Lecture 2: Shannon and Perfect Secrecy

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Spring 2018 (CSE390)

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- We discussed some historical ciphers
- ...and how to break them

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- Volunteer for today's scribes?

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$$\forall k, \forall m : \mathsf{Dec}(k, \mathsf{Enc}(k, m)) = m.$$

- The set of all messages m is called *message space* $\mathcal{M}$;
- c is called the *ciphertext* and set of all ciphertexts *ciphertext space* $\mathcal{C}$;
- The set of all keys k is called the *key space* $\mathcal{K}$.

Security of a Cipher

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What should it mean **intuitively**?

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- All ciphers in the frequency analysis recover the key...

  What if we just guarantee that key remains completely hidden?
- No reason why plaintext should be hidden!
- Example from Caesar Cipher:
  ATTACK = BUUBDL and DEFEND = EFGFOE

  Broken by checking patterns! don't need the key!

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- Dangerous: May be enough to find out if the army will attack or defend?
- Hide *everything* about the message: all possible functions of the message.
    - Good starting point but impossible! Something about the message may already be known!
      (E.g., it is in English, starts with “Hello” and today’s date, etc.)

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- How to capture it mathematically?

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  - m chosen according to D
  - k is chosen randomly (according to KG)
  - Enc may also use some randomness
  - These induce a distribution C over the ciphertexts c.
- The adversary only observes c
  (for some $m \overset{D}{\leftarrow} \mathcal{M}$ and $k \overset{\mathsf{KG}}{\leftarrow} \mathcal{K}$, but m, k themselves)

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- Knowledge about m **after** observing the output of C is captured by: $D|C$
- **Shannon secrecy**: distribution D and $D|C$ must be **identical**.
- Intuitively, this means that:

  C contains **no NEW information** about m

  ...in the standard sense of information theory.

Shannon Secrecy

Definition (Shannon Secrecy)

A cipher $(\mathcal{M}, \mathcal{K}, \mathsf{KG}, \mathsf{Enc}, \mathsf{Dec})$ is **Shannon secure w.r.t a distribution** D over $\mathcal{M}$ if for all $m' \in \mathcal{M}$ and for all c,

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$$\Pr\left[m \leftarrow D : m = m'\right] = \Pr\left[k \leftarrow \mathsf{KG}, m \leftarrow D : m = m' | \mathsf{Enc}(m,k) = c\right]$$

It is Shannon secure if it is Shannon secure w.r.t. all distributions D over $\mathcal{M}$.

Questions?

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- Suppose you have two messages: $m_1 \in \mathcal{M}$ and $m_2 \in \mathcal{M}$.

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- **Perfect secrecy:**
  C_1 and C_2 must be **identical for every pair** of m_1, m_2.

  $\Rightarrow$ Ciphertexts are independent of the plaintext(s)!

Perfect Secrecy (conitinued)

Definition (Perfect Secrecy)

Scheme $(\mathcal{M}, \mathcal{K}, \mathsf{KG}, \mathsf{Enc}, \mathsf{Dec})$ is **perfectly secure** for every pair of messages m_1, m_2 in $\mathcal{M}$ and for all c,

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$$\Pr\left[k \leftarrow \mathsf{KG} : \mathsf{Enc}(m_1, k) = c\right] = \Pr\left[k \leftarrow \mathsf{KG} : \mathsf{Enc}(m_2, k) = c\right]$$

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- So much simpler than Shannon Secrecy!
- No mention of distributions, a priori or posteriori.
- Much easier to work with...

Which notion is better?

- OK, so we have two definitions: perfect secrecy and Shannon secrecy.
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- OK, so we have two definitions: perfect secrecy and Shannon secrecy.
- Both of them intuitively seem to guarantee great security!

- Is one better than the other?
- If our intuition is right, shouldn't they offer "same level" of security?

Equivalence Theorem

Theorem (Equivalence Theorem)

A private-key encryption scheme is perfectly secure if and only if it is Shannon secure.

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- For example:
  - $\Pr_m[\ldots]$ means $\Pr[m \leftarrow D : \ldots]$
  - $\Pr_k[\ldots]$ means $\Pr[k \leftarrow \mathsf{KG} : \ldots]$
  - $\Pr_{k,m}[\ldots]$ means $\Pr[k \leftarrow \mathsf{KG}, m \leftarrow D : \ldots]$

Proof: Perfect Secrecy $\Rightarrow$ Shannon Secrecy

Proof: Perfect Secrecy ⇒ Shannon Secrecy

Given: $\forall(m_1, m_2) \in \mathcal{M} \times \mathcal{M}$ and every $c \in \mathcal{C}$:

$$\Pr_k[\mathsf{Enc}_k(m_1) = c] = \Pr_k[\mathsf{Enc}_k(m_2) = c]$$

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Show: for every D over $\mathcal{M}$, $m' \in \mathcal{M}$, and $c \in \mathcal{C}$:

$$\Pr_{k,m}[m = m' | \mathsf{Enc}_k(m) = c] = \Pr_m[m = m']$$

Proof: Perfect Secrecy ⇒ Shannon Secrecy (continued)

$$\text{L.H.S.} \quad = \quad \Pr_{k,m}[m = m' | \mathsf{Enc}_k(m) = c]$$

Proof: Perfect Secrecy $\Rightarrow$ Shannon Secrecy (continued)

$$\begin{aligned} \text{L.H.S.} &= \Pr_{k,m}[m = m' | \mathsf{Enc}_k(m) = c] \\ &= \frac{\Pr_{k,m}[m{=}m' \quad \cap \quad \mathsf{Enc}_k(m){=}c]}{\Pr_{k,m}[\mathsf{Enc}_k(m){=}c]} \end{aligned}$$

Proof: Perfect Secrecy ⇒ Shannon Secrecy (continued)

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\text{L.H.S.} &= \Pr_{k,m}[m = m' | \mathsf{Enc}_k(m) = c] \\
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&= \frac{\Pr_{m}[m{=}m'] \cdot \Pr_{k}[\mathsf{Enc}_k(m'){=}c]}{\Pr_{k,m}[\mathsf{Enc}_k(m){=}c]}
\end{aligned}$$

Proof: Perfect Secrecy $\Rightarrow$ Shannon Secrecy (continued)

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&= \frac{\Pr_{k,m}[m=m' \quad \cap \quad \mathsf{Enc}_k(m')=c]}{\Pr_{k,m}[\mathsf{Enc}_k(m)=c]} \\
&= \frac{\Pr_m[m=m'] \cdot \Pr_k[\mathsf{Enc}_k(m')=c]}{\Pr_{k,m}[\mathsf{Enc}_k(m)=c]} \\
&= \text{R.H.S.} \quad \times \quad \frac{\Pr_k[\mathsf{Enc}_k(m')=c]}{\Pr_{k,m}[\mathsf{Enc}_k(m)=c]}
\end{aligned}
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Proof: Perfect Secrecy ⇒ Shannon Secrecy (continued)

Show:

$$\frac{\Pr_k[\mathsf{Enc}_k(m') = c]}{\Pr_{k,m}[\mathsf{Enc}_k(m) = c]} = 1$$

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&= \Pr_k[\mathsf{Enc}_k(m') = c] \times 1. \quad \text{(QED)}
\end{aligned}$$

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We have to show: $\forall (m_1, m_2) \in \mathcal{M} \times \mathcal{M}$ and $\forall c$:

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Fix any m_1, m_2, c as above.
Let D be the uniform distribution over $\{m_1, m_2\}$ so that:

$$\Pr_m[m = m_1] = \Pr_m[m = m_2] = 1/2.$$

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By definition, the scheme is Shannon secure w.r.t. this D. Therefore,

$$\begin{aligned} \Pr_{k,m}[m = m_1 | \mathsf{Enc}_k(m) = c] &= \Pr_m[m = m_1], \quad \text{and} \\ \Pr_{k,m}[m = m_2 | \mathsf{Enc}_k(m) = c] &= \Pr_m[m = m_2] \end{aligned}$$

Proof: Perfect Secrecy $\Leftarrow$ Shannon Secrecy (continued)

Therefore: $\Pr_{k,m}[m = m_1 | \mathsf{Enc}_k(m) = c] = \Pr_{k,m}[m = m_2 | \mathsf{Enc}_k(m) = c]$

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Therefore: $\Pr_{k,m}[m = m_1 | \mathsf{Enc}_k(m) = c] = \Pr_{k,m}[m = m_2 | \mathsf{Enc}_k(m) = c]$

Consider the LHS:

$$\Pr_{k,m}[m = m_1 | \mathsf{Enc}_k(m) = c] \quad =$$

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Cancel and rearrange. (QED)

Should we go over this proof again?

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  - $\mathsf{Dec}(c,k)$: XOR bit-by-bit.
    Return m where $m_i = c_i \oplus k_i$ for every i.

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$\Rightarrow \forall (m_1, m_2) \in \{0,1\}^{n \times n}$ and $\forall c :$
$\Pr_k[\mathsf{Enc}_k(m_1) = c] = \Pr_k[\mathsf{Enc}_k(m_2) = c]. \quad (QED)$

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  - You can work modulo any number n
- As the name suggests, one key can be used only once.
- The key must be:
  - sampled uniformly every time, and
  - be as long as the message.

Key Length in Perfectly Secure Encryption

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  - This is never done in practice...
- OTP looks naïve, quite elementary: can't we design a more sophisticated scheme with shorter keys?

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Some Remarks:

- Message length is $n = \lg |\mathcal{M}|$ and key length is $\ell = \lg |\mathcal{K}|$.
- It follows that $\ell \geqslant n$, i.e., keys must be as long as the messages.

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- Can you construct such examples?

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- It means we really have to give up on perfect secrecy for practical applications, unless we absolutely need it.
- This is really the dawn of modern cryptography: we want to construct something that is "just as good for practical purposes."