Generalized Josephus Problem

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1 Problem Statement

- Given n people numbered $0..n-1$ in a circle waiting to be executed. Starting with the first person, we eliminate every k^{th} person in a circular manner until only 1 survivor is left.
- Given the values of n and k, design an efficient algorithm to find the survivor's number $J(n,k)$ in the initial circle.

2 Definitions

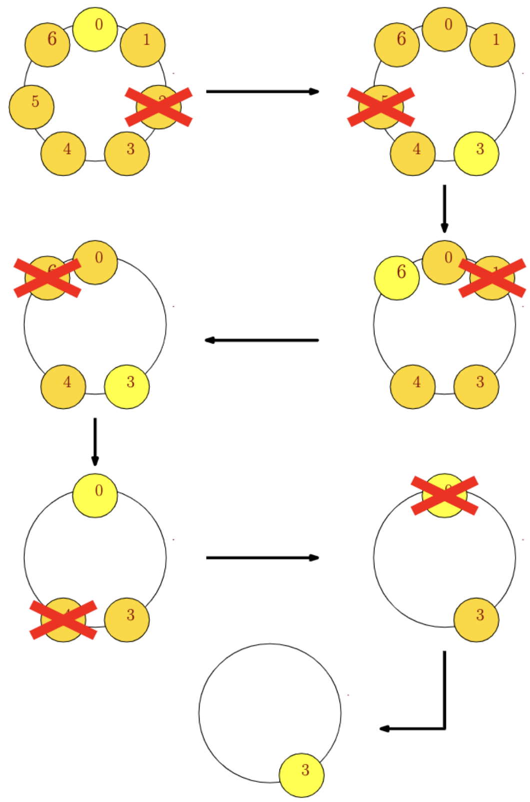
J(n,k) = Survivor's position in a circle of n people
with every k^{th} person being executed

3 Approach

- Instead of executing 1 person in an iteration, we execute as many number of people we can execute in a single iteration.
- If n is the number of people in the circle, we can remove $\lfloor \frac{n}{k} \rfloor$ in a single iteration. This process is repeated till n becomes less than k (step size).
- When $n < k$, we follow the naive algorithm given by the recurrence:

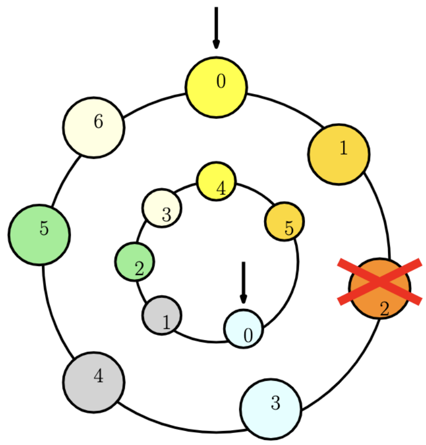
$$J(n,k) = (J(n-1,k) + k)\%n$$

- When $n >= k$, we follow the generalized approach which computes $J(n,k)$ in a more efficient way.



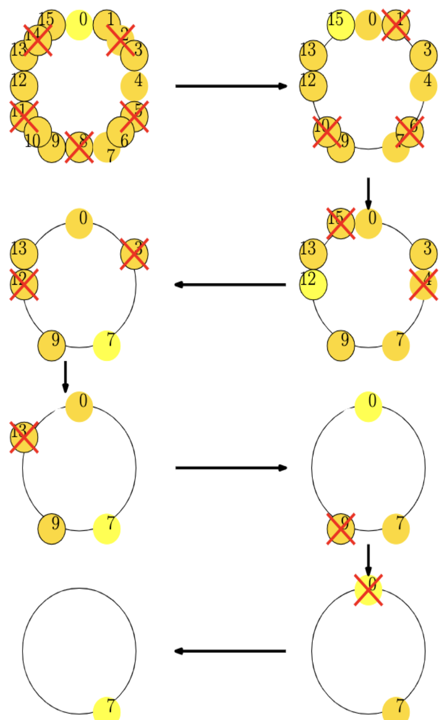
$$J(7,3) = (J(6,3) + 3) \mod 7$$

Naive Approach



$$J(n,k) = (J(n-1,k)+k) \mod n$$

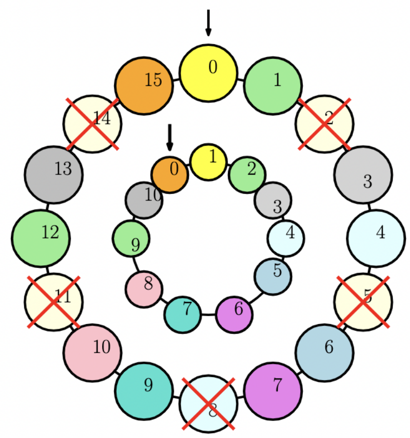
Naive Approach : Mapping & General Formula



$$J(16,3) = \lfloor \frac{3(J(16-\lfloor \frac{16}{3} \rfloor,3)-16 \mod 3) \mod 16}{3-1} \rfloor$$

$$J(16,3) = \lfloor \frac{3(J(11,3)-1) \mod 16}{2} \rfloor$$

Efficient Approach



$$J(n,k) = \lfloor \frac{k(J(n - \lfloor \frac{n}{k} \rfloor, k) - n \mod k) \mod n}{k-1} \rfloor$$

Efficient Approach : Mapping & General Formula

4 Derivation

- Consider a circle having j people (say $j = 7$ as an example) waiting to be executed. Then after the first iteration exactly j - $\lfloor \frac{j}{k} \rfloor$ people are left in the circle.

- We define this new circle to contain i people where:

$$i = j - \lfloor \frac{j}{k} \rfloor$$

- To reiterate, we are considering two circles now - one having j people and the other circle formed after 1 iteration of the execution process, having exactly i people.

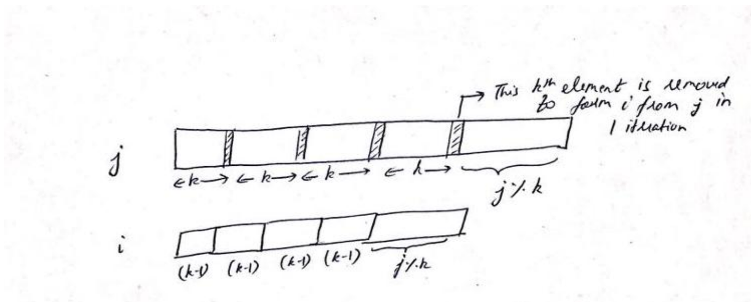
- Thus, j can be written as:

$$j = \lfloor \frac{j}{k} \rfloor * k + j\%k$$

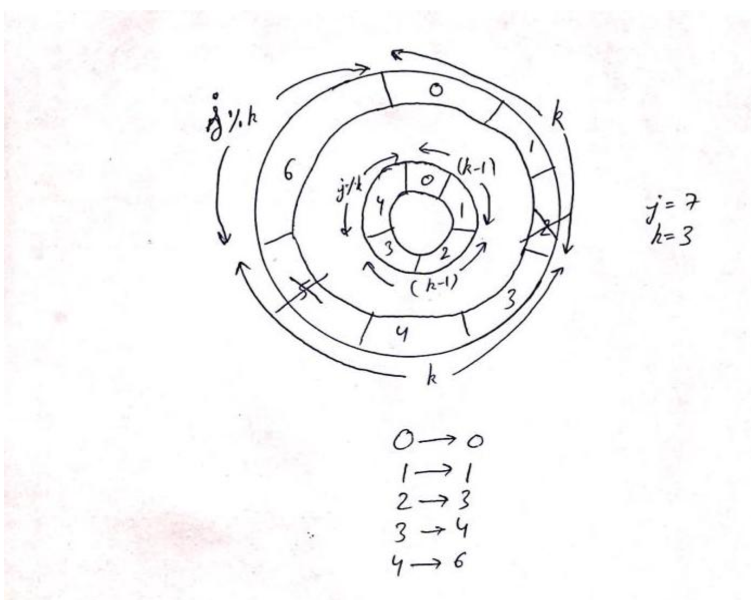
  And i can we written as:

$$i = \lfloor \frac{j}{k} \rfloor * (k-1) + j\%k$$

- Therefore, we can visualize j to be made up of $\lfloor \frac{j}{k} \rfloor$ batches of k length and a length j%k. Similarly, i can be seen as made up of $\lfloor \frac{j}{k} \rfloor$ batches of $(k-1)$ length and a length j%k.

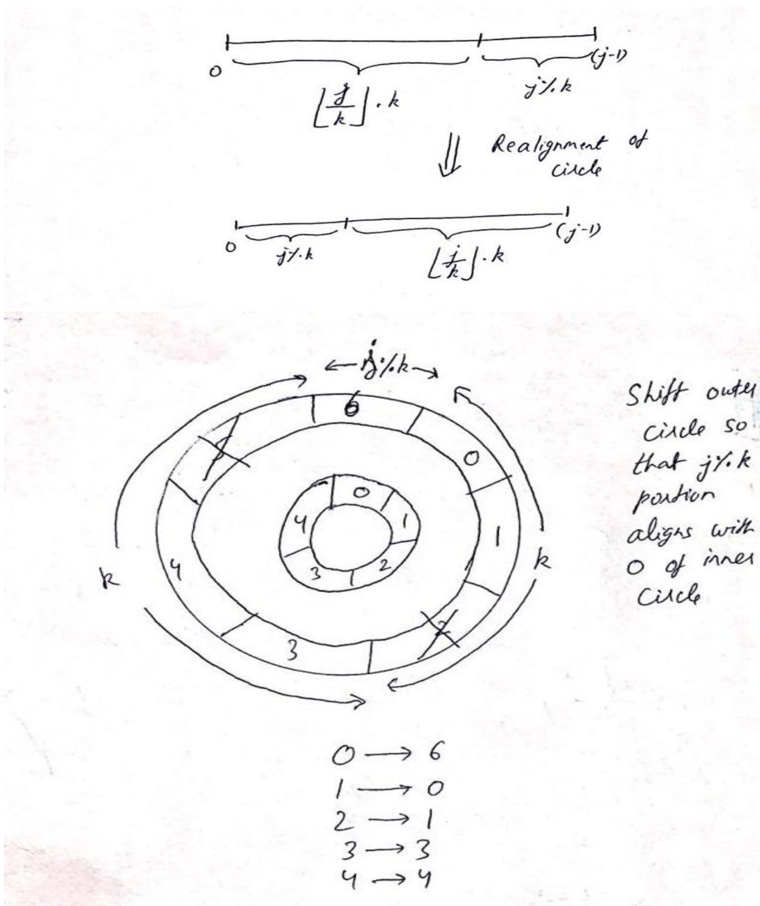


- It is clear from the above diagram that a portion of length of the circle (j%k) remains constant while the $k * \lfloor \frac{j}{k} \rfloor$ part of the circle reduces to $(k-1) * \lfloor \frac{j}{k} \rfloor$.
- Now, let us take the example of a circle having 7 people (i.e. $n = 7$) and $k = 3$. Since $j - \lfloor \frac{j}{k} \rfloor$ = 7 - $\lfloor \frac{7}{3} \rfloor$ = 5. There are 5 people in the second circle. This can be visualized as follows:



- In the above diagram, we see that every kth person is executed and a mapping between the new (i circle) and the old circle (j circle) is done.
- Currently, the circles has the 0^{th} (or the first person) being on the side of the $\lfloor \frac{j}{k} \rfloor * k$ length portion and the $(j-1)^{th}$ person aligned with the $j\%k$ length portion of the circle.

- Since, it is a circle, we can realign this axis by moving clockwise or anti-clockwise. We can move the j circle by $n\%k$ clockwise or $\lfloor \frac{j}{k} \rfloor * k$ anti-clockwise so that it align the $j\%k$ portion of the outer circle (j circle) with the 0 of the inner circle (i circle). This will lead to a shift in the mapping between the two circles.

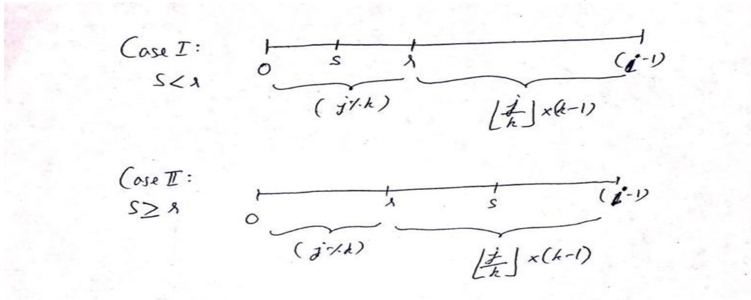


- We did this because we observe that there are two portions of each circle. The $j\%k$ portion does not change in size while the $\lfloor \frac{j}{k} \rfloor * k$ portion changes in size. Depending upon the location of the last survivor in the i circle, we can calculate the corresponding location in the j circle. In doing so, the difference in the growth of the two portions of the circle play a big role. We aligned the same length portion of the two circles together so that we can compare the lengths of the other portions easily.

- Let us define s as the position of the survivor in the i circle and $r = j\%k$. Therefore,

$$s = J(j - \lfloor \frac{j}{k} \rfloor, k)$$

- s can be anywhere in the circle i and hence we get two cases as below:



Case I - $(s < r)$:
Since, s lies in the portion $j\%k$ which remains of the same length in the two circles, there is no other additional shift in the position of s in the j circle. However, since we realigned the axis of the outer j circle, a shift of $\lfloor \frac{j}{k} \rfloor * k$ is introduced (since we rotated by that amount).

Therefore,
$J(j, k) = s + \lfloor \frac{j}{k} \rfloor * k = s + j - j\%k$

Case II - $(s >= r)$:
Since, s lies in the portion that changes in length, there would be an additional shift while mapping from the i circle to the j circle. This would be accompanied by the shift caused by the realignment of the axis.

We know that, $\lfloor \frac{j}{k} \rfloor = \lfloor \frac{i}{k-1} \rfloor$ = number of blocks of size k (j circle) or (k-1)(i circle).

Therefore, for a distance of $(s - r)$ in the i circle, a distance of $\frac{k*(s-r)}{k-1}$ is there in the j circle. Thus, the additional shift is:
$= \frac{k*(s-r)}{k-1} - (s - r)$
$= (s - r)[\frac{k}{k-1} - 1]$
$= \frac{(s-r)}{(k-1)}$

Therefore,
$J(j, k) = (s + \lfloor \frac{j}{k} \rfloor * k + \frac{(s-r)}{(k-1)})\%j = (s + j - j\%k + \frac{(s-r)}{(k-1)})\%j$
but $j\%k = r$.
Hence, $J(j, k) = ((s - r)(1 + \frac{1}{(k-1)}))\%j$
$= \frac{k*(s-r)}{(k-1)}\%j$

5 Final Recurrence

$$J(n,k) = \begin{Bmatrix} 0 & \text{if } n = 1, \\ (J(n-1,k)+k) \mod n & \text{if } 1 < n < k, \\ \begin{Bmatrix} N + n \text{ if } N < 0, \\ \lfloor \frac{k \times N}{k-1} \rfloor \mod n \text{ if } N \geq 0 \end{Bmatrix} & \text{if } k \leq n \end{Bmatrix}$$

where $N = J(n - \lfloor \frac{n}{k} \rfloor, k) - n \mod k$

6 Pseudo Code

Input: Given the number of people in a circle n and the k^{th} person being executed in every iteration.
Output: Survivor's position in the initial circle
1. **if** n = 1 **then return** 0
2. **if** k = 1 **then return** $n - 1$
3. **if** $k > n$ **then return** $(J(n-1,k)+k) \mod n$
4. $N \leftarrow J(n - \lfloor \frac{n}{k} \rfloor, k) - n \mod k$
5. **if** $N < 0$ **then** $N \leftarrow N + n$ **else** $N \leftarrow N + \frac{N}{(k-1)}$
6. **return** N

7 Complexity

- During each iteration, the number of people decreases by a factor of $(1-\frac{1}{k})$. Thus, at the end when only 1 survivor remains:

$$n * (1 - \tfrac{1}{k})^x = 1$$

- Taking natural log both sides,

$$\ln(n) + x * \ln(1 - \tfrac{1}{k}) = 0$$
$$x = \frac{-\ln(n)}{ln(1-\frac{1}{k})}$$

- Expanding using taylor series,

$$x = k \ln(n)$$

- Therefore, time complexity is O(klog n).
- Space complexity is O(n).

8 References

- https://cp-algorithms.com/others/josephus_problem.html
- https://developer.aliyun.com/article/602333