Relations

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

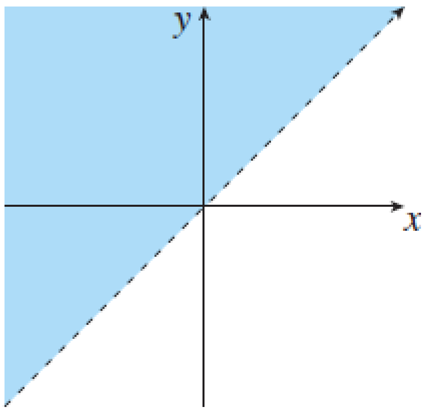
Relations on Sets

- **A relation is a collection ordered pairs.**
- The Less-than Relation for Real Numbers: a relation L from $\mathbf{R}$ to $\mathbf{R}$: for all real numbers x and y,

$$x \, L \, y \Leftrightarrow x < y$$

$$(-17) \, L \, (-14), \qquad (-17) \, L \, (-10), \qquad (-35) \, L \, 1, \ldots$$

- The graph of L as a subset of the Cartesian plane $\mathbf{R} \times \mathbf{R}$:
  - All the points (x, y) with y > x are on the graph. I.e., all the points above the line x = y.



Relations on Sets

- The Congruence Modulo 2 Relation: a relation E from **Z** to **Z**:
  - for all $(m, n) \in \mathbf{Z} \times \mathbf{Z}$

$$m \ E \ n \Leftrightarrow m - n \text{ is even.}$$

$4 \ E \ 0$ because $4 - 0 = 4$ and 4 is even.

$2 \ E \ 6$ because $2 - 6 = -4$ and -4 is even.

$3 \ E \ (-3)$ because $3 - (-3) = 6$ and 6 is even.

- If n is any odd integer, then n E 1.

Proof: Suppose n is any odd integer.

Then $n = 2k + 1$ for some integer k.

By definition of E, n E 1 if, and only if, $n - 1$ is even.

By substitution, $n - 1 = (2k + 1) - 1 = 2k$, and since k is an integer, 2k is even. Hence **n E 1.**

Relations on Sets

- A Relation on a Power Set:

P({a, b, c})={∅, {a}, {b}, {c}, {a, b}, {a, c}, {b,c},{a,b,c}}

relation S from P({a, b, c}): for all sets A and B in P({a,b,c})

A S B ⇔ A has at least as many elements as B.

{a, b} S {b, c}

{a} S ∅ because {a} has one element and ∅ has zero elements, and $1 \geq 0$.

{c} S {a}

Relations on Sets

- The Inverse of a Relation: let R be a relation from A to B.

The inverse relation R^{-1} from B to A:

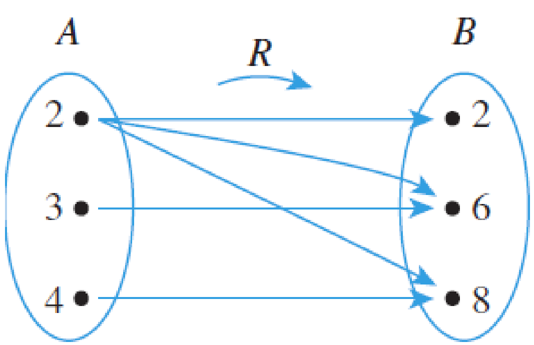
$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

For all x ∈ A and y ∈ B, (y, x) ∈ R^{-1} ⇔ (x, y) ∈ R.

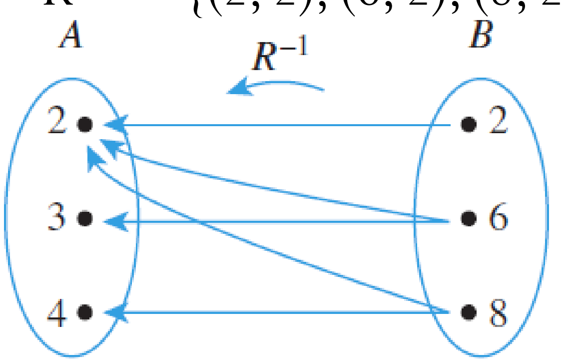
Example: Let A = {2, 3, 4} and B = {2, 6, 8} and let R be the "divides" relation from A to B: for all (x, y) ∈ A × B,

$$x \; R \; y \Leftrightarrow x \mid y \quad (x \text{ divides } y).$$

R = {(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)}



R^{-1} = {(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)}



For all (y, x) ∈ B × A, y R^{-1} x ⇔ y is a multiple of x.

Relations on Sets

- The Inverse of an Infinite Relation: a relation R from **R** to **R** as follows: for all (x, y) ∈ **R** × **R**,

$$x\,R\,y \Leftrightarrow y = 2 * |x|.$$

R and R^{-1} in the Cartesian plane:

$R = \{(x, y) \mid y = 2|x|\}$

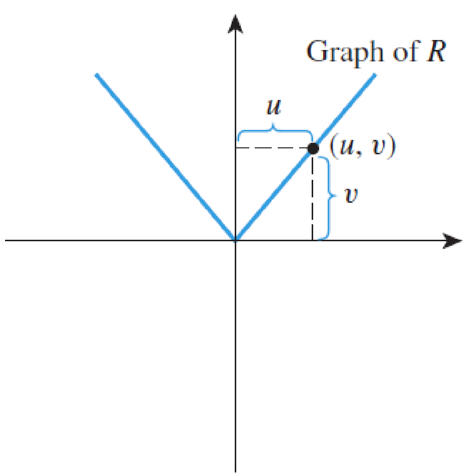
x	y
0	0
1	2
−1	2
2	4
−2	4

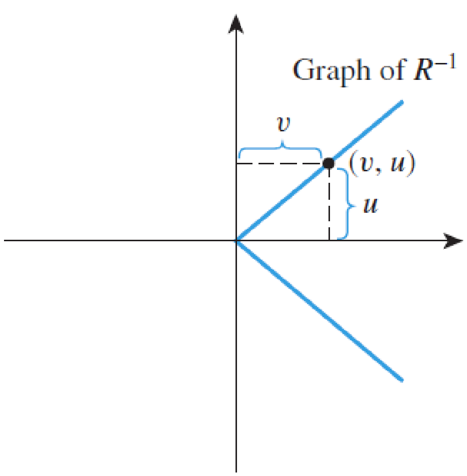
1st coordinate — 2nd coordinate

$R^{-1} = \{(y, x) \mid y = 2|x|\}$

y	x
0	0
2	1
2	−1
4	2
4	−2

1st coordinate — 2nd coordinate





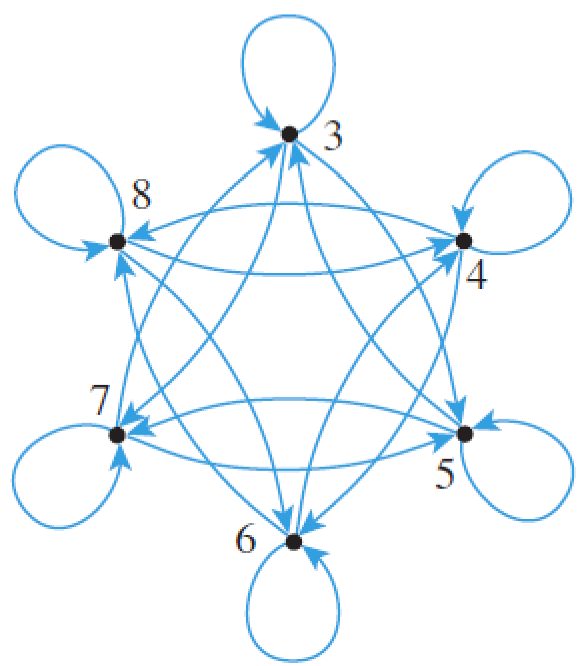
R^{-1} is not a function because, for instance, both (2, 1) and (2, −1) are in R^{-1}.

Relations on Sets

- A relation on a set A is a relation from A to A:
  - the arrow diagram of the relation becomes a **directed graph**
    - For all points x and y in A, there is an arrow from x to y ⇔xRy⇔(x,y)∈R

Example: let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A:

$$\text{for all } x, y \in A,\ xRy \Leftrightarrow 2 \mid (x-y)$$



N-ary Relations and Relational Databases

- Given sets $A_1, A_2, \ldots, A_n$, an *n-ary relation* R on $A_1 \times A_2 \times \cdots A_n$ is a subset of $A_1 \times A_2 \times \cdots A_n$.
  - The special cases of 2-ary, 3-ary, and 4-ary relations are called binary, ternary, and quaternary relations, respectively.
  - A Simple Database: $(a_1, a_2, a_3, a_4) \in R \Leftrightarrow$ a patient with patient ID number a_1, named a_2, was admitted on date a_3, with primary diagnosis a_4

    (011985, John Schmidt, 120111, asthma)

    (244388, Sarah Wu, 010310, broken leg)

    (574329, Tak Kurosawa, 120111, pneumonia)
  - In the database language SQL:

    SELECT Patient−ID#, Name FROM S WHERE

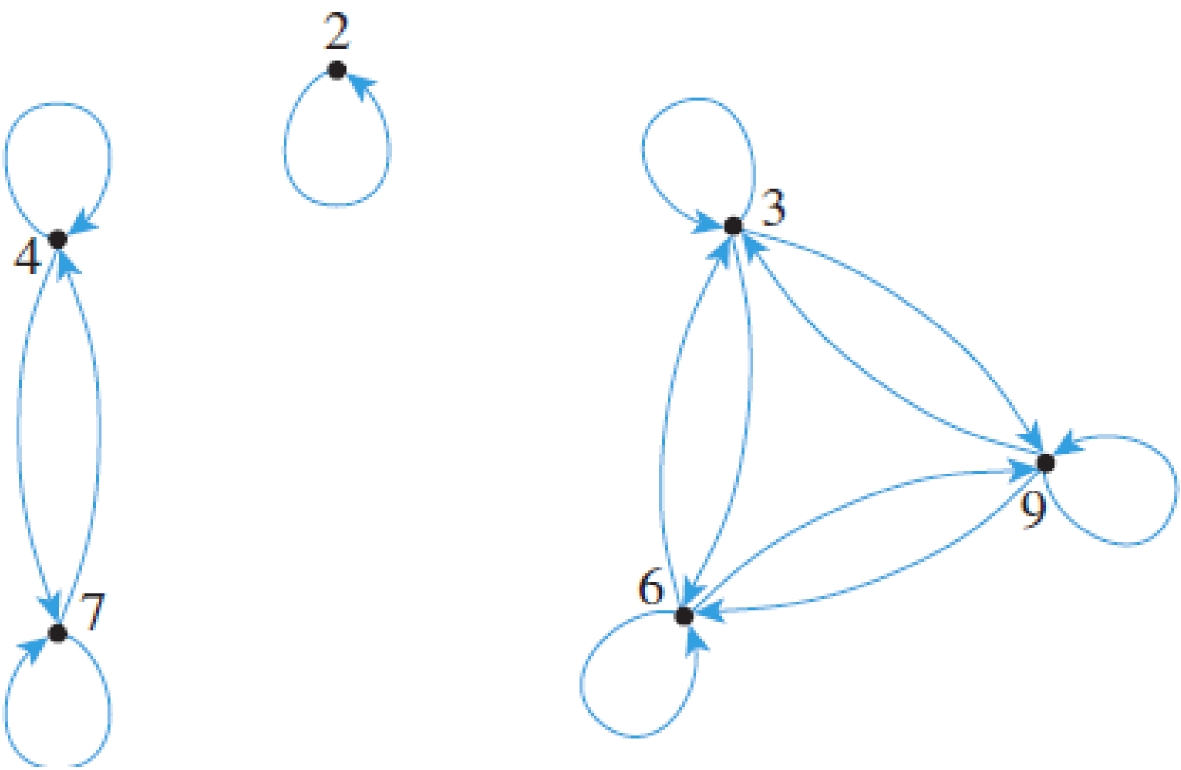
    Admission−Date = 120111

    011985 John Schmidt, 574329 Tak Kurosawa

Reflexivity, Symmetry, and Transitivity

- Let A = {2, 3, 4, 6, 7, 9} and define a relation R on A as follows:

$$\text{for all } x, y \in A,\ x\,R\,y \Leftrightarrow 3 \mid (x - y).$$



R is reflexive, symmetric and transitive.

Reflexivity, Symmetry, and Transitivity

- Let R be a relation on a set A.

1. R is reflexive if, and only if, for all x ∈ A, xRx ((x,x)∈R).
2. R is symmetric if, and only if, for all x, y ∈ A, if xRy then yRx
3. R is transitive if, and only if, for all x, y, z ∈ A, if xRy and yRz then xRz.

- Direct graph properties:

1. Reflexive: each point of the graph has an arrow looping around from it back to itself.
2. Symmetric: in each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
3. Transitive: in each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third.

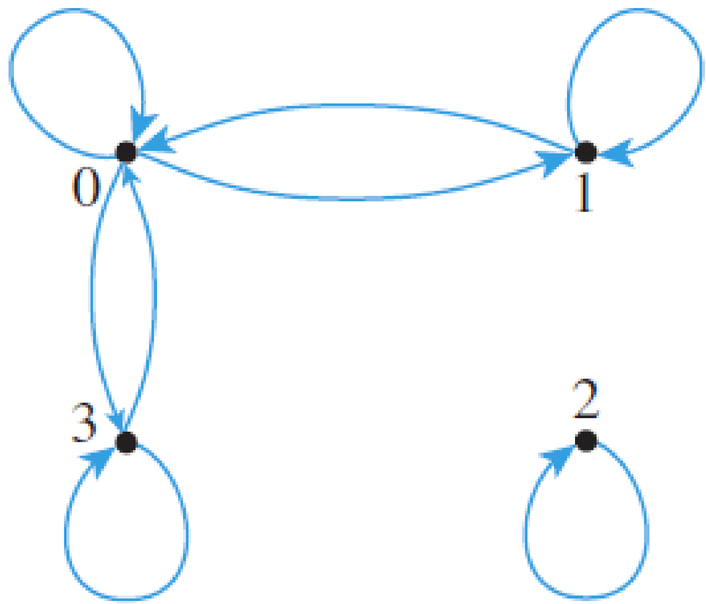
Reflexivity, Symmetry, and Transitivity

- R is not reflexive ⇔ there is an element x in A such that x $\not R$ x [that is, such that (x, x) ∉ R].
- R is not symmetric ⇔ there are elements x and y in A such that x R y but y $\not R$ x [that is, such that (x, y) ∈ R but (y, x)∉ R].
- R is not transitive ⇔ there are elements x, y and z in A such that x R y and y R z but x $\not R$ z [that is, such that (x,y) ∈ R and (y,z) ∈ R but (x,z)∉R]

Relations on Sets

- Let $A = \{0, 1, 2, 3\}$.

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$



R is reflexive: There is a loop at each point of the directed graph.

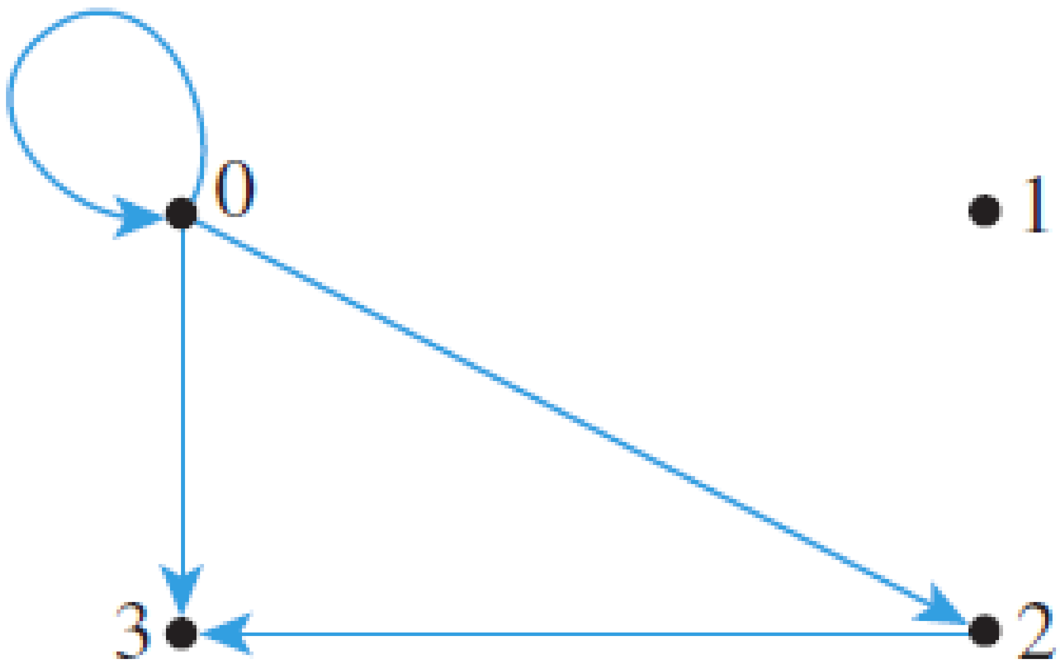
R is symmetric: In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

R is not transitive: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3.

Relations on Sets

- Let $A = \{0, 1, 2, 3\}$.

$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$



S is not reflexive: There is no loop at 1.

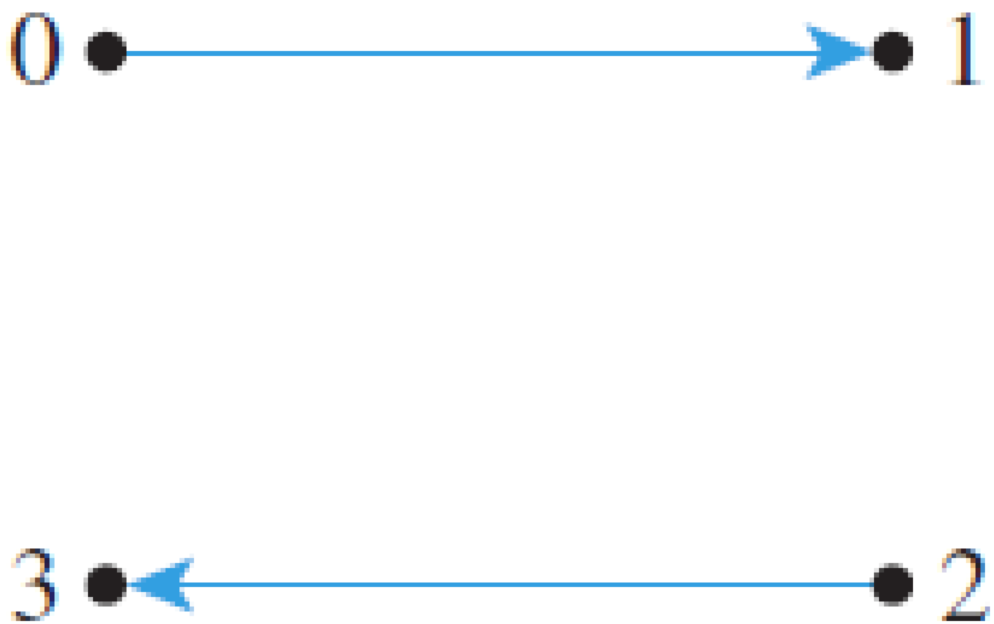
S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0.

S is transitive!

Relations on Sets

- Let A = {0, 1, 2, 3}.

T = {(0, 1), (2, 3)}



T is not reflexive: There is no loop at 0.

T is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0.

T is transitive: The transitivity condition is vacuously true for T.

Relations on Sets

- Properties of Relations on Infinite Sets:
  - Suppose a relation R is defined on an infinite set A:
    - Reflexivity: $\forall x \in A$, x R x.
    - Symmetry: $\forall x, y \in A$, if x R y then y R x.
    - Transitivity: $\forall x, y, z \in A$, if x R y and y R z then x R z.
- Example: property of equality
  - R is a relation on **R**, for all real numbers x and y:

$$x \; R \; y \Leftrightarrow x = y$$

R is reflexive: For all $x \in R$, x R x ($x=x$).

R is symmetric: For all $x, y \in R$, if x R y then y R x.
if $x = y$ then $y = x$.

R is transitive: For all $x, y, z \in R$, if x R y and y R z then x R z
if $x = y$ and $y = z$ then $x = z$.

Relations on Sets

- Example: properties of "Less Than"

For all $x, y \in \mathbb{R}$, $x\ R\ y \Leftrightarrow x < y$.

R is not reflexive: R is reflexive if, and only if, $\forall x \in \mathbb{R}, x\ R\ x$. By definition of R, this means that $\forall x \in \mathbb{R}, x < x$.

This is false: $\exists x=0 \in \mathbb{R}$ such that $x \not< x$.

R is not symmetric: R is symmetric if, and only if, $\forall x, y \in \mathbb{R}$, if $x\ R\ y$ then $y\ R\ x$.

By definition of R, this means that $\forall x, y \in \mathbb{R}$, if $x<y$ then $y<x$

This is false: $\exists x=0, y=1 \in \mathbb{R}$ such that $x < y$ and $y \not< x$.

R is transitive: R is transitive if, and only if, for all $x, y, z \in \mathbb{R}$, if xRy and $y\ R\ z$, then $x\ R\ z$.

By definition of R, this means that for all $x, y, z \in \mathbb{R}$, if $x<y$ and $y<z$, then $x < z$.

Relations on Sets

- Example: congruence modulo 3

  For all m, n ∈ Z, $m\,T\,n \Leftrightarrow 3 \mid (m - n)$.

T is reflexive: Suppose m is a particular but arbitrarily chosen integer. [*We must show that m T m.*] Now $m - m = 0$. But $3 \mid 0$ since $0 = 3 \cdot 0$. Hence $3 \mid (m - m)$. Thus, by definition of T, mTm

T is symmetric: Suppose m and n are particular but arbitrarily chosen integers that satisfy the condition m T n. [*We must show that n T m.*] By definition of T, since m T n then $3 \mid (m - n)$. By definition of "divides," this means that $m - n = 3k$, for some integer k. Multiplying both sides by -1 gives $n - m = 3(-k)$. Since $-k$ is an integer, this equation shows that $3 \mid (n - m)$. Hence, by definition of T, n T m.

Relations on Sets

- Example: congruence modulo 3

  For all x, y ∈ Z, $m\,T\,n \Leftrightarrow 3 \mid (m - n)$.

T is transitive: Suppose m, n, and p are particular but arbitrarily chosen integers that satisfy the condition $m\,T\,n$ and $n\,T\,p$. [*We must show that m T p*.] By definition of T , since $m\,T\,n$ and $n\,T\,p$, then $3 \mid (m - n)$ and $3 \mid (n - p)$. By definition of "divides," this means that $m - n = 3r$ and $n - p = 3s$, for some integers r and s. Adding the two equations gives $(m - n) + (n - p) = 3r + 3s$, and simplifying gives that $m - p = 3(r + s)$. Since $r + s$ is an integer, this equation shows that $3 \mid (m - p)$. Hence, by definition of T , mTp.

The Transitive Closure of a Relation

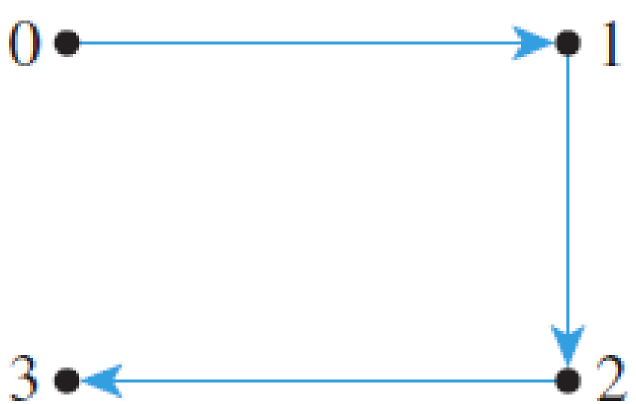
- Let A be a set and R a relation on A. The transitive closure of R is the relation R^t on A that satisfies the following three properties:

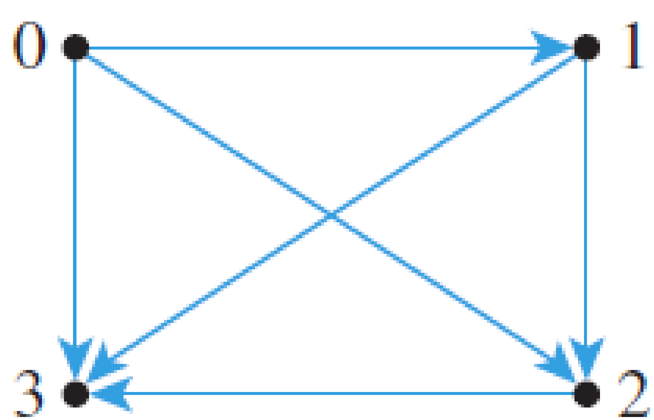
1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$

Example: Let A = {0, 1, 2, 3}

R = {(0, 1), (1, 2), (2, 3)}

R^t = {(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)}





Equivalence Relation

- Let A be a set and R a relation on A.

R is an *equivalence relation* ⇔ R is reflexive, symmetric, and transitive

- Example: X = {{1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

A relation R on X: A R B ⇔ the least element of A equals the least element of B

R is an equivalence relation on X:

R is reflexive: Suppose A is a nonempty subset of {1, 2, 3} *[We must show that A R A]*

By definition of R, A R A : the least element of A equals the least element of A.

R is symmetric: Suppose A and B are nonempty subsets of {1, 2, 3} and A R B.

[We must show that B R A] By A R B, the least element of A equals the least element of B. Thus, by symmetry of equality, B R A.

R is transitive: Suppose A, B, and C are nonempty subsets of {1, 2, 3}, A R B, and B R C.

[We must show that A R C.] By A R B, the least element of A equals the least element of B

By B R C, the least element of B equals the least element of C.

By transitivity of equality, the least element of A equals the least element of C: A R C.

The Relation Induced by a Partition

- Example: The Relation Induced by a Partition: given a partition of a set A, the relation induced by the partition, R, is defined on A as follows: for all x, y ∈ A, x R y ⇔ there is a subset A_i of the partition such that both x and y are in A_i.
  - Example: Let A = {0, 1, 2, 3, 4} and consider the following partition of A: {0, 3, 4}, {1}, {2}.

0 R 3 because both 0 and 3 are in {0, 3, 4}
3 R 0 because both 3 and 0 are in {0, 3, 4}
0 R 4 because both 0 and 4 are in {0, 3, 4}
4 R 0 because both 4 and 0 are in {0, 3, 4}
3 R 4 because both 3 and 4 are in {0, 3, 4}
4 R 3 because both 4 and 3 are in {0, 3, 4}
0 R 0 because both 0 and 0 are in {0, 3, 4}
3 R 3 because both 3 and 3 are in {0, 3, 4}
4 R 4 because both 4 and 4 are in {0, 3, 4}

1 R 1 because both 1 and 1 are in {1}
2 R 2 because both 2 and 2 are in {2}

R = {(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)}.

The Relation Induced by a Partition

- Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Proof: Suppose A is a set with a partition (finite): $A_1, A_2, \ldots, A_n$

$A_i \cap A_j \neq \emptyset$ whenever $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

For all x, y ∈ A, x R y ⇔ there is a set A_i of the partition such that x ∈ A_i and y ∈ A_i.

Proof that R is reflexive: Suppose x ∈ A. Since $A_1, A_2, \ldots, A_n$ is a partition of A, $A_1 \cup A_2 \cup \cdots \cup A_n = A$, it follows that x ∈ A_i for some i.

There is a set A_i of the partition such that x ∈ A_i.

By definition of R, x R x.

The Relation Induced by a Partition

Proof that R is symmetric: Suppose x and y are elements of A such that x R y. Then there is a subset A_i of the partition such that $x \in A_i$ and $y \in A_i$ by definition of R. It follows that the statement there is a subset A_i of the partition such that $y \in A_i$ and $x \in A_i$ is also true. By definition of R, y R x.

The Relation Induced by a Partition

Proof that **R is transitive:** Suppose x, y, and z are in A and xRy and yRz. By definition of R, there are subsets A_i and A_j of the partition such that x and y are in A_i and y and z are in A_j.

Suppose $A_i \neq A_j$. [We will deduce a contradiction.] Then $A_i \cap A_j = \emptyset$ since $\{A_1, A_2, A_3, \ldots, A_n\}$ is a partition of A. But y is in A_i and y is in A_j also. Hence $A_i \cap A_j \neq \emptyset$. [This contradicts the fact that $A_i \cap A_j = \emptyset$.] Thus $A_i = A_j$. It follows that x, y, and z are all in A_i, and so in particular, x and z are in A_i.

Thus, by definition of R, x R z.

Equivalence Classes

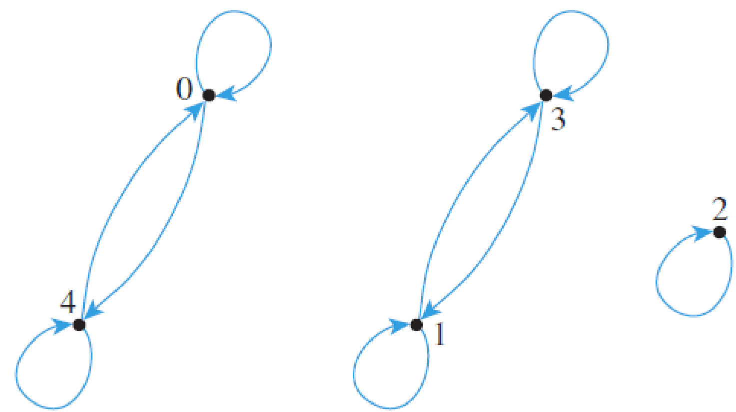
- Let A be a set and R an equivalence relation on A. For each element a in A, *the equivalence class of a* (*the class of a*) is the set of all elements x in A such that x is related to a by R.

$$[a] = \{x \in A \mid x \, R \, a\}$$

- Example: Let A = {0, 1, 2, 3, 4} and R be a relation on A:

R = {(0, 0),(0, 4),(1, 1),(1, 3),(2, 2),(3, 1),(3, 3),(4,0),(4,4)}

R is an equivalence relation



$[0] = \{x \in A \mid x \, R \, 0\} = \{0, 4\} = [4]$

$[1] = \{x \in A \mid x \, R \, 1\} = \{1, 3\} = [3]$

$[2] = \{x \in A \mid x \, R \, 2\} = \{2\}$

{0, 4}, {1, 3} and {2} are *distinct equivalence classes*

Equivalence Classes of a Relation on a Set of Subsets

- $X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

A R B $\Leftrightarrow$ the least element of A equals the least element of B

$[\{1\}] = \{\{1\},\{1,2\},\{1,3\},\{1,2,3\}\}=[\{1,2\}]=[\{1,3\}]=$
$= [\{1,2,3\}]$

$[\{2\}] = \{\{2\}, \{2, 3\}\} =[\{2, 3\}]$

$[\{3\}] = \{\{3\}\}$

Equivalence Classes of the Identity Relation

- Let A be any set and R a relation on A: For all x and y in A,

$$x \; R \; y \Leftrightarrow x = y$$

Given any a in A, the class of a is:

$$[a] = \{x \in A \mid x \; R \; a\} = \{a\}$$

since the only element of A that equals a is a.

Equivalence Classes

- Let A be a set and R an equivalence relation on A.

For any a and b elements of A, if a R b, then [a] = [b].

Proof: [a] = [b] ⇔ [a] ⊆ [b] and [b] ⊆ [a].

1. [a] ⊆ [b]

   Let x ∈ [a] iff then x R a.

   a R b by hypothesis ➔ by transitivity of R, x R b ➔ x ∈ [b]

2. [b] ⊆ [a]

   Let x ∈ [b] iff then x R b.

   b R a by hypothesis and symmetry ➔ by transitivity of R, xRa ➔ x ∈ [a]

Equivalence Classes

- If A is a set, R is an equivalence relation on A, and a and b are elements of A, then either $[a] \cap [b] = \emptyset$ or $[a] = [b]$.

Proof:

Suppose A is a set, R is an equivalence relation on A, a and b are elements of A:

Case1: a R b: by the previous theorem, $[a] = [b]$.

Therefore, $[a] \cap [b] = \emptyset$ or $[a] = [b]$ is true.

Case 2: a $\not R$ b (we will prove the $[a] \cap [b] = \emptyset$).

By element method, by contradiction,there exists an element x in A s.t. $x \in [a] \cap [b]$ ➔ $x \in [a]$ and $x \in [b]$ ➔ so x R a and x R b

By symmetry and transitivity, a R b (contradiction).

Congruence Modulo 3

- Let R be the relation of congruence modulo 3 on the set Z of all integers: for all integers m and n,

$$m \, R \, n \Leftrightarrow 3 \mid (m - n) \Leftrightarrow m \equiv n \pmod 3.$$

Solution For each integer a,

$[a] = \{x \in Z \mid 3 \mid (x-a)\} = \{x \in Z \mid x-a=3k, \text{ for some integer } k\}$

$= \{x \in Z \mid x = 3k + a, \text{ for some integer } k\}.$

$[0] = \{x \in Z \mid x = 3k + 0, \text{ for some int } k\} = \{x \in Z \mid x=3k, \text{ for some integer } k\}$

$= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\} = [3] = [-3] = [6] = [-6] = \ldots$

$[1] = \{x \in Z \mid x = 3k + 1, \text{ for some integer } k\}$

$= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\} = [4] = [-2] = [7] = [-5] = \ldots$

$[2] = \{x \in Z \mid x = 3k + 2, \text{ for some integer } k\}$

$= \{\ldots -4, -1, 2, \ldots\} = [5] = [-1] = [8] = [-4] = \ldots$

Congruence Modulo

- Let m and n be integers and let d be a positive integer.

m is congruent to n modulo d:

$$m \equiv n \pmod{d} \Leftrightarrow d \mid (m - n)$$

Example:

$12 \equiv 7 \pmod{5}$ because $12 - 7 = 5 = 5 \cdot 1$

➔

$5 \mid (12 - 7)$.

Rational Numbers Are Equivalence Classes

- Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero: $A = Z \times (Z - \{0\})$

R is a relation on A: for all (a, b), (c, d) ∈ A,

$$(a, b)\ R\ (c, d) \Leftrightarrow a/b = c/d$$

R is an equivalence relation

Example equivalence class:

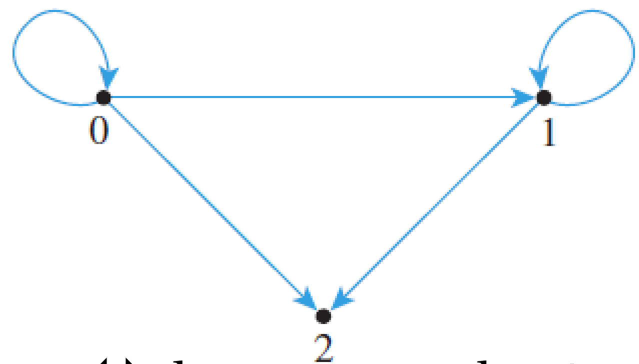
$$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6), (-3, -6), \ldots\}$$

$$\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6} \text{ and so forth.}$$

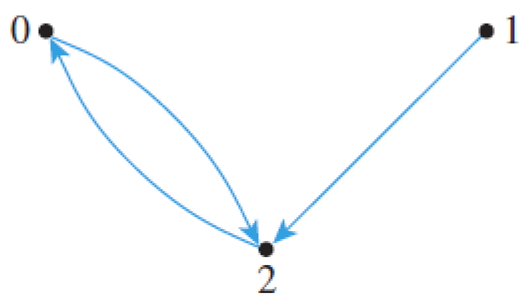
Antisymmetry

- Let R be a relation on a set A.

R is *antisymmetric* ⇔ for all a,b∈A, if aRb and bRa then a=b



R is *not antisymmetric* ⇔ there exist a,b∈A s.t. aRb, bRa, but a≠b



0 R 2 and 2 R 0 but 0≠2

Antisymmetry of "Divides" Relations

- For all a, b $\in Z^+$, a R_1 b $\Leftrightarrow$ a | b.

R_1 is antisymmetric: Suppose a, b $\in Z^+$ such that aR_1b and bR_1a. [We must show that a = b]

By definition of R_1, a|b and b|a $\rightarrow$ b=k_1a and a=k_2b, k_1,$k_2$$\in$ Z (and both are positive since a and b are positive) $\rightarrow$ b=k_1k_2b $\rightarrow$

Dividing both sides by b gives k_1k_2=1 (and both >0) $\rightarrow$ k_1=k_2=1 $\rightarrow$ a=b

- For all a, b $\in$ Z, a R_2 b $\Leftrightarrow$ a | b.

R_2 is not antisymmetric:

Counterexample: a = 2 and b = −2 $\rightarrow$ a $\neq$ b

a | b since −2 = (−1) · 2 $\rightarrow$ a R_2 b

b | a since 2 = (−1)(−2) $\rightarrow$ b R_2 a

Partial Order Relations

- Let R be a relation defined on a set A.

R is a *partial order relation* ⇔ R is reflexive, antisymmetric and transitive.

- Example: The "Subset" Relation

Let A be any collection of sets and ⊆ (the "subset") relation on A:

For all U, V ∈ A,   U ⊆ V ⇔ for all x, if x ∈ U then x ∈ V.

⊆ is a partial order (reflexive, transitive and antisymmetric)

Proof that ⊆ is antisymmetric: for all sets U and V in A

if U ⊆ V and V ⊆ U then U = V (by definition of equality of sets)

The "Less Than or Equal to" Relation

- The "less than or equal to" relation $\leq$ on **R** (reals): for all x,y∈**R**

$$x \leq y \Leftrightarrow x < y \text{ or } x = y.$$

$\leq$ is a partial order relation:

$\leq$ is reflexive: $x \leq x$ for all real numbers. $x \leq x$ means that $x < x$ or $x = x$, and $x = x$ is always true.

$\leq$ is antisymmetric: for all x,y∈**R**, if $x \leq y$ and $y \leq x$ then $x = y$.

$\leq$ is transitive: for all x,y,z∈**R**, if $x \leq y$ and $y \leq z$ then $x \leq z$.

Lexicographic Order

- Order in an English dictionary: compare letters one by one from left to right in words.
- Let A be a set with a partial order relation R, and let S be a set of strings over A. $\preccurlyeq$ is a relation on S: for any 2 strings in S, $a_1a_2\ldots a_m$ and $b_1b_2\ldots b_n$, $m,n \in Z^+$:

1. If $m \leq n$ and $a_i=b_i$ for all $i=1,2,\ldots,m$, then $a_1a_2\ldots a_m \preccurlyeq b_1b_2\ldots b_n$
2. If for some integer k with $k\leq m$, $k\leq n$, and $k\geq 1$, $a_i=b_i$ for all $i=1,2,\ldots,k-1$, and $a_k\neq b_k$, but a_kRb_k then $a_1a_2\ldots a_m \preccurlyeq b_1b_2\ldots b_n$.
3. If ε is the null string and s is any string in S, then $\varepsilon \preccurlyeq s$.

If no strings are related other than by these three conditions, then $\preccurlyeq$ is a partial order relation (**lexicographic order for S**).

Lexicographic Order

- Let $A = \{x, y\}$ and R the partial order relation on A:

$R = \{(x, x), (x, y), (y, y)\}$.

Let S be the set of all strings over A, and $\preccurlyeq$ the lexicographic order for S that corresponds to R.

$x \preccurlyeq xx$

$x \preccurlyeq xy$

$yxy \preccurlyeq yxyxxx$

$x \preccurlyeq y$

$xx \preccurlyeq xyx$

$xxxy \preccurlyeq xy$

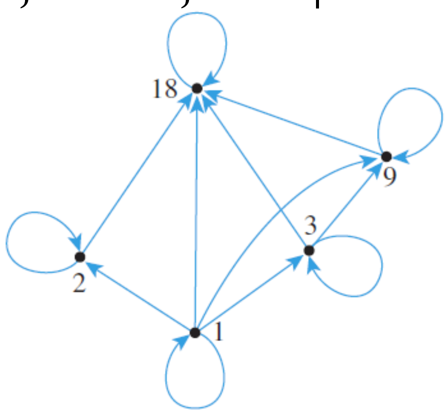
$\varepsilon \preccurlyeq x$

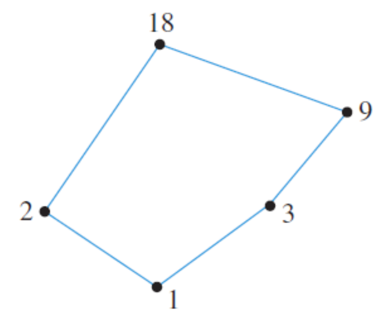
$\varepsilon \preccurlyeq xyxyyx$

Hasse Diagrams

- A Hasse Diagram is a simpler graph with a partial order relation defined on a finite set
- Example: let A = {1, 2, 3, 9, 18} and the "divides" relation | on A:

for all a, b ∈ A, $a \mid b \Leftrightarrow b = ka$ for some integer k.





- Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Eliminate:

1. the loops at all the vertices
2. all arrows whose existence is implied by the transitive property
3. the direction indicators on the arrows

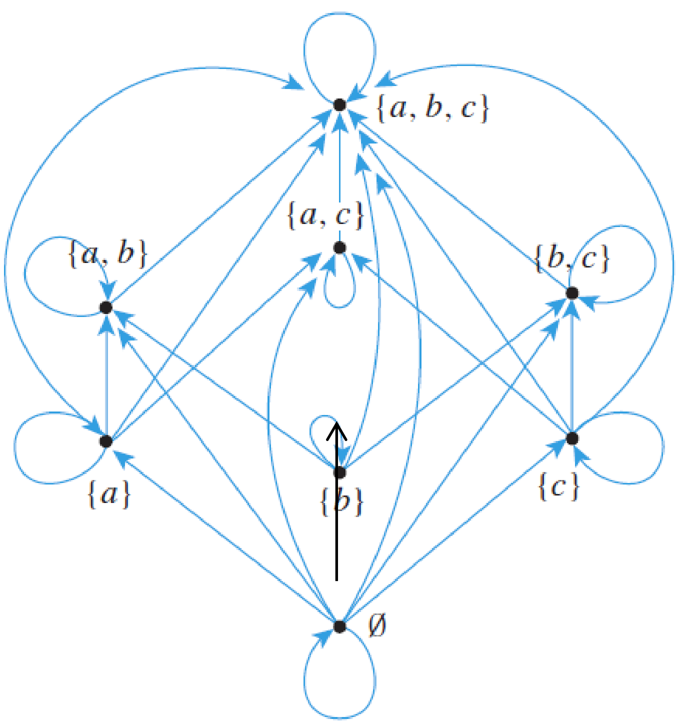
Hasse Diagrams

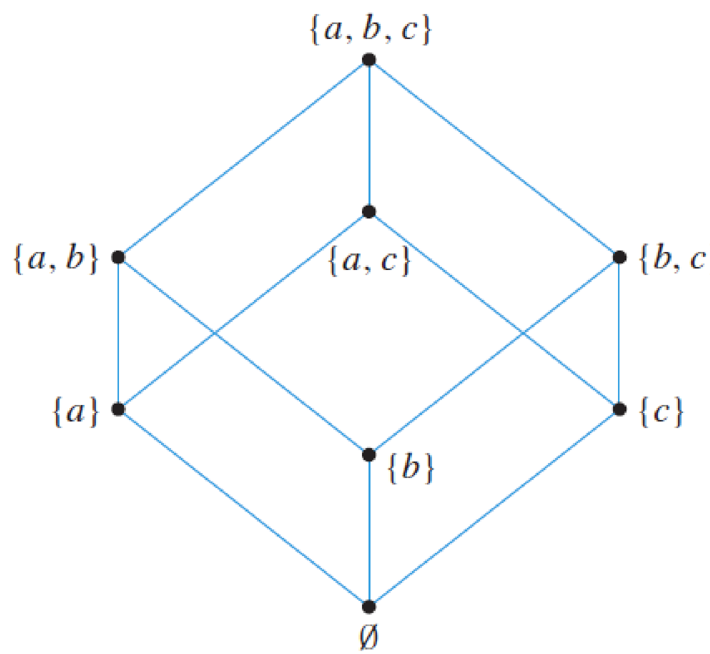
- The "subset" relation ⊆ on the set P({a, b, c}): for all sets U and V in P({a, b, c})

$$U \subseteq V \Leftrightarrow \forall x, \text{ if } x \in U \text{ then } x \in V$$

Draw the directed graph of the relation in such a way that all arrows except loops point upward.

Strip away all loops, unnecessary arrows, and direction indicators to obtain the Hasse diagram.

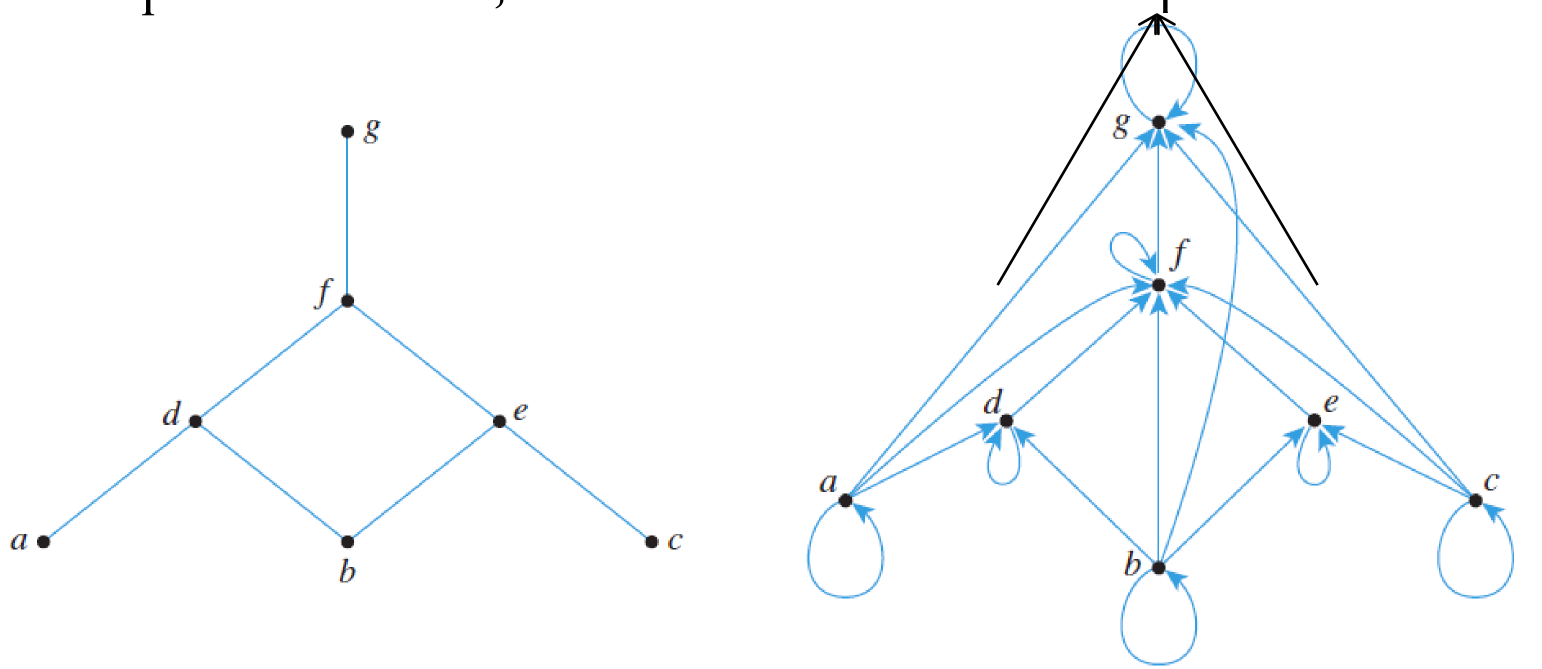




Hasse Diagrams

- Obtain the original directed graph from the Hasse diagram:

1. Reinsert the direction markers on the arrows making all arrows point upward.
2. Add loops at each vertex.
3. For each sequence of arrows from one point to a second and from that second point to a third, add an arrow from the first point to the third.



Partially and Totally Ordered Sets

- Let $\preccurlyeq$ be a partial order relation on a set A. Elements a and b of A are *comparable* $\Leftrightarrow$ either a $\preccurlyeq$ b or b $\preccurlyeq$ a. Otherwise, a and b are *noncomparable*.
- If R is a partial order relation on a set A, and any two elements a and b in A are comparable, then R is a *total order relation on* A.
  - The Hasse diagram for a total order relation can be drawn as a single vertical "chain."
- A set A is called a *partially ordered set* (or *poset*) with respect to a relation $\preccurlyeq$ $\Leftrightarrow$ $\preccurlyeq$ is a partial order relation on A.
- A set A is called a *totally ordered set* with respect to a relation $\preccurlyeq$ $\Leftrightarrow$ A is partially ordered with respect to $\preccurlyeq$ and $\preccurlyeq$ is a total order.

Partially and Totally Ordered Sets

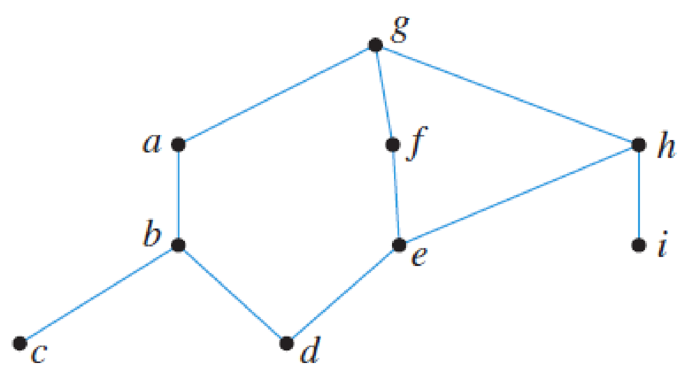
- Let A be a set that is partially ordered with respect to a relation $\preccurlyeq$. A subset B of A is called a *chain* $\Leftrightarrow$ the elements in each pair of elements in B is comparable.
- The *length* of a chain is one less than the number of elements in the chain.
- Example: Chain of Subsets

The set $P(\{a, b, c\})$ is partially ordered with respect to $\subseteq$.

Chain of length 3: $\emptyset \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$

Partially and Totally Ordered Sets

- An element a in A is called a *maximal element* ⇔ for all b in A, either b≼a or b and a are not comparable.
- An element a in A is called a *greatest element* of A ⇔ for all b in A, b≼a.
- An element a in A is called a *minimal element* ⇔ for all b in A, either a≼b or b and a are not comparable.
- An element a in A is called a *least element* of A ⇔ for all b in A, a≼b.
- Example:



  - one maximal element = g = also the greatest element
  - minimal elements: c, d and i
  - there is no least element

Topological Sorting

- Given partial order relations ≼ and ≼' on a set A, ≼' is *compatible* with ≼ ⇔ for all a and b in A, if a ≼ b then a≼'b
- Given partial order relations ≼ and ≼' on a set A, ≼' is a *topological sorting* for ≼ ⇔ ≼' is a total order that is compatible with ≼.
- Example: P({a, b, c}) with partial order ⊆ (any element in P({a,b,c}) we can either compare them or not, e.g., {a,b} with {a,c}

  Total order:

  ∅≼'{a}≼'{b}≼'{c}≼'{a, b}≼'{a, c}≼' {b, c}≼'{a, b, c}

Topological Sorting

- Constructing a Topological Sorting:

1. Pick any minimal element x in A with respect to $\preccurlyeq$.

   [Such an element exists since A is nonempty.]

2. Set A' = A − {x}

3. Repeat steps a–c while A' ≠ ∅:

   a. Pick any minimal element y in A'.

   b. Define x $\preccurlyeq$' y.

   c. Set A' = A' − {y} and x = y.