Relations

CSE 215, Foundations of Computer Science
Stony Brook University

http://www.cs.stonybrook.edu/~cse215
Relations on Sets

- A relation is a collection ordered pairs.
- The Less-than Relation for Real Numbers: a relation $L$ from $\mathbb{R}$ to $\mathbb{R}$: for all real numbers $x$ and $y$,
  $$ x \ L \ y \iff x < y $$
  
  $(-17) \ L \ (-14), \quad (-17) \ L \ (-10), \quad (-35) \ L \ 1, \ldots$

- The graph of $L$ as a subset of the Cartesian plane $\mathbb{R} \times \mathbb{R}$:
  - All the points $(x, y)$ with $y > x$ are on the graph. I.e., all the points above the line $x = y$. 

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Relations on Sets

- The Congruence Modulo 2 Relation: a relation $E$ from $\mathbb{Z}$ to $\mathbb{Z}$:
  
  - for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$

    $$m \ E \ n \iff m - n \text{ is even}.$$ 

  4 $\ E \ 0$ because $4 - 0 = 4$ and 4 is even.

  2 $\ E \ 6$ because $2 - 6 = -4$ and $-4$ is even.

  3 $\ E \ (-3)$ because $3 - (-3) = 6$ and 6 is even.

  - If $n$ is any odd integer, then $n \ E \ 1$.

Proof: Suppose $n$ is any odd integer.
Then $n = 2k + 1$ for some integer $k$.
By definition of $E$, $n \ E \ 1$ if, and only if, $n - 1$ is even.
By substitution, $n - 1 = (2k + 1) - 1 = 2k$, and since $k$ is an integer, $2k$ is even. Hence $n \ E \ 1$. 
Relations on Sets

• A Relation on a Power Set:

\[ P(\{a, b, c\}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \} \]

relation \( S \) from \( P(\{a, b, c\}) \): for all sets \( A \) and \( B \) in \( P(\{a, b, c\}) \)

\[ A \ S \ B \iff A \ has \ at \ least \ as \ many \ elements \ as \ B. \]

\[ \{a, b\} \ S \ \{b, c\} \]

\[ \{a\} \ S \ \emptyset \ \text{because} \ \{a\} \ \text{has one element and} \ \emptyset \ \text{has zero elements,} \]

and \( 1 \geq 0 \).

\[ \{c\} \ S \ \{a\} \]
Relations on Sets

- **The Inverse of a Relation**: let \( R \) be a relation from \( A \) to \( B \).

The inverse relation \( R^{-1} \) from \( B \) to \( A \):

\[
R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.
\]

For all \( x \in A \) and \( y \in B \), \( (y, x) \in R^{-1} \iff (x, y) \in R \).

**Example**: Let \( A = \{2, 3, 4\} \) and \( B = \{2, 6, 8\} \) and let \( R \) be the “divides” relation from \( A \) to \( B \): for all \((x, y) \in A \times B\),

\[
x R y \iff x \mid y \quad (x \text{ divides } y).
\]

\[
R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\} \quad R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}
\]

For all \((y, x) \in B \times A\), \( y R^{-1} x \iff y \text{ is a multiple of } x \).
The Inverse of an Infinite Relation: A relation $R$ from $\mathbb{R}$ to $\mathbb{R}$ as follows: for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$x \, R \, y \iff y = 2 \cdot |x|.$$ 

$R$ and $R^{-1}$ in the Cartesian plane:

$R = \{(x, y) \mid y = 2 |x|\}$

$R^{-1} = \{(y, x) \mid y = 2 |x|\}$

$R^{-1}$ is not a function because, for instance, both $(2, 1)$ and $(2, -1)$ are in $R^{-1}$. 
Relations on Sets

- A relation on a set $A$ is a relation from $A$ to $A$:
  - the arrow diagram of the relation becomes a **directed graph**
  - For all points $x$ and $y$ in $A$, there is an arrow from $x$ to $y$ $\iff xRy \iff (x,y) \in R$

Example: let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation $R$ on $A$:

for all $x, y \in A$, $xRy \iff 2 \mid (x-y)$
N-ary Relations and Relational Databases

- Given sets $A_1, A_2, \ldots, A_n$, an \textit{n-ary relation} $R$ on $A_1 \times A_2 \times \cdots A_n$ is a subset of $A_1 \times A_2 \times \cdots A_n$.
- The special cases of 2-ary, 3-ary, and 4-ary relations are called binary, ternary, and quaternary relations, respectively.
- A Simple Database: $(a_1, a_2, a_3, a_4) \in R \iff$ a patient with patient ID number $a_1$, named $a_2$, was admitted on date $a_3$, with primary diagnosis $a_4$
  
  (011985, John Schmidt, 120111, asthma)  
  (244388, Sarah Wu, 010310, broken leg)  
  (574329, Tak Kurosawa, 120111, pneumonia)

- In the database language SQL:

  ```sql
  SELECT Patient-ID#, Name FROM S WHERE Admission-Date = 120111
  ```

  011985 John Schmidt, 574329 Tak Kurosawa
Let $A = \{2, 3, 4, 6, 7, 9\}$ and define a relation $R$ on $A$ as follows:

for all $x, y \in A$, $x R y \iff 3 \mid (x - y)$.

$R$ is reflexive, symmetric and transitive.
Let R be a relation on a set A.

1. R is reflexive if, and only if, for all \( x \in A \), \( xRx \) \( (x,x) \in R \).
2. R is symmetric if, and only if, for all \( x, y \in A \), if \( xRy \) then \( yRx \).
3. R is transitive if, and only if, for all \( x, y, z \in A \), if \( xRy \) and \( yRz \) then \( xRz \).

Direct graph properties:

1. Reflexive: each point of the graph has an arrow looping around from it back to itself.
2. Symmetric: in each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
3. Transitive: in each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third.
Reflexivity, Symmetry, and Transitivity

- R is not reflexive $\iff$ there is an element $x$ in $A$ such that $x \not\sim x$ [that is, such that $(x, x) \notin R$].
- R is not symmetric $\iff$ there are elements $x$ and $y$ in $A$ such that $x R y$ but $y \not\sim x$ [that is, such that $(x, y) \in R$ but $(y, x) \notin R$].
- R is not transitive $\iff$ there are elements $x$, $y$ and $z$ in $A$ such that $x R y$ and $y R z$ but $x \not\sim z$ [that is, such that $(x,y) \in R$ and $(y,z) \in R$ but $(x,z) \notin R$].
Relations on Sets

Let $A = \{0, 1, 2, 3\}$.

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$

$R$ is reflexive: There is a loop at each point of the directed graph.

$R$ is symmetric: In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

$R$ is not transitive: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3.
Let \( A = \{0, 1, 2, 3\} \).

\[ S = \{(0, 0), (0, 2), (0, 3), (2, 3)\} \]

\( S \) is not reflexive: There is no loop at 1.

\( S \) is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0.

\( S \) is transitive!
Relations on Sets

• Let $A = \{0, 1, 2, 3\}$.

$T = \{(0, 1), (2, 3)\}$

$T$ is not reflexive: There is no loop at 0.

$T$ is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0.

$T$ is transitive: The transitivity condition is vacuously true for $T$. 
Properties of Relations on Infinite Sets:

Suppose a relation R is defined on an infinite set A:

- Reflexivity: \( \forall x \in A, x R x. \)
- Symmetry: \( \forall x, y \in A, \text{ if } x R y \text{ then } y R x. \)
- Transitivity: \( \forall x, y, z \in A, \text{ if } x R y \text{ and } y R z \text{ then } x R z. \)

Example: property of equality

- R is a relation on \( \mathbb{R} \), for all real numbers x and y:
  \[ x R y \iff x = y \]
- R is reflexive: For all \( x \in \mathbb{R} \), \( x R x \) (\( x = x \)).
- R is symmetric: For all \( x, y \in \mathbb{R} \), if \( x R y \) then \( y R x \).
  \[ \text{if } x = y \text{ then } y = x. \]
- R is transitive: For all \( x, y, z \in \mathbb{R} \), if \( x R y \) and \( y R z \) then \( x R z \)
  \[ \text{if } x = y \text{ and } y = z \text{ then } x = z. \]
Relations on Sets

- Example: properties of “Less Than”

For all \( x, y \in \mathbb{R} \), \( x \leq y \) \iff \( x < y \).

**R is not reflexive:** \( R \) is reflexive if, and only if, \( \forall x \in \mathbb{R}, x \leq x \). By definition of \( R \), this means that \( \forall x \in \mathbb{R}, x < x \).

This is false: \( \exists x=0 \in \mathbb{R} \) such that \( x \not< x \).

**R is not symmetric:** \( R \) is symmetric if, and only if, \( \forall x, y \in \mathbb{R}, if x \leq y then y \leq x \).

By definition of \( R \), this means that \( \forall x, y \in \mathbb{R}, if x < y then y < x \)

This is false: \( \exists x=0, y=1 \in \mathbb{R} \) such that \( x < y \) and \( y \not< x \).

**R is transitive:** \( R \) is transitive if, and only if, for all \( x, y, z \in \mathbb{R} \), if \( x \leq y \) and \( y \leq z \), then \( x \leq z \).

By definition of \( R \), this means that for all \( x, y, z \in \mathbb{R} \), if \( x < y \) and \( y < z \), then \( x < z \).
Relations on Sets

• Example: congruence modulo 3

For all $m, n \in \mathbb{Z}$, $m \equiv n \mod 3$ if $3 | (m - n)$.

**T is reflexive:** Suppose $m$ is a particular but arbitrarily chosen integer. [We must show that $m \equiv m \mod 3$.] Now $m - m = 0$. But $3 | 0$ since $0 = 3 \cdot 0$. Hence $3 | (m - m)$. Thus, by definition of $T$, $m \equiv m \mod 3$.

**T is symmetric:** Suppose $m$ and $n$ are particular but arbitrarily chosen integers that satisfy the condition $m \equiv n \mod 3$. [We must show that $n \equiv m \mod 3$.] By definition of $T$, since $m \equiv n \mod 3$ then $3 | (m - n)$. By definition of “divides,” this means that $m - n = 3k$, for some integer $k$. Multiplying both sides by $-1$ gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 | (n - m)$. Hence, by definition of $T$, $n \equiv m \mod 3$. 

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Relations on Sets

- Example: congruence modulo 3

For all $x, y \in \mathbb{Z}$, $m \sim n \iff 3 \mid (m - n)$.

T is transitive: Suppose $m, n,$ and $p$ are particular but arbitrarily chosen integers that satisfy the condition $m \sim n$ and $n \sim p$. [We must show that $m \sim p$.] By definition of $\sim$, since $m \sim n$ and $n \sim p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$. By definition of “divides,” this means that $m - n = 3r$ and $n - p = 3s$, for some integers $r$ and $s$. Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of $\sim$, $m \sim p$. 

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The Transitive Closure of a Relation

- Let A be a set and R a relation on A. The transitive closure of R is the relation $R^t$ on A that satisfies the following three properties:
  1. $R^t$ is transitive
  2. $R \subseteq R^t$
  3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Example: Let $A = \{0, 1, 2, 3\}$
$R = \{(0, 1), (1, 2), (2, 3)\}$
$R^t = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$
Equivalence Relation

Let A be a set and R a relation on A.

R is an equivalence relation \( \iff \) R is reflexive, symmetric, and transitive

Example: \( X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \)

A relation R on X: \( A R B \iff \) the least element of A equals the least element of B

R is an equivalence relation on X:

R is reflexive: Suppose A is a nonempty subset of \( \{1, 2, 3\} \) [We must show that \( A RA \)]

By definition of R, \( A RA \): the least element of A equals the least element of A.

R is symmetric: Suppose A and B are nonempty subsets of \( \{1, 2, 3\} \) and \( A RB \).

[We must show that \( B RA \)] By \( A RB \), the least element of A equals the least element of B. Thus, by symmetry of equality, \( B RA \).

R is transitive: Suppose A, B, and C are nonempty subsets of \( \{1, 2, 3\} \), \( A RB \), and \( B RC \).

[We must show that \( A RC \)] By \( A RB \), the least element of A equals the least element of B

By \( B RC \), the least element of B equals the least element of C.

By transitivity of equality, the least element of A equals the least element of C: \( A RC \).
The Relation Induced by a Partition

- Example: The Relation Induced by a Partition: given a partition of a set \( A \), the relation induced by the partition, \( R \), is defined on \( A \) as follows: for all \( x, y \in A \), \( x \ R \ y \) \iff there is a subset \( A_i \) of the partition such that both \( x \) and \( y \) are in \( A_i \).

- Example: Let \( A = \{0, 1, 2, 3, 4\} \) and consider the following partition of \( A \): \( \{0, 3, 4\}, \{1\}, \{2\} \).

\[
\begin{align*}
0 & \ R \ 3 \text{ because both 0 and 3 are in } \{0, 3, 4\} \\
3 & \ R \ 0 \text{ because both 3 and 0 are in } \{0, 3, 4\} \\
0 & \ R \ 4 \text{ because both 0 and 4 are in } \{0, 3, 4\} \\
4 & \ R \ 0 \text{ because both 4 and 0 are in } \{0, 3, 4\} \\
3 & \ R \ 4 \text{ because both 3 and 4 are in } \{0, 3, 4\} \\
4 & \ R \ 3 \text{ because both 4 and 3 are in } \{0, 3, 4\} \\
0 & \ R \ 0 \text{ because both 0 and 0 are in } \{0, 3, 4\} \\
3 & \ R \ 3 \text{ because both 3 and 3 are in } \{0, 3, 4\} \\
4 & \ R \ 4 \text{ because both 4 and 4 are in } \{0, 3, 4\} \\
1 & \ R \ 1 \text{ because both 1 and 1 are in } \{1\} \\
2 & \ R \ 2 \text{ because both 2 and 2 are in } \{2\} \\
R & = \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)\}.
\end{align*}
\]
The Relation Induced by a Partition

Let $A$ be a set with a partition and let $R$ be the relation induced by the partition. Then $R$ is reflexive, symmetric, and transitive.

Proof: Suppose $A$ is a set with a partition (finite): $A_1, A_2, \ldots, A_n$. $A_i \cap A_j \neq \emptyset$ whenever $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

For all $x, y \in A$, $x \mathrel{R} y \iff$ there is a set $A_i$ of the partition such that $x \in A_i$ and $y \in A_i$.

**Proof that $R$ is reflexive:** Suppose $x \in A$. Since $A_1, A_2, \ldots, A_n$ is a partition of $A$, $A_1 \cup A_2 \cup \cdots \cup A_n = A$, it follows that $x \in A_i$ for some $i$.

There is a set $A_i$ of the partition such that $x \in A_i$.

By definition of $R$, $x \mathrel{R} x$. 
The Relation Induced by a Partition

Proof that \( R \) is symmetric: Suppose \( x \) and \( y \) are elements of \( A \) such that \( x \ R \ y \). Then there is a subset \( A_i \) of the partition such that \( x \in A_i \) and \( y \in A_i \) by definition of \( R \). It follows that the statement there is a subset \( A_i \) of the partition such that \( y \in A_i \) and \( x \in A_i \) is also true. By definition of \( R \), \( y \ R \ x \).
Proof that \( R \) is transitive: Suppose \( x, y, \) and \( z \) are in \( A \) and \( xRy \) and \( yRz \). By definition of \( R \), there are subsets \( A_i \) and \( A_j \) of the partition such that \( x \) and \( y \) are in \( A_i \) and \( y \) and \( z \) are in \( A_j \).

Suppose \( A_i \neq A_j \). [We will deduce a contradiction.] Then \( A_i \cap A_j = \emptyset \) since \( \{A_1, A_2, A_3, \ldots, A_n\} \) is a partition of \( A \). But \( y \) is in \( A_i \) and \( y \) is in \( A_j \) also. Hence \( A_i \cap A_j \neq \emptyset \). [This contradicts the fact that \( A_i \cap A_j = \emptyset \).] Thus \( A_i = A_j \). It follows that \( x, y, \) and \( z \) are all in \( A_i \), and so in particular, \( x \) and \( z \) are in \( A_i \).

Thus, by definition of \( R \), \( x R z \).
Equivalence Classes

- Let $A$ be a set and $R$ an equivalence relation on $A$. For each element $a$ in $A$, the equivalence class of $a$ (the class of $a$) is the set of all elements $x$ in $A$ such that $x$ is related to $a$ by $R$.

  $$[a] = \{x \in A \mid x R a\}$$

- Example: Let $A = \{0, 1, 2, 3, 4\}$ and $R$ be a relation on $A$:
  
  $$R = \{(0, 0),(0, 4),(1, 1),(1, 3),(2, 2),(3, 1),(3, 3),(4,0),(4,4)\}$$

  $R$ is an equivalence relation

  $[0] = \{x \in A \mid x R 0\} = \{0, 4\} = [4]$

  $[1] = \{x \in A \mid x R 1\} = \{1, 3\} = [3]$

  $[2] = \{x \in A \mid x R 2\} = \{2\}$

  $\{0, 4\}$, $\{1, 3\}$ and $\{2\}$ are distinct equivalence classes
Equivalence Classes of a Relation on a Set of Subsets

- \( X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \)

- \( A \sim B \iff \) the least element of \( A \) equals the least element of \( B \)

- \([\{1\}] = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\} = [\{1, 2\}] = [\{1, 3\}] = [\{1, 2, 3\}]\)

- \([\{2\}] = \{\{2\}, \{2, 3\}\} = [\{2, 3\}]\)

- \([\{3\}] = \{\{3\}\}\)
Equivalence Classes of the Identity Relation

- Let $A$ be any set and $R$ a relation on $A$: For all $x$ and $y$ in $A$,
  \[ x \sim y \iff x = y \]
  Given any $a$ in $A$, the class of $a$ is:
  \[ [a] = \{ x \in A \mid x \sim a \} = \{a\} \]
  since the only element of $A$ that equals $a$ is $a$. 

Equivalence Classes

- Let $A$ be a set and $R$ an equivalence relation on $A$. For any $a$ and $b$ elements of $A$, if $a R b$, then $[a] = [b]$.

Proof: $[a] = [b] \iff [a] \subseteq [b]$ and $[b] \subseteq [a]$.

1. $[a] \subseteq [b]$
   - Let $x \in [a]$ iff then $x R a$.
     - $a R b$ by hypothesis $\implies$ by transitivity of $R$, $x R b \implies x \in [b]$

2. $[b] \subseteq [a]$
   - Let $x \in [b]$ iff then $x R b$.
     - $b R a$ by hypothesis and symmetry $\implies$ by transitivity of $R$, $xRa \implies x \in [a]$

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If $A$ is a set, $R$ is an equivalence relation on $A$, and $a$ and $b$ are elements of $A$, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

Proof:

Suppose $A$ is a set, $R$ is an equivalence relation on $A$, $a$ and $b$ are elements of $A$:

Case 1: $a \mathrel{R} b$: by the previous theorem, $[a] = [b]$. Therefore, $[a] \cap [b] = \emptyset$ or $[a] = [b]$ is true.

Case 2: $a \not\mathrel{R} b$ (we will prove the $[a] \cap [b] = \emptyset$).

By element method, by contradiction, there exists an element $x$ in $A$ s.t.

$x \in [a] \cap [b] \implies x \in [a]$ and $x \in [b] \implies$ so $x \mathrel{R} a$ and $x \mathrel{R} b$

By symmetry and transitivity, $a \mathrel{R} b$ (contradiction).
Let $R$ be the relation of congruence modulo 3 on the set $\mathbb{Z}$ of all integers: for all integers $m$ and $n$,

$$m \sim n \iff 3 \mid (m - n) \iff m \equiv n \pmod{3}.$$ 

Solution For each integer $a$,

$$[a] = \{x \in \mathbb{Z} \mid 3 \mid (x-a)\} = \{x \in \mathbb{Z} \mid x-a=3k, \text{ for some integer } k\}$$

$$= \{x \in \mathbb{Z} \mid x = 3k + a, \text{ for some integer } k\}.$$

$[0] = \{x \in \mathbb{Z} \mid x = 3k + 0, \text{ for some integer } k\} = \{x \in \mathbb{Z} \mid x = 3k, \text{ for some integer } k\}$

$$= \{...-9,-6,-3,0,3,6,9,...\} = [3] = [-3] = [6] = [-6] = ...$$

$[1] = \{x \in \mathbb{Z} \mid x = 3k + 1, \text{ for some integer } k\}$

$$= \{...-8,-5,-2,1,4,7,10,...\} = [4] = [-2] = [7] = [-5] = ...$$

$[2] = \{x \in \mathbb{Z} \mid x = 3k + 2, \text{ for some integer } k\}$

$$= \{...-4,-1,2,...\} = [5] = [-1] = [8] = [-4] = ...$$
Congruence Modulo

- Let $m$ and $n$ be integers and let $d$ be a positive integer.

$m$ is congruent to $n$ modulo $d$:

$$m \equiv n \pmod{d} \iff d \mid (m - n)$$

Example:

$12 \equiv 7 \pmod{5}$ because $12 - 7 = 5 = 5 \cdot 1$

$\Rightarrow$

$5 \mid (12 - 7)$. 
Rational Numbers Are Equivalence Classes

- Let $A$ be the set of all ordered pairs of integers for which the second element of the pair is nonzero: $A = \mathbb{Z} \times (\mathbb{Z} - \{0\})$

$R$ is a relation on $A$: for all $(a, b), (c, d) \in A$,

$$(a, b) R (c, d) \iff \frac{a}{b} = \frac{c}{d}$$

$R$ is an equivalence relation

Example equivalence class:

$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

$$\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6} \text{ and so forth.}$$
Antisymmetry

- Let $R$ be a relation on a set $A$.

$R$ is *antisymmetric* $\iff$ for all $a, b \in A$, if $aRb$ and $bRa$ then $a = b$

$R$ is *not antisymmetric* $\iff$ there exist $a, b \in A$ s.t. $aRb$, $bRa$, but $a \neq b$

0 R 2 and 2 R 0 but 0 \neq 2
Antisymmetry of “Divides” Relations

• For all $a, b \in \mathbb{Z}^+$, $a R_1 b \iff a \mid b$.

$R_1$ is antisymmetric: Suppose $a, b \in \mathbb{Z}^+$ such that $a R_1 b$ and $b R_1 a$. [We must show that $a = b$]

By definition of $R_1$, $a \mid b$ and $b \mid a \Rightarrow b = k_1 a$ and $a = k_2 b$, $k_1, k_2 \in \mathbb{Z}$ (and both are positive since $a$ and $b$ are positive) $\Rightarrow b = k_1 k_2 b$.

Dividing both sides by $b$ gives $k_1 k_2 = 1$ (and both $> 0$) $\Rightarrow k_1 = k_2 = 1$ $\Rightarrow a = b$

• For all $a, b \in \mathbb{Z}$, $a R_2 b \iff a \mid b$.

$R_2$ is not antisymmetric:

Counterexample: $a = 2$ and $b = -2 \Rightarrow a \neq b$

$a \mid b$ since $-2 = (-1) \cdot 2 \Rightarrow a R_2 b$

$b \mid a$ since $2 = (-1)(-2) \Rightarrow b R_2 a$
Partial Order Relations

- Let $R$ be a relation defined on a set $A$.

$R$ is a **partial order relation** $\iff$ $R$ is reflexive, antisymmetric and transitive.

- Example: The “Subset” Relation

Let $A$ be any collection of sets and $\subseteq$ (the “subset”) relation on $A$:

For all $U, V \in A$, $U \subseteq V \iff$ for all $x$, if $x \in U$ then $x \in V$.

$\subseteq$ is a partial order (reflexive, transitive and antisymmetric)

Proof that $\subseteq$ is antisymmetric: for all sets $U$ and $V$ in $A$

if $U \subseteq V$ and $V \subseteq U$ then $U = V$ (by definition of equality of sets)
The “Less Than or Equal to” Relation

- The “less than or equal to” relation \( \leq \) on \( \mathbb{R} \) (reals): for all \( x,y \in \mathbb{R} \)
  \[
x \leq y \iff x < y \text{ or } x = y.
  \]

\( \leq \) is a partial order relation:
- \( \leq \) is reflexive: \( x \leq x \) for all real numbers. \( x \leq x \) means that \( x < x \) or \( x = x \), and \( x = x \) is always true.
- \( \leq \) is antisymmetric: for all \( x,y \in \mathbb{R} \), if \( x \leq y \) and \( y \leq x \) then \( x = y \).
- \( \leq \) is transitive: for all \( x,y,z \in \mathbb{R} \), if \( x \leq y \) and \( y \leq z \) then \( x \leq z \).
Lexicographic Order

• Order in an English dictionary: compare letters one by one from left to right in words.

• Let A be a set with a partial order relation R, and let S be a set of strings over A. "\leq" is a relation on S: for any 2 strings in S, $a_1a_2...a_m$ and $b_1b_2...b_n$, $m,n \in \mathbb{Z}^+$:

  1. If $m \leq n$ and $a_i=b_i$ for all $i=1,2,...,m$, then $a_1a_2...a_m \leq b_1b_2...b_n$

  2. If for some integer $k$ with $k \leq m$, $k \leq n$, and $k \geq 1$, $a_i=b_i$ for all $i=1,2,...,k-1$, and $a_k \neq b_k$, but $a_k R b_k$ then $a_1a_2...a_m \leq b_1b_2...b_n$.

  3. If $\varepsilon$ is the null string and $s$ is any string in S, then $\varepsilon \leq s$.

If no strings are related other than by these three conditions, then "\leq" is a partial order relation (lexicographic order for S).
Lexicographic Order

- Let $A = \{x, y\}$ and $R$ the partial order relation on $A$:
  
  $R = \{(x, x), (x, y), (y, y)\}$. 

Let $S$ be the set of all strings over $A$, and $\preceq$ the lexicographic order for $S$ that corresponds to $R$.

\[
\begin{align*}
  x & \preceq xx & x & \preceq xy \\
  yxy & \preceq yxyxxx & x & \preceq y \\
  xx & \preceq xyx & xxxy & \preceq xy \\
  \varepsilon & \preceq x & \varepsilon & \preceq xyxyyx
\end{align*}
\]
A Hasse Diagram is a simpler graph with a partial order relation defined on a finite set.

Example: let $A = \{1, 2, 3, 9, 18\}$ and the “divides” relation $|$ on $A$:
for all $a, b \in A$, $a | b \iff b = ka$ for some integer $k$.

Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Eliminate:
1. the loops at all the vertices
2. all arrows whose existence is implied by the transitive property
3. the direction indicators on the arrows
The “subset” relation $\subseteq$ on the set $P(\{a, b, c\})$:

For all sets $U$ and $V$ in $P(\{a, b, c\})$

$$U \subseteq V \iff \forall x, \text{ if } x \in U \text{ then } x \in V$$

Draw the directed graph of the relation in such a way that all arrows except loops point upward.

Strip away all loops, unnecessary arrows, and direction indicators to obtain the Hasse diagram.

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Obtain the original directed graph from the Hasse diagram:

1. Reinsert the direction markers on the arrows making all arrows point upward.
2. Add loops at each vertex.
3. For each sequence of arrows from one point to a second and from that second point to a third, add an arrow from the first point to the third.
Partially and Totally Ordered Sets

• Let $\preceq$ be a partial order relation on a set $A$. Elements $a$ and $b$ of $A$ are comparable $\iff$ either $a \preceq b$ or $b \preceq a$. Otherwise, $a$ and $b$ are noncomparable.

• If $R$ is a partial order relation on a set $A$, and any two elements $a$ and $b$ in $A$ are comparable, then $R$ is a total order relation on $A$.

• The Hasse diagram for a total order relation can be drawn as a single vertical “chain.”

• A set $A$ is called a partially ordered set (or poset) with respect to a relation $\preceq$ if $\preceq$ is a partial order relation on $A$.

• A set $A$ is called a totally ordered set with respect to a relation $\preceq$ if $A$ is partially ordered with respect to $\preceq$ and $\preceq$ is a total order.
Partially and Totally Ordered Sets

- Let $A$ be a set that is partially ordered with respect to a relation $\leq$. A subset $B$ of $A$ is called a *chain* if the elements in each pair of elements in $B$ is comparable.

- The *length* of a chain is one less than the number of elements in the chain.

- Example: Chain of Subsets

  The set $P\{{a, b, c}\}$ is partially ordered with respect to $\subseteq$.

  Chain of length 3: $\emptyset \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$
An element $a$ in $A$ is called a **maximal element** $\iff$ for all $b$ in $A$, either $b \leq a$ or $b$ and $a$ are not comparable.

An element $a$ in $A$ is called a **greatest element** of $A$ $\iff$ for all $b$ in $A$, $b \leq a$.

An element $a$ in $A$ is called a **minimal element** $\iff$ for all $b$ in $A$, either $a \leq b$ or $b$ and $a$ are not comparable.

An element $a$ in $A$ is called a **least element** of $A$ $\iff$ for all $b$ in $A$, $a \leq b$.

Example:

- one maximal element = $g$ = also the greatest element
- minimal elements: $c$, $d$ and $i$
- there is no least element
Topological Sorting

- Given partial order relations \( \preceq \) and \( \preceq ' \) on a set \( A \), \( \preceq ' \) is compatible with \( \preceq \) if for all \( a \) and \( b \) in \( A \), if \( a \preceq b \) then \( a \preceq ' b \)

- Given partial order relations \( \preceq \) and \( \preceq ' \) on a set \( A \), \( \preceq ' \) is a topological sorting for \( \preceq \) if \( \preceq ' \) is a total order that is compatible with \( \preceq \).

- Example: \( P(\{a, b, c\}) \) with partial order \( \subseteq \) (any element in \( P(\{a,b,c\}) \) we can either compare them or not, e.g., \( \{a,b\} \) with \( \{a,c\} \)

  Total order:

  \[
  \emptyset \preceq ' \{a\} \preceq ' \{b\} \preceq ' \{c\} \preceq ' \{a, b\} \preceq ' \{a, c\} \preceq ' \{b, c\} \preceq ' \{a, b, c\}
  \]
Topological Sorting

- Constructing a Topological Sorting:
  1. Pick any minimal element x in A with respect to $\preceq$.
     [Such an element exists since A is nonempty.]
  2. Set $A' = A - \{x\}$
  3. Repeat steps a–c while $A' \neq \emptyset$:
     a. Pick any minimal element y in A’.
     b. Define $x \preceq y$.
     c. Set $A' = A' - \{y\}$ and $x = y$. 