AVL Trees

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Objectives

- To know what an ***AVL tree*** is
- To understand how to ***rebalance*** a tree using the ***LL rotation***, ***LR rotation***, ***RR rotation***, and ***RL rotation***
- To know how to design the **`AVLTree`** class
- To ***insert*** elements into an AVL tree
- To implement node ***rebalancing***
- To ***delete*** elements from an AVL tree
- To implement and test the **`AVLTree`** class
- To analyze the ***complexity*** of search, insert, and delete operations in AVL trees

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Why AVL Trees?

- The search, insertion, and deletion time for a binary search tree is dependent on the **height of the tree**
  - In the worst case, the height is **n - 1**, so worse time complexity is **O(n)**
- If a tree is ***perfectly balanced***, i.e., a **complete binary tree**, its height is **log n (e.g., sorting heaps)**
  - So, search, insertion, and deletion time would be **O(log n)**
- Can we maintain a perfectly balanced tree?
  - Yes, but it will be costly to do so
  - The compromise is to maintain a ***well-balanced tree***, i.e., the heights of two subtrees for every node are about the same

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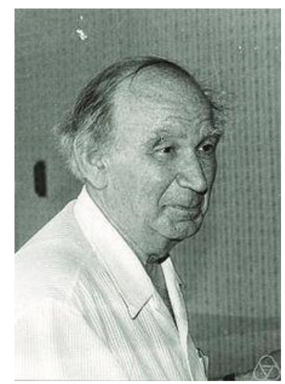
What are AVL Trees?

- ***AVL trees*** are well-balanced binary search trees were invented by two Russian computer scientists: **Georgy Adelson-Velsky** and **Evgenii Landis** in 1962 at the Institute for Theoretical and Experimental Physics in Moscow
  - In an AVL tree, the difference between the heights of two subtrees **for every node** is **0** or **1**
    - Can be proved that the **maximum height** of an AVL tree is **O(log n)**

Adelson-Velsky (left) also developed Kaissa, the first computer chess champion (1974)



Evgenii Mikhailovich Landis

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Balance Factor/Left-Heavy/Right-Heavy

- The process for inserting or deleting an element in an AVL tree is the same as in a regular binary search tree
  - The difference is that you may have to ***rebalance*** the tree after an insertion or deletion operation
- The ***balance factor*** of a node is the height of its right subtree minus the height of its left subtree
- A node is said to be ***balanced*** if its balance factor is -1, 0, or 1
  - A node is said to be ***left-heavy*** if its balance factor is -1
  - A node is said to be ***right-heavy*** if its balance factor is +1

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Balancing Trees

- If a node is **not balanced** (i.e., its balance factor is not -1, 0, or 1) after an insertion or deletion operation, you need to rebalance it:
  - The process of rebalancing a node is called a ***rotation***
- There are four possible rotations:
  - ***LL rotation*** (***left-heavy left-heavy rotation***)
  - ***RR rotation*** (***right-heavy right-heavy rotation***)
  - ***LR rotation*** (***left-heavy right-heavy rotation***)
  - ***RL rotation*** (***right-heavy left-heavy rotation***)

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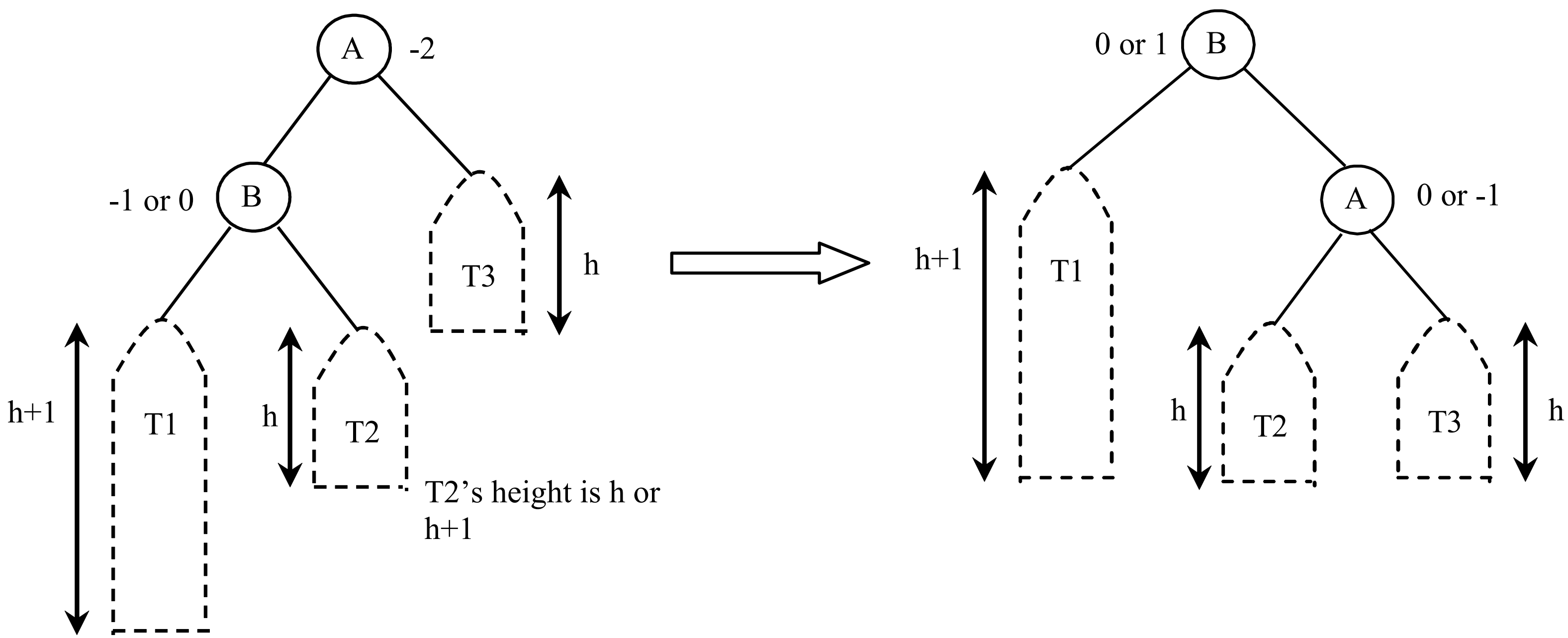
LL imbalance and LL rotation

- ***LL Rotation***: An ***left-heavy, left-heavy imbalance*** occurs at a node **A** if **A** has a balance factor **-2 (left-heavy)** and its left child **B** has a balance factor **-1 (left-heavy)** or **0**
  - This type of imbalance can be fixed by performing a single **right rotation** at **A:**



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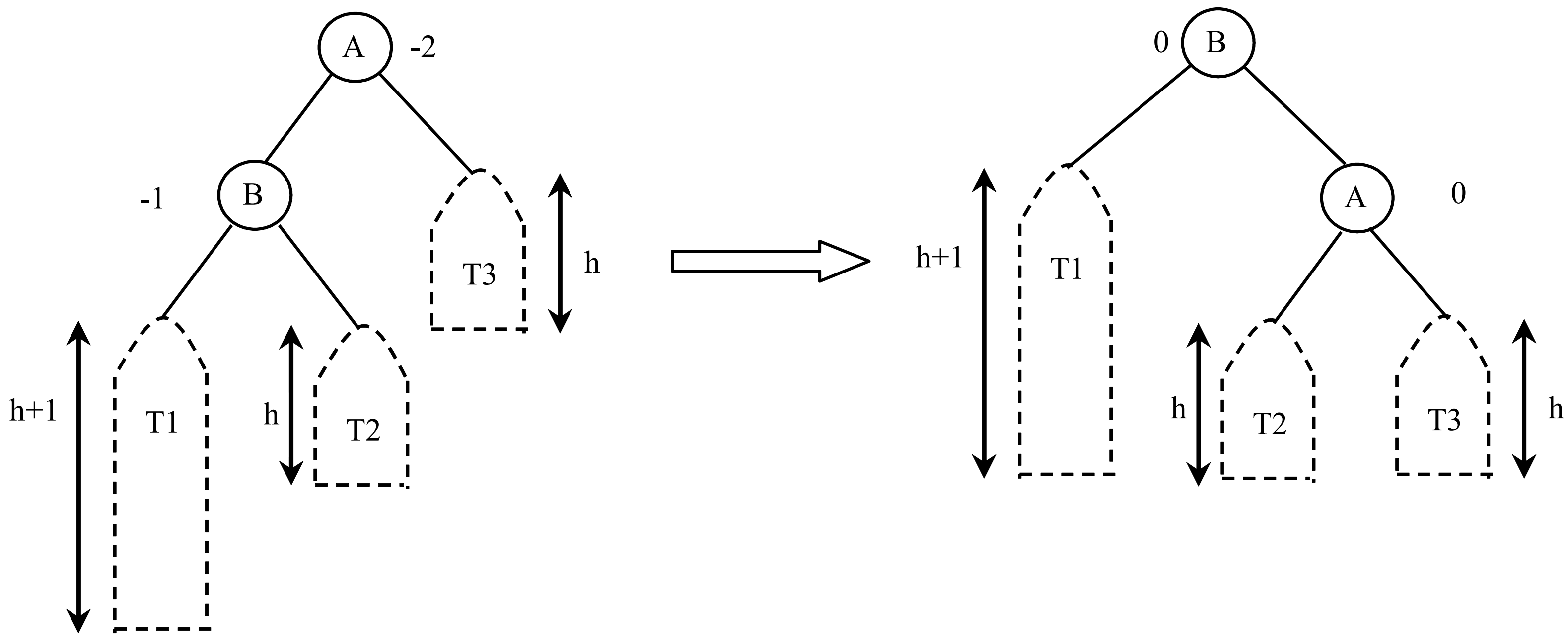
LL imbalance and LL rotation

- **LL case 1:** If the left child **B** has a balance factor **-1**



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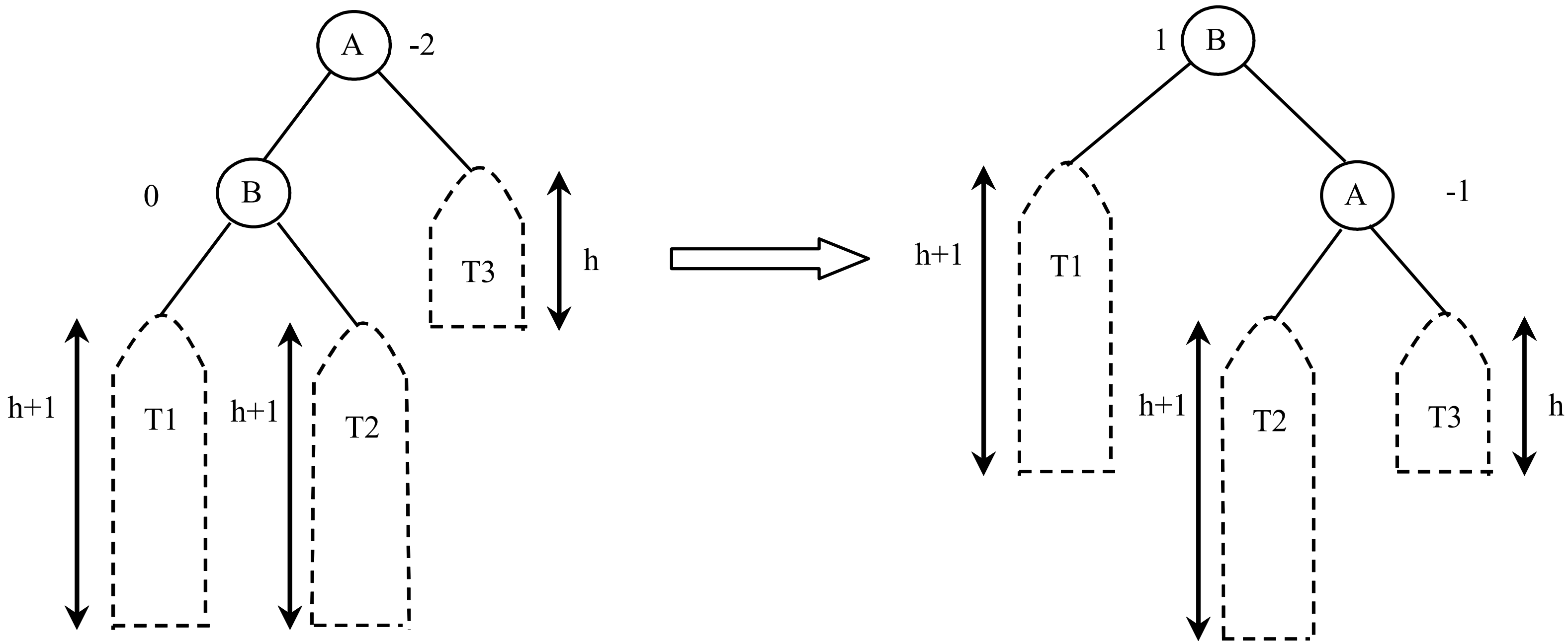
LL imbalance and LL rotation

- **LL case 2:** If the left child **B** has a balance factor of **0**



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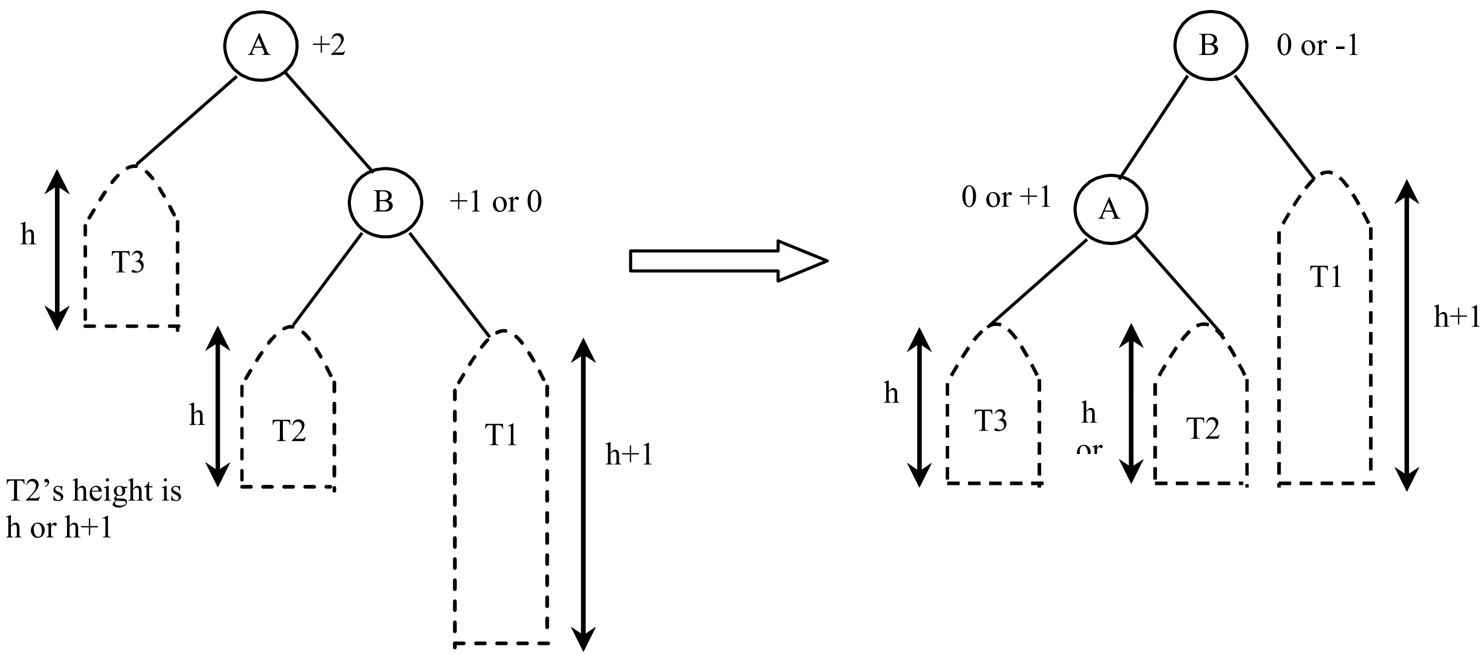
RR imbalance and RR rotation

- ***RR Rotation***: An ***RR imbalance*** occurs at a node **A** if **A** has a balance factor **+2 (right-heavy)** and a right child **B** has a balance factor **+1 (right-heavy)** or **0**
  - This type of imbalance can be fixed by performing a single **left rotation** at **A**:



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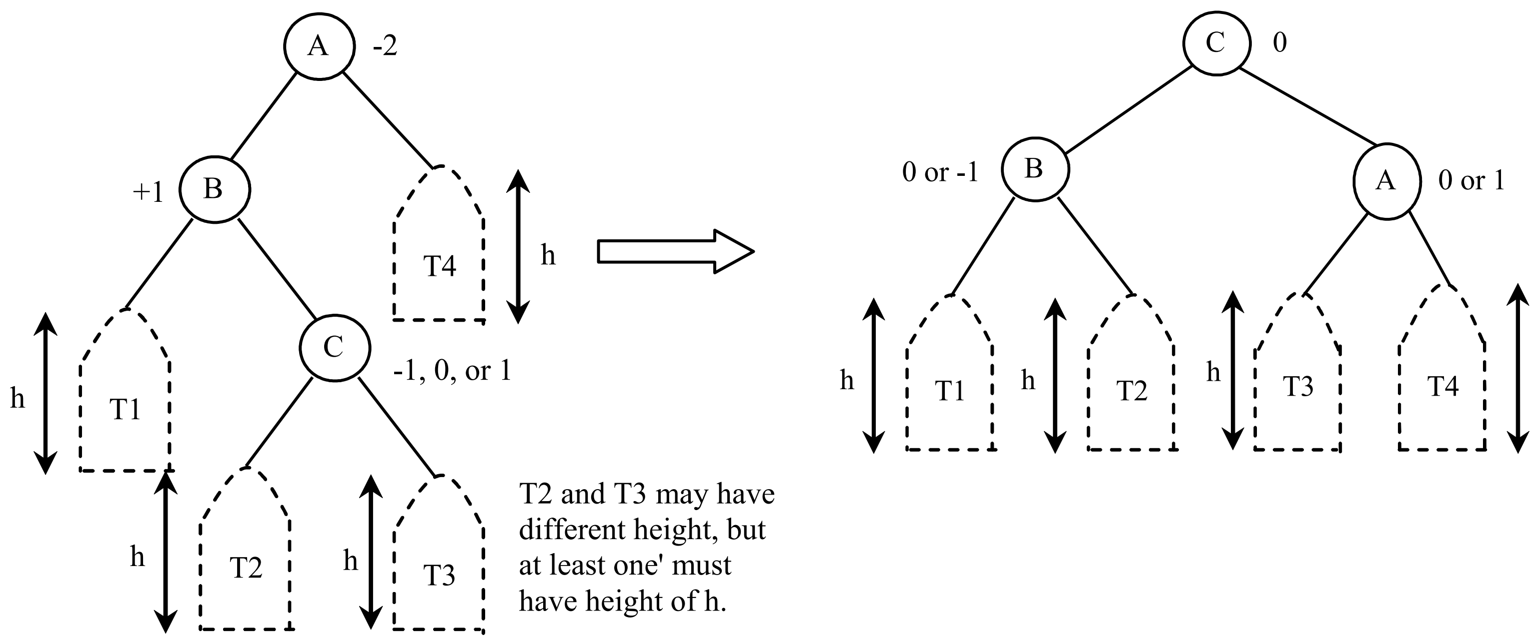
LR imbalance and LR rotation

- ***LR Rotation***: An ***LR imbalance*** occurs at a node **A** if **A** has a balance factor **-2 (left-heavy)** and a left child **B** has a balance factor **+1 (right-heavy)**
  - Assume **B**'s right child is **C**
  - This type of imbalance can be fixed by performing a double rotation: first a single left rotation at **B** and then a single right rotation at **A**



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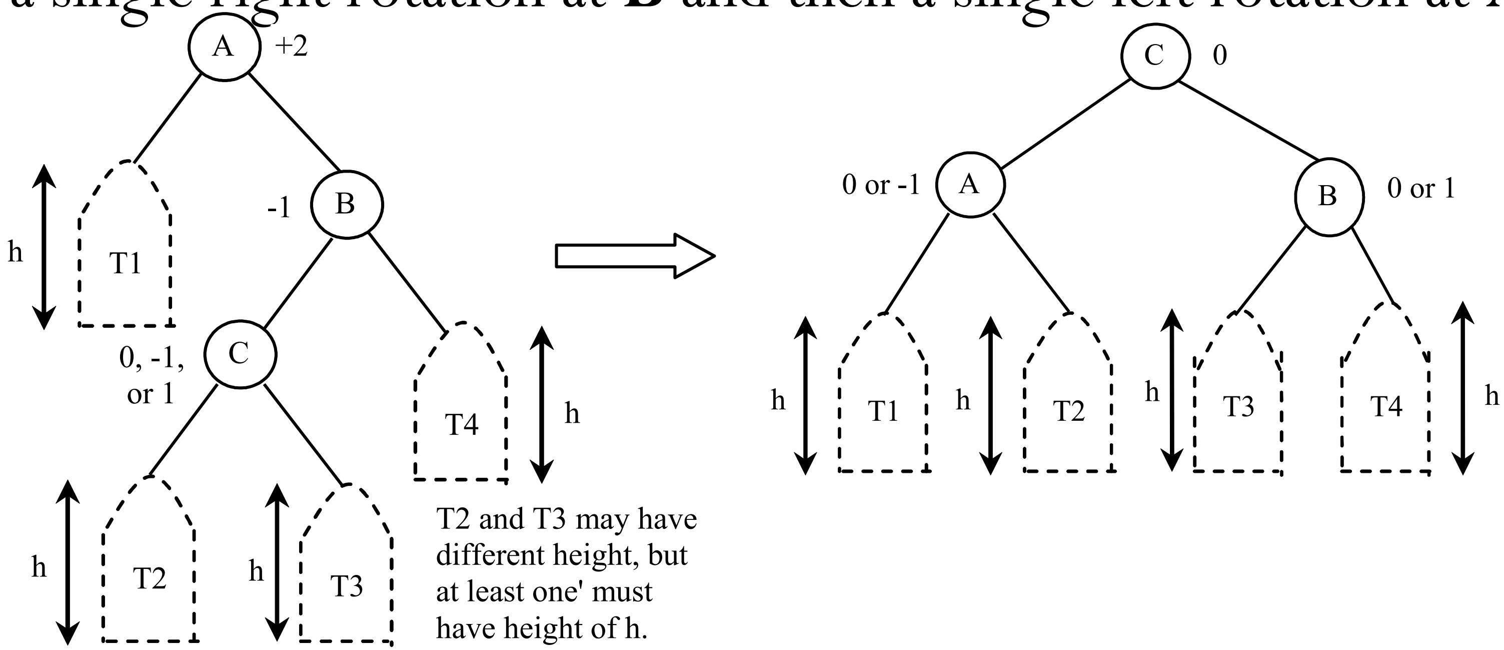
RL imbalance and RL rotation

- ***RL Rotation***: An ***RL imbalance*** occurs at a node **A** if **A** has a balance factor **+2 (right-heavy)** and a right child **B** has a balance factor **-1 (left-heavy)**
  - Assume B's left child is C
  - This type of imbalance can be fixed by performing a double rotation: first a single right rotation at **B** and then a single left rotation at **A**



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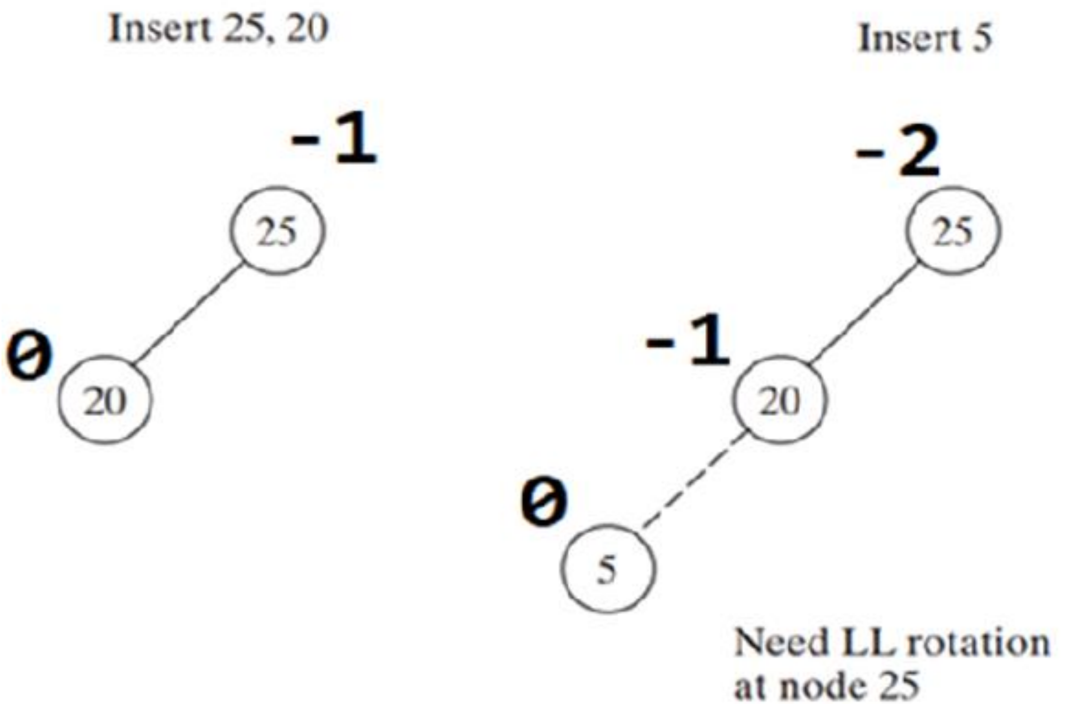
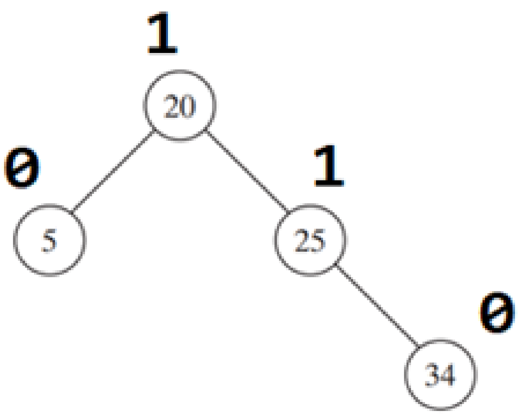
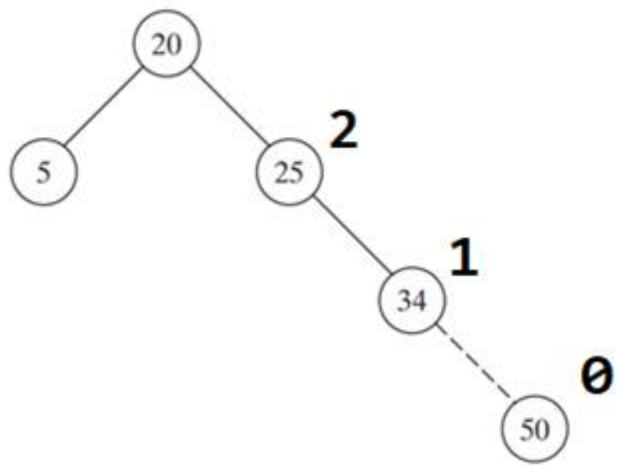

Insert 25, 20
-1
25
0
20
Insert 5
-2
25
-1
20
0
5
Need LL rotation
at node 25
0
20
0
5
0
25
Balanced

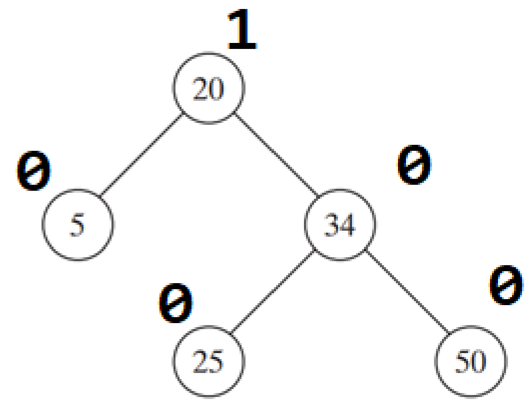
Insert 34



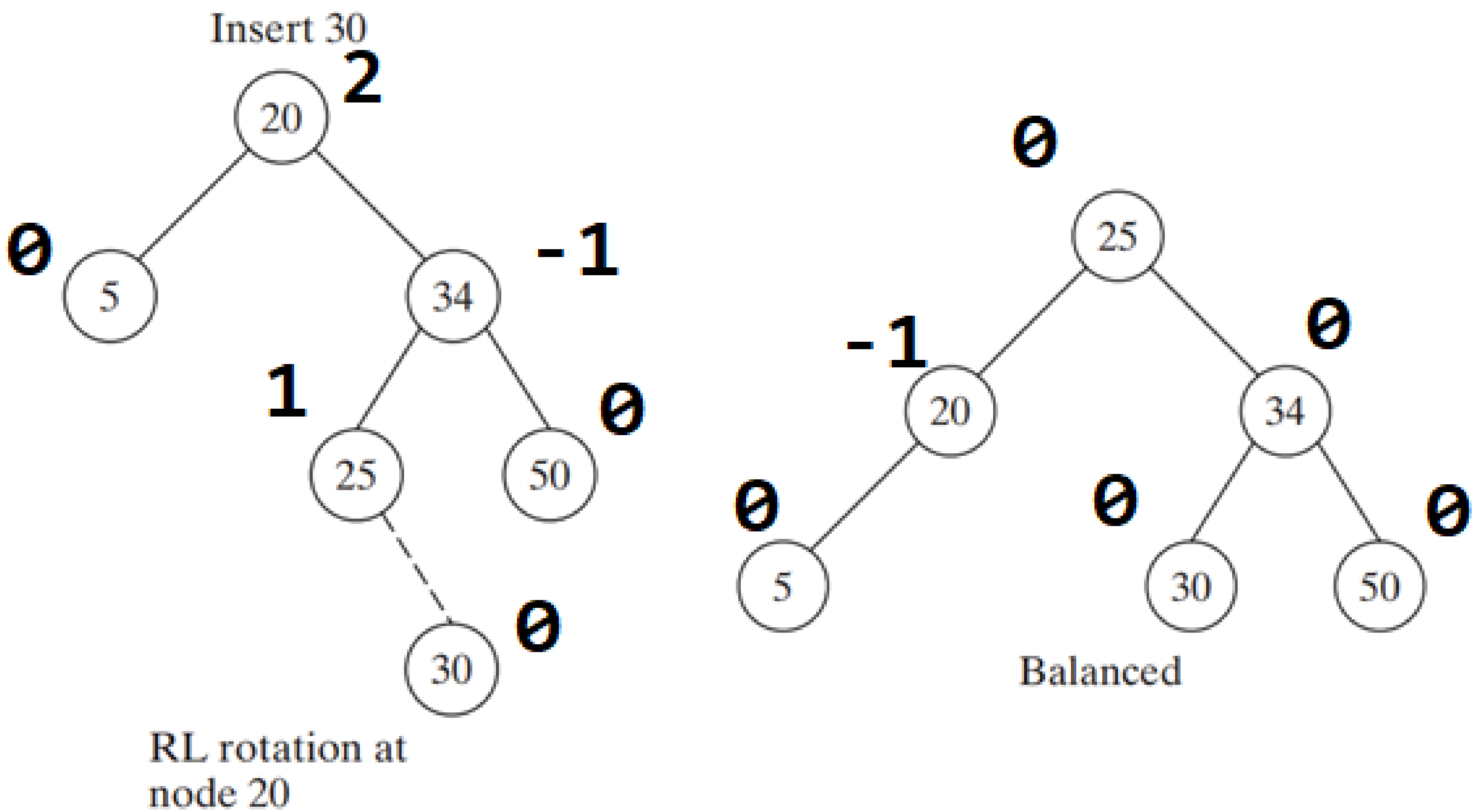
Insert 50



Need RR rotation
at node 25



Balanced



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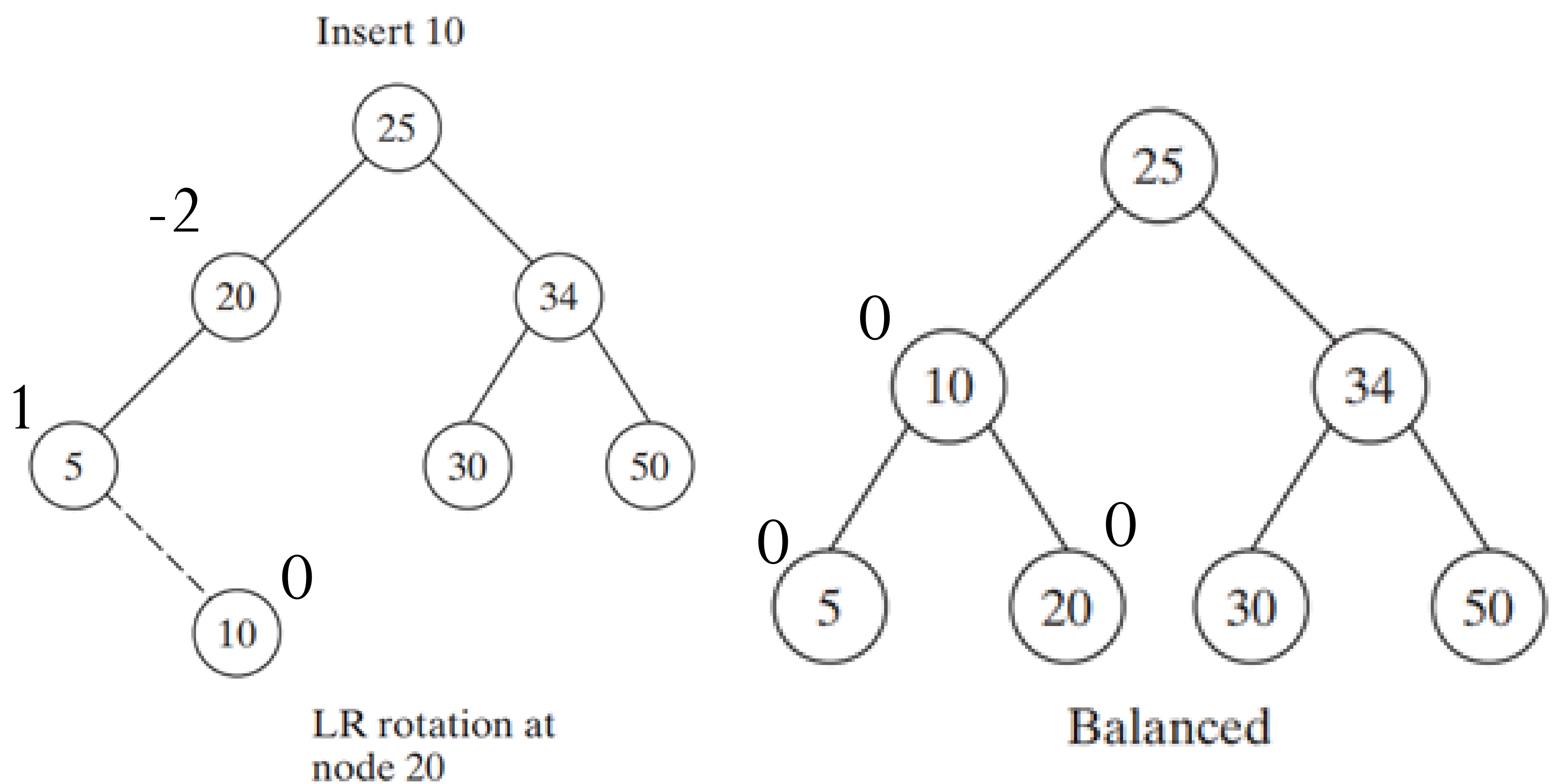

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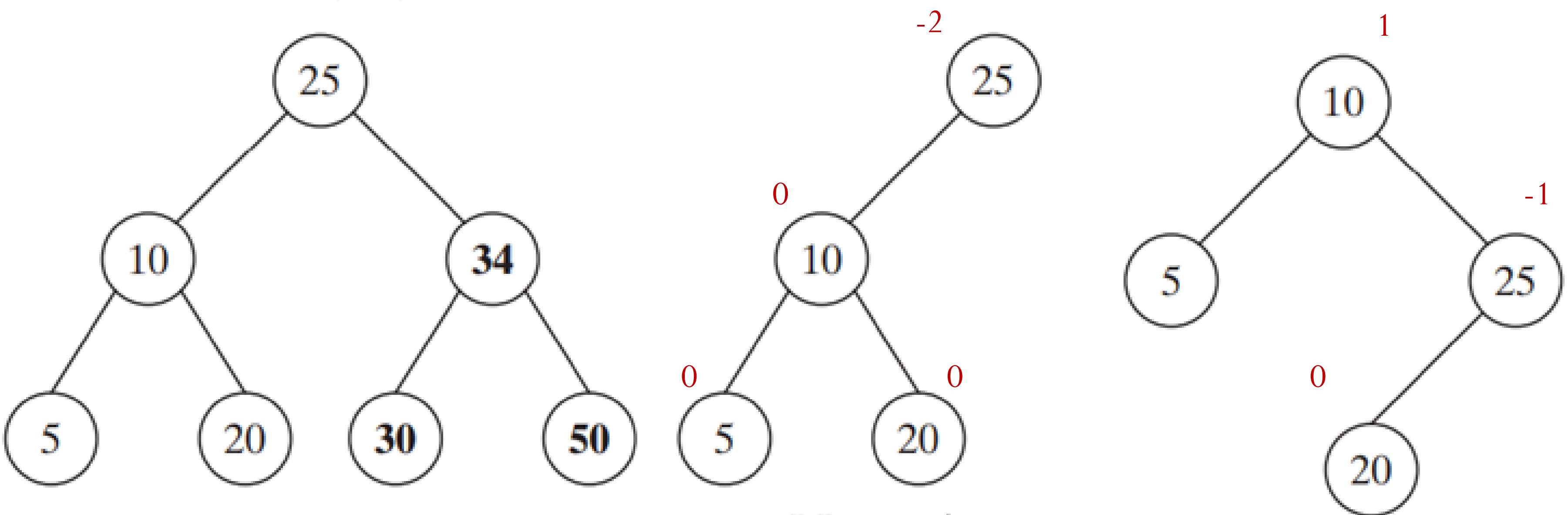
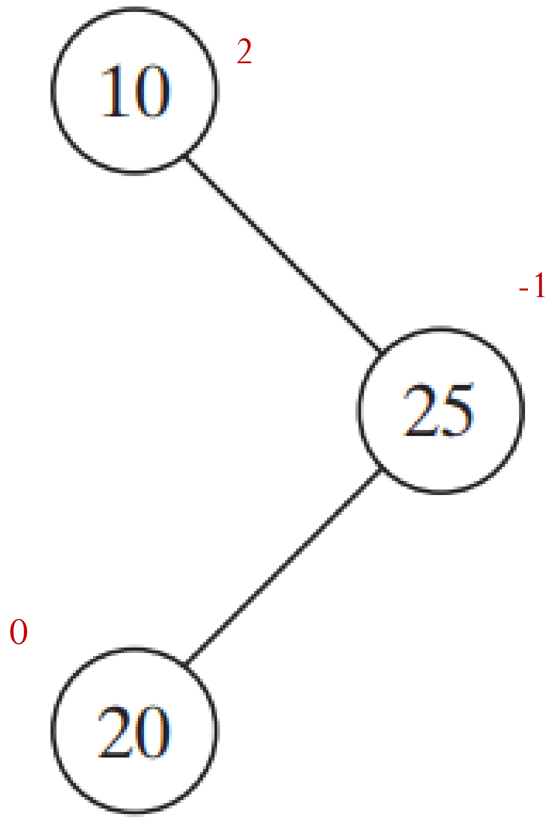
Delete 34, 30, 50



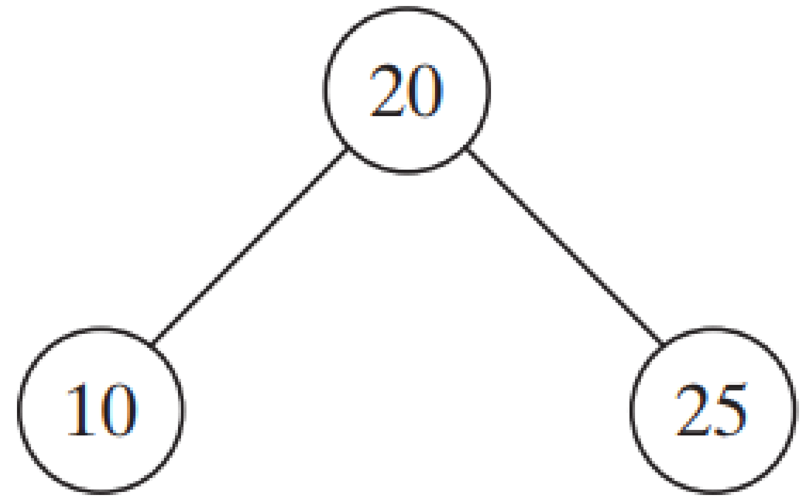
LL rotation
at node 25

Balanced

After 5 is deleted



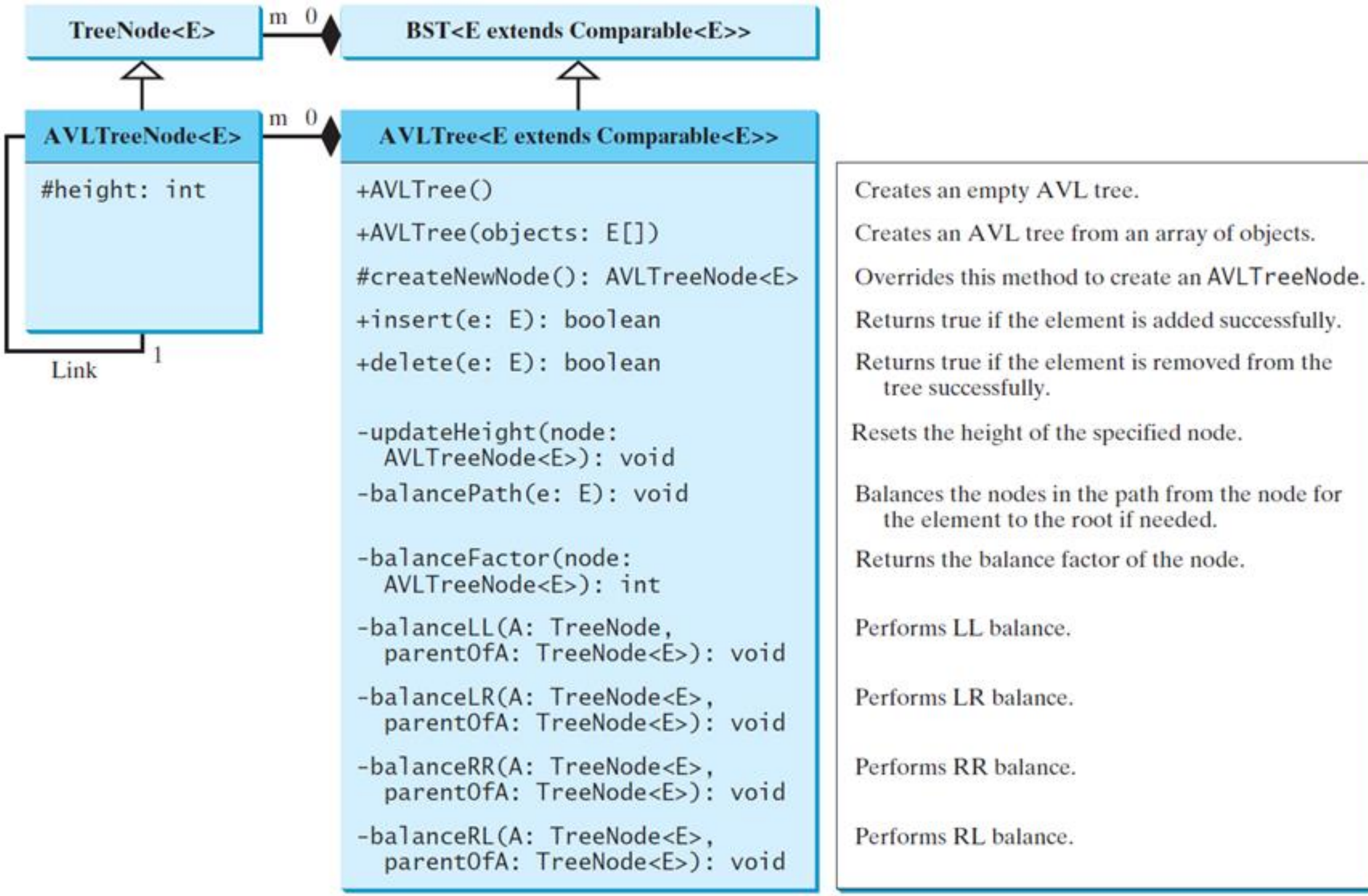
RL rotation at 10



Balanced

Designing Classes for AVL Trees

- An AVL is a binary search tree, so we can define the **AVLTree** class to extend the **BST** class



```java
public class AVLTree<E extends Comparable<E>> extends BST<E> {
  /** AVLTreeNode is TreeNode plus height */
  protected static class AVLTreeNode<E extends Comparable<E>>
      extends BST.TreeNode<E> {
    protected int height = 0; // New data field
    public AVLTreeNode(E o) {
      super(o);
    }
  }

  @Override /** Override createNewNode to create an AVLTreeNode */
  protected AVLTreeNode<E> createNewNode(E e) {
    return new AVLTreeNode<E>(e);
  }

  /** Create a default AVL tree */
  public AVLTree() {
  }

  /** Create an AVL tree from an array of objects */
  public AVLTree(E[] objects) {
    super(objects);
  }
```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```java
@Override /** Insert an element and rebalance if necessary */
public boolean insert(E e) {
  boolean successful = super.insert(e);
  if (!successful)
    return false; // e is already in the tree
  else {
    balancePath(e); // Balance from e to the root if necessary
  }
  return true; // e is inserted
}
```


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```java
/** Balance the nodes in the path from the specified
 * node to the root if necessary
 */
private void balancePath(E e) {
  java.util.ArrayList<TreeNode<E>> path = path(e);// from root to e
  for (int i = path.size() - 1; i >= 0; i--) {
    AVLTreeNode<E> A = (AVLTreeNode<E>)(path.get(i));
    updateHeight(A);
    AVLTreeNode<E> parentOfA = (A == root) ? null :
      (AVLTreeNode<E>)(path.get(i - 1));
    switch(balanceFactor(A)) {
      case -2:
        if (balanceFactor((AVLTreeNode<E>)A.left) <= 0) {
          balanceLL(A, parentOfA); // Perform LL rotation
        } else {
          balanceLR(A, parentOfA); // Perform LR rotation
        }
        break;
      case +2:
        if (balanceFactor((AVLTreeNode<E>)A.right) >= 0) {
          balanceRR(A, parentOfA); // Perform RR rotation
        } else {
          balanceRL(A, parentOfA); // Perform RL rotation
        }
    }
  }
}
```


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```java
/** Update the height of a specified node */
private void updateHeight(AVLTreeNode<E> node) {
  if (node.left == null && node.right == null) // node is a leaf
    node.height = 0;
  else if (node.left == null) // node has no left subtree
    node.height = 1 + ((AVLTreeNode<E>)(node.right)).height;
  else if (node.right == null) // node has no right subtree
    node.height = 1 + ((AVLTreeNode<E>)(node.left)).height;
  else
    node.height = 1 +
      Math.max(((AVLTreeNode<E>)(node.right)).height,
      ((AVLTreeNode<E>)(node.left)).height);
}

/** Return the balance factor of the node */
private int balanceFactor(AVLTreeNode<E> node) {
  if (node.right == null) // node has no right subtree
    return -node.height;
  else if (node.left == null) // node has no left subtree
    return +node.height;
  else
    return ((AVLTreeNode<E>)node.right).height -
      ((AVLTreeNode<E>)node.left).height;
}
```


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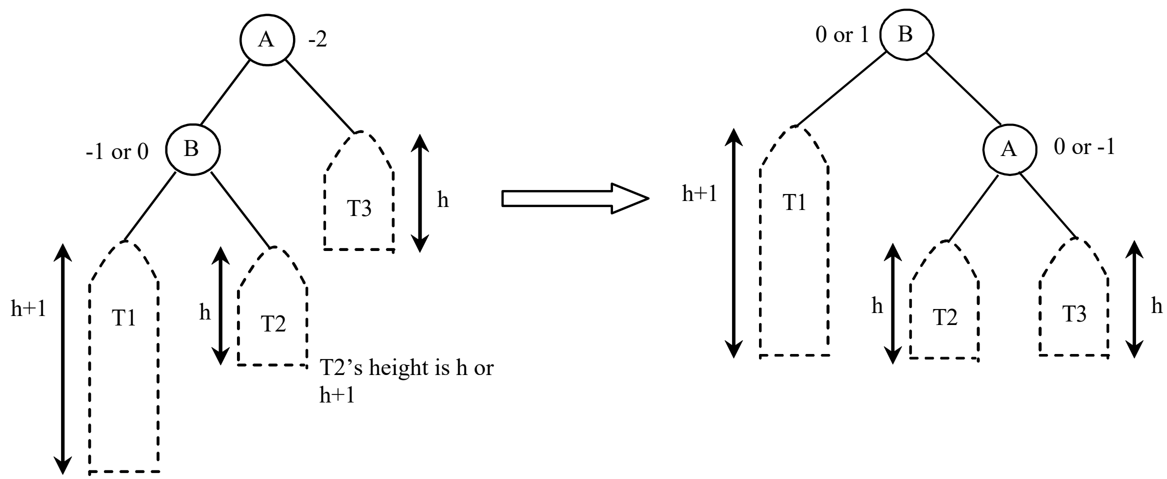
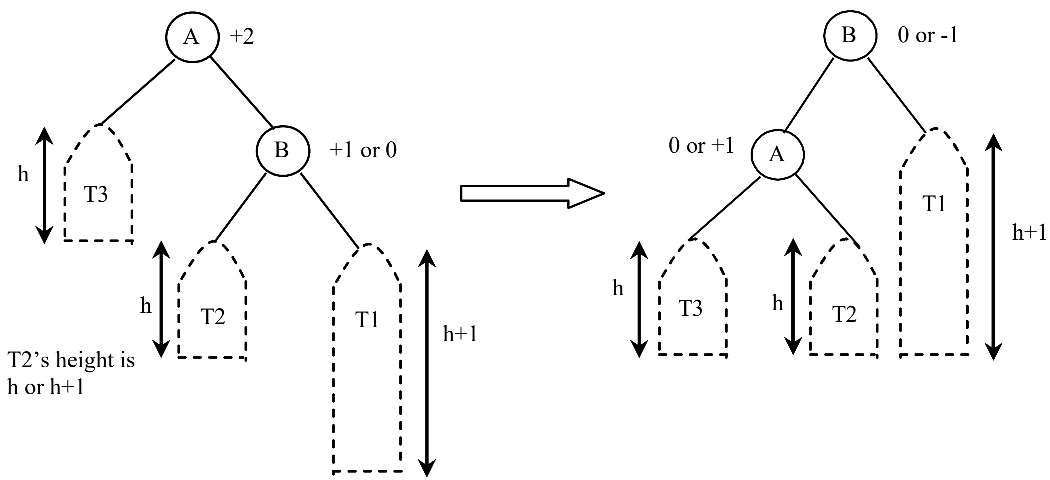
```
/** Balance LL */
private void balanceLL(TreeNode<E> A, TreeNode<E> parentOfA) {
  TreeNode<E> B = A.left; // A is left-heavy and B is left-heavy
  if (A == root) {
    root = B;
  } else {
    if (parentOfA.left == A) {
      parentOfA.left = B;
    } else {
      parentOfA.right = B;
    }
  }
  A.left = B.right; // Make T2 the left subtree of A
  B.right = A; // Make A the left child of B
  updateHeight((AVLTreeNode<E>)A);
  updateHeight((AVLTreeNode<E>)B);
}
```



```java
/** Balance RR */
private void balanceRR(TreeNode<E> A, TreeNode<E> parentOfA) {
  TreeNode<E> B = A.right; // A is right-heavy and B is right-heavy
  if (A == root) {
    root = B;
  } else {
    if (parentOfA.left == A) {
      parentOfA.left = B;
    } else {
      parentOfA.right = B;
    }
  }
  A.right = B.left; // Make T2 the right subtree of A
  B.left = A;
  updateHeight((AVLTreeNode<E>)A);
  updateHeight((AVLTreeNode<E>)B);
}
```



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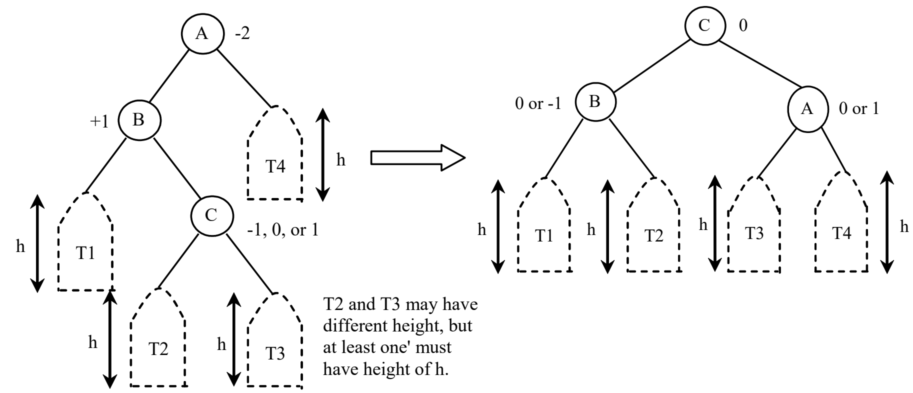
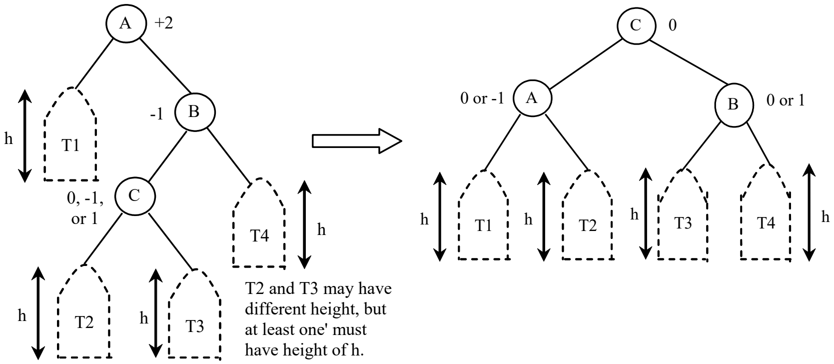
```java
/** Balance LR */
private void balanceLR(TreeNode<E> A, TreeNode<E> parentOfA) {
  TreeNode<E> B = A.left; // we know that A is left-heavy
  TreeNode<E> C = B.right; // we know that B is right-heavy
  if (A == root) {
    root = C;
  } else {
    if (parentOfA.left == A) {
      parentOfA.left = C;
    } else {
      parentOfA.right = C;
    }
  }
  A.left = C.right; // Make T3 the left subtree of A
  B.right = C.left; // Make T2 the right subtree of B
  C.left = B;
  C.right = A;
  // Adjust heights
  updateHeight((AVLTreeNode<E>)A);
  updateHeight((AVLTreeNode<E>)B);
  updateHeight((AVLTreeNode<E>)C);
}
```



```java
/** Balance RL */
private void balanceRL(TreeNode<E> A, TreeNode<E> parentOfA) {
  TreeNode<E> B = A.right; // we know that A is right-heavy
  TreeNode<E> C = B.left; // we know that B is left-heavy
  if (A == root) {
    root = C;
  } else {
    if (parentOfA.left == A) {
      parentOfA.left = C;
    } else {
      parentOfA.right = C;
    }
  }
  A.right = C.left; // Make T2 the right subtree of A
  B.left = C.right; // Make T3 the left subtree of B
  C.left = A;
  C.right = B;
  // Adjust heights
  updateHeight((AVLTreeNode<E>)A);
  updateHeight((AVLTreeNode<E>)B);
  updateHeight((AVLTreeNode<E>)C);
}
```



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```
@Override /** Delete an element from the binary tree.
 * Return true if the element is deleted successfully
 * Return false if the element is not in the tree */
public boolean delete(E element) {
  if (root == null)
    return false; // Element is not in the tree
  // Locate the node to be deleted and also locate its parent node
  TreeNode<E> parent = null;
  TreeNode<E> current = root;
  while (current != null) {
    if (element.compareTo(current.element) < 0) {
      parent = current;
      current = current.left;
    } else if (element.compareTo(current.element) > 0) {
      parent = current;
      current = current.right;
    } else
      break; // Element is in the tree pointed by current
  }
  if (current == null)
    return false; // Element is not in the tree
  // Case 1: current has no left children
  if (current.left == null) {
    // Connect the parent with the right child of the current node
    if (parent == null) {
      root = current.right;
    }
```

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```
      else {
        if (element.compareTo(parent.element) < 0)
          parent.left = current.right;
        else
          parent.right = current.right;
        // Balance the tree if necessary
        balancePath(parent.element);
      }
    } else {
      // Case 2: The current node has a left child
      // Locate the rightmost node in the left subtree of
      // the current node and also its parent
      TreeNode<E> parentOfRightMost = current;
      TreeNode<E> rightMost = current.left;
      while (rightMost.right != null) {
        parentOfRightMost = rightMost;
        rightMost = rightMost.right; // Keep going to the right
      }
      // Replace the element in current by the element in rightMost
      current.element = rightMost.element;
      // Eliminate rightmost node
      if (parentOfRightMost.right == rightMost)
        parentOfRightMost.right = rightMost.left;
      else
        // Special case: parentOfRightMost is current
        parentOfRightMost.left = rightMost.left;
```


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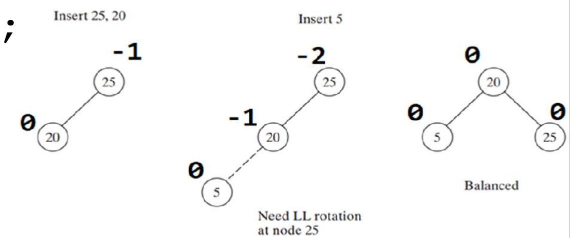
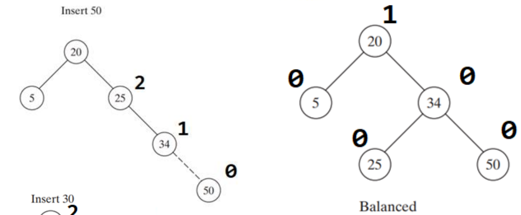
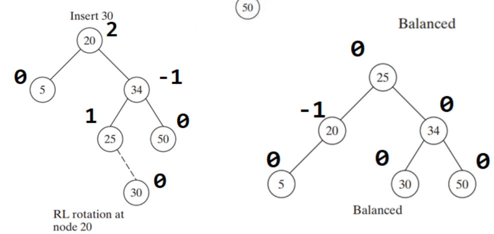
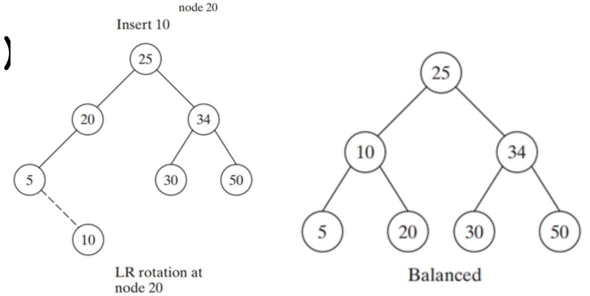
```
      // Balance the tree if necessary
      balancePath(parentOfRightMost.element);
    }
    size--;
    return true; // Element deleted
  }
}
```

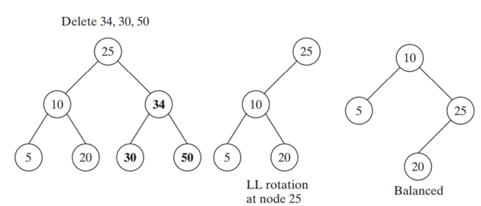
```java
public class TestAVLTree {
  public static void main(String[] args) {
    // Create an AVL tree
    AVLTree<Integer> tree = new AVLTree<>(new Integer[]{25, 20, 5});
    System.out.print("After inserting 25, 20, 5:");
    printTree(tree);

    tree.insert(34);
    tree.insert(50);
    System.out.print("\nAfter inserting 34, 50:")
    printTree(tree);

    tree.insert(30);
    System.out.print("\nAfter inserting 30");
    printTree(tree);

    tree.insert(10);
    System.out.print("\nAfter inserting 10")
    printTree(tree);

    tree.delete(34);
    tree.delete(30);
    tree.delete(50);
    System.out.print("\nAfter removing 34, 30, 50:");
    printTree(tree);
```











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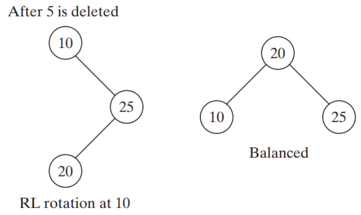
```
    tree.delete(5);
    System.out.print("\nAfter removing 5:");
    printTree(tree);

    System.out.print("\nTraverse the elements in the tree: ");
    for (int e: tree) { // inorder: 10 20 25
      System.out.print(e + " ");
    }
  }

  public static void printTree(BST tree) {
    // Traverse tree
    System.out.print("\nPreorder: ");
    tree.preorder();
    System.out.print("\nInorder (sorted): ");
    tree.inorder();
    System.out.print("\nPostorder: ");
    tree.postorder();
    System.out.print("\nThe number of nodes is " + tree.getSize());
    System.out.println();
  }
}
```



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```
After inserting 25, 20, 5:
Preorder: 20 5 25
Inorder (sorted): 5 20 25
Postorder: 5 25 20
The number of nodes is 3

After inserting 34, 50:
Preorder: 20 5 34 25 50
Inorder (sorted): 5 20 25 34 50
Postorder: 5 25 50 34 20
The number of nodes is 5

After inserting 30
Preorder: 25 20 5 34 30 50
Inorder (sorted): 5 20 25 30 34 50
Postorder: 5 20 30 50 34 25
The number of nodes is 6

After inserting 10
Preorder: 25 10 5 20 34 30 50
Inorder (sorted): 5 10 20 25 30 34 50
Postorder: 5 20 10 30 50 34 25
The number of nodes is 7

After removing 34, 30, 50:
Preorder: 10 5 25 20
Inorder (sorted): 5 10 20 25
Postorder: 5 20 25 10
The number of nodes is 4

After removing 5:
Preorder: 20 10 25
Inorder (sorted): 10 20 25
Postorder: 10 25 20
The number of nodes is 3

Traverse the elements in the tree:
10 20 25
```


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AVL Tree Time Complexity Analysis

- Let **G(h)** denote the minimum number of the nodes in an AVL tree with height **h**
  - **G(1) = 1**
  - **G(2) = 2**
  - The minimum number of nodes in an AVL tree with height **h>2** must have two minimum subtrees: one with height **h-1** and the other with height **h-2**
  - Thus, **G(h) = G(h - 1) + G(h - 2) + 1**
    - A Fibonacci number at index **i** can be described using the recurrence relation: **F(i) = F(i - 1) + F(i - 2)**
    - Therefore, the function **G(h)** is essentially the same as **F(i)**
    - It can be proven that **h < 1.4405 log(n + 2) - 1.3277** where **n** is the number of nodes in the tree
    - Hence, the height of an AVL tree is **O(log n)**

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AVL Tree Time Complexity Analysis

- The search, insert, and delete methods involve only the nodes along a single path in the tree
  - The **`updateHeight`** and **`balanceFactor`** methods are executed in a constant time for each node in the path
  - The **`balancePath`** method is executed in a constant time for a node in the path
  - Thus, the time complexity for the search, insert, and delete methods is **O(log n)**

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