i++)
        cost[i] = Double.POSITIVE_INFINITY; // Initial cost
    cost[startingVertex] = 0; // Cost of source is 0
    int[] parent = new int[getSize()]; // Parent of a vertex
    parent[startingVertex] = -1; // startingVertex is the root
    double totalWeight = 0; // Total weight of the tree thus far
    List<Integer> T = new ArrayList<>();

// Expand T
while (T.size() < getSize()) {
  // Find smallest cost v in V - T
  int u = -1; // Vertex to be determined
  double currentMinCost = Double.POSITIVE_INFINITY;

  for (int i = 0; i < getSize(); i++)
    if (!T.contains(i) && cost[i] < currentMinCost) {
      currentMinCost = cost[i];
      u = i;
    }
  T.add(u); // Add a new vertex to T
  totalWeight += cost[u]; // Add cost[u] to the tree

  // Adjust cost[v] for v that is adjacent to u and v in V - T
  for (Edge e : neighbors.get(u))
    if (!T.contains(e.v) && cost[e.v] > ((WeightedEdge)e).weight) {
      cost[e.v] = ((WeightedEdge)e).weight;
      parent[e.v] = u;
    }
} // End of while

return new MST(startingVertex, parent, T, totalWeight);
public class TestMinimumSpanningTree {
    public static void main(String[] args) {
        String[] vertices = {
            "Seattle", "San Francisco", "Los Angeles",
            "Atlanta", "Miami", "Dallas", "Houston"};

        int[][] edges = {
            {0, 1, 807}, {0, 3, 1331}, {0, 5, 2097},
            {1, 0, 807}, {1, 2, 381}, {1, 3, 1267},
            {2, 1, 381}, {2, 3, 1015}, {2, 4, 1663}, {2, 10, 1435},
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            {11, 8, 810}, {11, 9, 1187}, {11, 10, 239}
        };

        WeightedGraph<String> graph1 = new WeightedGraph<>(vertices, edges);

        WeightedGraph<String>.MST tree1 = graph1.getMinimumSpanningTree();

        System.out.println("Total weight is " + tree1.getTotalWeight());
        tree1.printTree();
    }
}
edges = new int[][]{
    {0, 1, 2}, {0, 3, 8},
    {1, 0, 2}, {1, 2, 7}, {1, 3, 3},
    {2, 1, 7}, {2, 3, 4}, {2, 4, 5},
    {3, 0, 8}, {3, 1, 3}, {3, 2, 4}, {3, 4, 6},
    {4, 2, 5}, {4, 3, 6}
};

WeightedGraph<Integer> graph2 = new WeightedGraph<>(edges, 5);

WeightedGraph<Integer>.MST tree2 = graph2.getMinimumSpanningTree(1);

System.out.println("\nTotal weight is " + tree2.getTotalWeight());

}
Total weight is 6513.0
Root is: Seattle
Edges: (Seattle, San Francisco) (San Francisco, Los Angeles) (Los Angeles, Denver) (Denver, Kansas City) (Kansas City, Chicago) (New York, Boston) (Chicago, New York) (Dallas, Atlanta) (Atlanta, Miami) (Kansas City, Dallas) (Dallas, Houston)

Total weight is 14.0
Root is: 1
Edges: (1, 0) (3, 2) (1, 3) (2, 4)
Shortest Path

- Find a shortest path between two vertices in the graph
- The *shortest path* between two vertices is a path with the minimum total weight
- A well-known algorithm for finding a shortest path between two vertices was discovered by Edsger Dijkstra, a Dutch computer scientist
- In order to find a shortest path from vertex \( s \) to vertex \( v \), Dijkstra’s algorithm finds the shortest path from \( s \) to all vertices

Edsger W. Dijkstra
Single Source Shortest Path Algorithm

Input: a graph G = (V, E) with non-negative weights
Output: a shortest path tree with the source vertex s as the root

ShortestPathTree getShortestPath(s) {
    Let T be a set that contains the vertices whose paths to s are known; Initially T is empty;
    Set cost[s] = 0; and cost[v] = infinity for all other vertices in V;
    while (size of T < n) {
        Find u not in T with the smallest cost[u];
        Add u to T;
        for (each (u, v) in E)
            if (v not in T and cost[v] > cost[u] + w(u, v)) {
                cost[v] = w(u, v) + cost[u];
                parent[v] = u;
            }
    }
}

This algorithm is very similar to Prim’s for finding a minimum spanning tree:
Both algorithms divide the vertices into two sets: T and V – T
    In the case of Prim’s algorithm, set T contains the vertices that are already added to the tree
    In the case of Dijkstra’s, set T contains the vertices whose shortest paths to the source have been found
Both algorithms repeatedly find a vertex from V – T and add it to T
    In the case of Prim’s algorithm, the vertex is adjacent to some vertex in the set with the minimum weight on the edge
    In Dijkstra’s algorithm, the vertex is adjacent to some vertex in the set with the minimum total cost to the source
Minimum Spanning Tree vs. Shortest Path Algorithm

Minimum Spanning Tree

\[ \text{cost}[v] = w(u, v) \]

Shortest Path from node A

\[ \text{cost}[v] = w(u, v) + \text{cost}[u] \]
Single Source Shortest Path Algorithm

T contains vertices whose shortest path to s have been found.

V - T contains vertices whose shortest path to s have not been found.

Adjust the cost for neighbors of u.
SP Algorithm Example (Step 0)

(a)

(b)

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<table>
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<tr>
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<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
SP Algorithm Example (Step 1)

(a) Graph with nodes and edges labeled with costs.

(b) Table showing costs and parent nodes.

- Cost Table:
  - 8 0 5 10 ∞ ∞ 9
  - 0 1 2 3 4 5 6

- Parent Table:
  - 1 -1 1 1
  - 0 1 2 3 4 5 6
SP Algorithm Example (Step 2)
SP Algorithm Example (Step 3)

(a) A graph with nodes and edges labeled with costs.

(b) A table showing the costs and parents for the algorithm.

The table includes columns for each node and the cost to reach that node.

Cost Table:

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<th>5</th>
<th>10</th>
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<tbody>
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<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
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</tbody>
</table>

Parents Table:

<table>
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<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>
SP Algorithm Example (Step 4)

(a) Graph representation with edges and costs.

(b) Cost and parent information tables.

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SP Algorithm Example (Step 5)

(a) Graph with edges and weights.

(b) Cost and parent matrices.

Cost:

<table>
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<th>6</th>
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<th>5</th>
<th>10</th>
<th>18</th>
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Parent:

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</table>
SP Algorithm Example (Step 6)

(a)

(b)

(cost)

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</tbody>
</table>

(parent)

<table>
<thead>
<tr>
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<td>3</td>
<td>4</td>
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<td>6</td>
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</tr>
</tbody>
</table>
SP Algorithm Example (Step 7)

(a)

(b)

(cost

\begin{array}{ccccccc}
6 & 0 & 5 & 10 & 15 & 10 & 8 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}

(parent

\begin{array}{ccccccc}
2 & -1 & 1 & 1 & 5 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}

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AbstractGraph.Tree

WeightedGraph.ShortestPathTree

- `cost: int[]`

+ `ShortestPathTree(source: int, parent: int[], searchOrder: List<Integer>, cost: int[])`
+ `getCost(v: int): int`
+ `printAllPaths(): void`

`cost[v]` stores the cost for the path from the source to `v`.

Constructs a shortest path tree with the specified `source`, `parent` array, `searchOrder`, and `cost` array.

Returns the cost for the path from the source to vertex `v`.

Displays all paths from the source.
/** Find single source shortest paths */

public ShortestPathTree getShortestPath(int sourceVertex) {
    // cost[v] stores the cost of the path from v to the source
    double[] cost = new double[getSize()];
    for (int i = 0; i < cost.length; i++)
        cost[i] = Double.POSITIVE_INFINITY; // Initial cost set to infinity
    cost[sourceVertex] = 0; // Cost of source is 0
    // parent[v] stores the previous vertex of v in the path
    int[] parent = new int[getSize()];
    parent[sourceVertex] = -1; // The parent of source is set to -1
    // T stores the vertices whose path found so far
    List<Integer> T = new ArrayList<>();
    // Expand T
    while (T.size() < getSize()) {
        // Find smallest cost v in V - T
        int u = -1; // Vertex to be determined
        double currentMinCost = Double.POSITIVE_INFINITY;
        for (int i = 0; i < getSize(); i++)
            if (!T.contains(i) && cost[i] < currentMinCost) {
                currentMinCost = cost[i];
                u = i;
            }
        T.add(u); // Add a new vertex to T
        // Adjust cost[v] for v that is adjacent to u and v in V - T
        for (Edge e : neighbors.get(u)) {
            if (!T.contains(e.v) && cost[e.v] > cost[u] + ((WeightedEdge)e).weight) {
                cost[e.v] = cost[u] + ((WeightedEdge)e).weight;
                parent[e.v] = u;
            }
        }
    } // End of while
    // Create a ShortestPathTree
    return new ShortestPathTree(sourceVertex, parent, T, cost);
}
/** ShortestPathTree is an inner class in WeightedGraph */
public class ShortestPathTree extends Tree {
  private double[] cost; // cost[v] is the cost from v to source

  /** Construct a path */
  public ShortestPathTree(int source, int[] parent,
      List<Integer> searchOrder, double[] cost) {
    super(source, parent, searchOrder);
    this.cost = cost;
  }

  /** Return the cost for a path from the root to vertex v */
  public double getCost(int v) {
    return cost[v];
  }

  /** Print paths from all vertices to the source */
  public void printAllPaths() {
    System.out.println("All shortest paths from " +
        vertices.get(getRoot()) + " are:");
    for (int i = 0; i < cost.length; i++) {
      printPath(i); // Print a path from i to the source
      System.out.println("(cost: " + cost[i] + ")"); // Path cost
    }
  }
}

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public class TestShortestPath {
    public static void main(String[] args) {
        String[] vertices = {
            "Seattle", "San Francisco", "Los Angeles", 
            "Atlanta", "Miami", "Dallas", "Houston"};

        int[][] edges = {
            {0, 1, 807}, {0, 3, 1331}, {0, 5, 2097},
            {1, 0, 807}, {1, 2, 381}, {1, 3, 1267},
            {2, 1, 381}, {2, 3, 1015}, {2, 4, 1663}, {2, 10, 1435},
            {3, 0, 1331}, {3, 1, 1267}, {3, 2, 1015}, {3, 4, 599}, {3, 5, 1003},
            {4, 2, 1663}, {4, 3, 599}, {4, 5, 533}, {4, 7, 1260}, {4, 8, 864}, {4, 10, 496},
            {5, 0, 2097}, {5, 3, 1003}, {5, 4, 533}, {5, 6, 983}, {5, 7, 787},
            {6, 5, 983}, {6, 7, 214},
            {7, 4, 1260}, {7, 5, 787}, {7, 6, 214}, {7, 8, 888},
            {8, 4, 864}, {8, 7, 888}, {8, 9, 661}, {8, 10, 781}, {8, 11, 810},
            {9, 8, 661}, {9, 11, 1187},
            {10, 2, 1435}, {10, 4, 496}, {10, 8, 781}, {10, 11, 239},
            {11, 8, 810}, {11, 9, 1187}, {11, 10, 239}
        };

        WeightedGraph<String> graph1 = new WeightedGraph<>(vertices, edges);

        WeightedGraph<String>.ShortestPathTree tree1 =
            graph1.getShortestPath(graph1.getIndex("Chicago"));

        tree1.printAllPaths();
    }
}
// Display shortest paths from Houston to Chicago
System.out.print("Shortest path from Houston to Chicago: ");
java.util.List<String> path = tree1.getPath(graph1.getIndex("Houston"));
for (String s: path) {
    System.out.print(s + " ");
}

edges = new int[][] {
    {0, 1, 2}, {0, 3, 8},
    {1, 0, 2}, {1, 2, 7}, {1, 3, 3},
    {2, 1, 7}, {2, 3, 4}, {2, 4, 5},
    {3, 0, 8}, {3, 1, 3}, {3, 2, 4}, {3, 4, 6},
    {4, 2, 5}, {4, 3, 6}
};
WeightedGraph<Integer> graph2 = new WeightedGraph<>(edges, 5);

WeightedGraph<Integer>.ShortestPathTree tree2 = graph2.getShortestPath(3);

System.out.println("\n");
tree2.printAllPaths();
}
All shortest paths from Chicago are:
A path from Chicago to Seattle: Chicago Seattle (cost: 2097.0)
A path from Chicago to San Francisco:
  Chicago Denver San Francisco (cost: 2270.0)
A path from Chicago to Los Angeles:
  Chicago Denver Los Angeles (cost: 2018.0)
A path from Chicago to Denver: Chicago Denver (cost: 1003.0)
A path from Chicago to Kansas City: Chicago Kansas City (cost: 533.0)
A path from Chicago to Chicago: Chicago (cost: 0.0)
A path from Chicago to Boston: Chicago Boston (cost: 983.0)
A path from Chicago to New York: Chicago New York (cost: 787.0)
A path from Chicago to Atlanta:
  Chicago Kansas City Atlanta (cost: 1397.0)
A path from Chicago to Miami:
  Chicago Kansas City Atlanta Miami (cost: 2058.0)
A path from Chicago to Dallas: Chicago Kansas City Dallas (cost: 1029.0)
A path from Chicago to Houston:
  Chicago Kansas City Dallas Houston (cost: 1268.0)
Shortest path from Houston to Chicago:
  Houston Dallas Kansas City Chicago

All shortest paths from 3 are:
A path from 3 to 0: 3 1 0 (cost: 5.0)
A path from 3 to 1: 3 1 (cost: 3.0)
A path from 3 to 2: 3 2 (cost: 4.0)
A path from 3 to 3: 3 (cost: 0.0)
A path from 3 to 4: 3 4 (cost: 6.0)
The Weighted Nine Tail Problem

- Reminder: the nine tail problem is to find the minimum number of the moves that lead to all coins face down
- Each move flips a head coin and its neighbors
- The *weighted nine tail problem* assigns the number of the flips as a weight on each move
  - For example, you can move the coin in the upper-left corner below by flipping the three coins, so the weight for this move is 3.

![Diagram of coin movement](image-url)
**WeightedNineTailModel**

### NineTailModel

- `#tree: AbstractGraph<Integer>.Tree`
- `+NineTailModel()`
- `+getShortestPath(nodeIndex: int): List<Integer>`
- `-getEdges(): List<AbstractGraph.Edge>`
- `+getNode(index: int): char[]`
- `+getIndex(node: char[]): int`
- `+getFlippedNode(node: char[], position: int): int`
- `+flipACell(node: char[], row: int, column: int): void`
- `+printNode(node: char[]): void`

A tree rooted at node 511.

Constructs a model for the nine tails problem and obtains the tree.

Returns a path from the specified node to the root. The path returned consists of the node labels in a list.

Returns a list of `Edge` objects for the graph.

Returns a node consisting of nine characters of H’s and T’s.

Returns the index of the specified node.

Flips the node at the specified position and returns the index of the flipped node.

Flips the node at the specified row and column.

Displays the node to the console.

### WeightedNineTailModel

- `+WeightedNineTailModel()`
- `+getNumberOfFlips(u: int): int`
- `-getNumberOfFlips(u: int, v: int): int`
- `-getEdges(): List<WeightedEdge>`

Constructs a model for the weighted nine tails problem and obtains a `ShortestPathTree` rooted from the target node.

Returns the number of flips from node u to the target node 511.

Returns the number of different cells between the two nodes.

Gets the weighted edges for the weighted nine tail problem.
import java.util.*;

public class WeightedNineTailModel extends NineTailModel {

    /** Construct a model */
    public WeightedNineTailModel() {
        // Create edges
        List<WeightedEdge> edges = getEdges();
        // Create a graph
        WeightedGraph<Integer> graph = new WeightedGraph<>(edges, NUMBER_OF_NODES);
        // Obtain a shortest path tree rooted at the target node
        tree = graph.getShortestPath(511);
    }

    /** Create all edges for the graph */
    private List<WeightedEdge> getEdges() {
        // Store edges
        List<WeightedEdge> edges = new ArrayList<>();
        for (int u = 0; u < NUMBER_OF_NODES; u++) {
            for (int k = 0; k < 9; k++) {
                char[] node = getNode(u); // Get the node for vertex u
                if (node[k] == 'H') {
                    int v = getFlippedNode(node, k);
                    int numberOfFlips = getNumberOfFlips(u, v);
                    // Add edge (v, u) for a legal move from node u to node v
                    edges.add(new WeightedEdge(v, u, numberOfFlips));
                }
            }
        }
        return edges;
    }
}
private static int getNumberOfFlips(int u, int v) {
    char[] node1 = getNode(u);
    char[] node2 = getNode(v);
    int count = 0; // Count the number of different cells
    for (int i = 0; i < node1.length; i++)
        if (node1[i] != node2[i]) count++;
    return count;
}

public int getNumberOfFlips(int u) {
    return (int)((WeightedGraph<Integer>.ShortestPathTree)tree).getCost(u);
}
import java.util.Scanner;

public class WeightedNineTail {
    public static void main(String[] args) {
        // Prompt the user to enter nine coins' Hs and Ts
        System.out.print("Enter an initial nine coins' Hs and Ts: ");
        Scanner input = new Scanner(System.in);
        String s = input.nextLine();
        char[] initialNode = s.toCharArray();

        WeightedNineTailModel model = new WeightedNineTailModel();
        java.util.List<Integer> path =
            model.getShortestPath(NineTailModel.getIndex(initialNode));

        System.out.println("The steps to flip the coins are ");
        for (int i = 0; i < path.size(); i++)
            NineTailModel.printNode(NineTailModel.getNode(path.get(i)));

        System.out.println("The number of flips is "+
            model.getNumberOfFlips(NineTailModel.getIndex(initialNode)));
    }
}