CSE 612: Advanced Visualization

Lecture 3: Fourier Transforms and Applications

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Fourier Transforms Overview

- Discussed in CSE 564
  - fundamentals of the Fourier Transform (FT)
  - various interpolation and gradient filters in time and frequency domain
  - causes of aliasing and its prevention
  - choice of filter width as a function of sampling rate
  - separabilty of common filters for higher dimensions (>1D)
- To be discussed now, in CSE 612
  - more details on the continuous and discrete FT
  - useful properties of the FT
  - the Fast Fourier Transform (FFT)
  - popular applications of the FFT
  - spherical harmonics ("FT" on a sphere)

From Continuous to Discrete

- FT of a non-periodic signal (Continuous FT):

  - infinite continuous signal and infinite continuous spectrum

$$F(u) = \int_{x=-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \qquad f(x) = \int_{u=-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

- FT of a periodic signal (Fourier Series):

  - periodic continuous signal (period=T), but infinite discrete spectrum (spacing=1/T)

$$F(k) = \frac{1}{T} \int_{x=0}^{T} f(x) e^{-j2\pi kx/T} dx \qquad f(x) = \frac{1}{T} \sum_{k=-\infty}^{\infty} F(k) e^{j2\pi kx/T}$$

- FT of a periodic sampled signal (Discrete FT or DFT):

  - periodic discrete signal (N samples) both in spatial and frequency domain

$$F(k) = \frac{1}{N} \sum_{i=0}^{N-1} f(i) e^{-j2\pi ik/N} \qquad f(i) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j2\pi ik/N}$$

- See “Introduction to the Discrete Fourier Transform” slides by van Vliet for more info

Fourier Transform Properties

- Linearity: $af(x) + bg(x) \leftrightarrow aF(u) + bG(v)$
- Time Shifting: $f(x - x_0) \leftrightarrow e^{-jux_0} F(u)$ and Frequency Shifting: $f(x)e^{jxu_0} \leftrightarrow F(u - u_0)$
- Differentiation: $\frac{df(x)}{dx} \leftrightarrow juF(u)$ and Integration: $\int f(x)dx \leftrightarrow \frac{1}{ju} F(u)$
- Scaling: $f(ax) \leftrightarrow \frac{1}{a} F\left(\frac{u}{a}\right)$
- Convolution: $f(x) \otimes g(x) \leftrightarrow F(u)G(u)$
- Time-Flip: $f(-x) \leftrightarrow F(-u)$
- Additional notes:

  - all properties work both ways, from space to frequency domain and vice versa

  - a zero-phase filter is a filter that leaves the phases alone, it has a phase slope of ϕ=0

  - a linear-phase filter scales the phases via a linear slope in ϕ, ϕ=k (shifts the signal in space)

  - all filters that are symmetric about 0 are zero-phase filters (box, linear, cubic, Gaussian,..)

  - Fourier Slice Theorem: “The FT of an X-ray projection is a slice across the f-domain”

The Fast Fourier Transform (FFT)

- Fast method to compute a DFT
- See presentation "FFT basics" for more detail

  - in this presentation: $W_N^{kn} = e^{-j2\pi kn/N}$

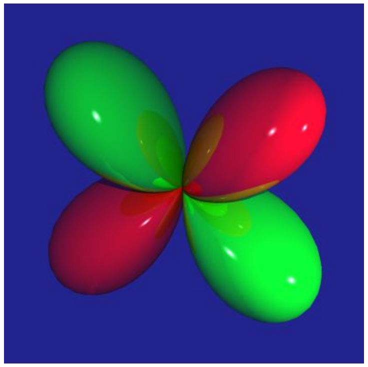
- Notes:

  - in FFT, the center data point is "aliased" and belongs to both the left and the right side of the spectrum

- Applications that take advantage of the FFT

  - convolution with a large filter
  - rotations with minimal error
  - patch matching using Euclidian distance
  - cross-correlation
  - resampling and zooming with minimal error

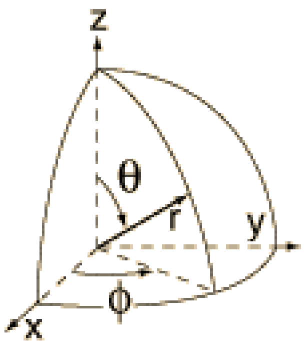
Spherical Harmonics



- A transform pair on the sphere:

$$C_{l,m} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} f(\theta,\phi) Y_{l,m}(\theta,\phi) \sin(\theta) d\theta d\phi$$

$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=0}^{l} C_{l,m} Y_{l,m}(\theta,\phi)$$



- The basis functions:

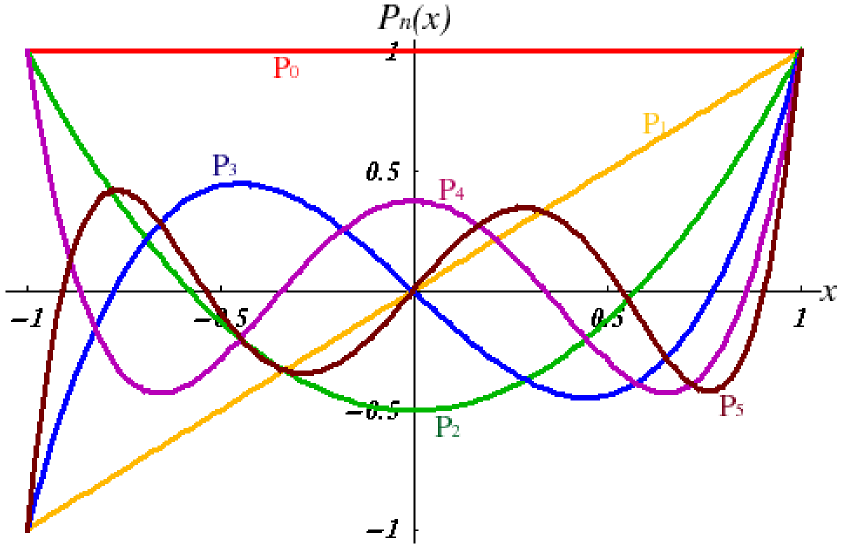
$$Y_{l,m}(\theta,\phi) = \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}} P_{m,l}(\cos\theta) e^{jm\phi}$$

where $Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$

and the $P_l(x)$ are the *Legendre polynomials*

and the $P_{l,m}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$ are the

associated Legendre polynomials

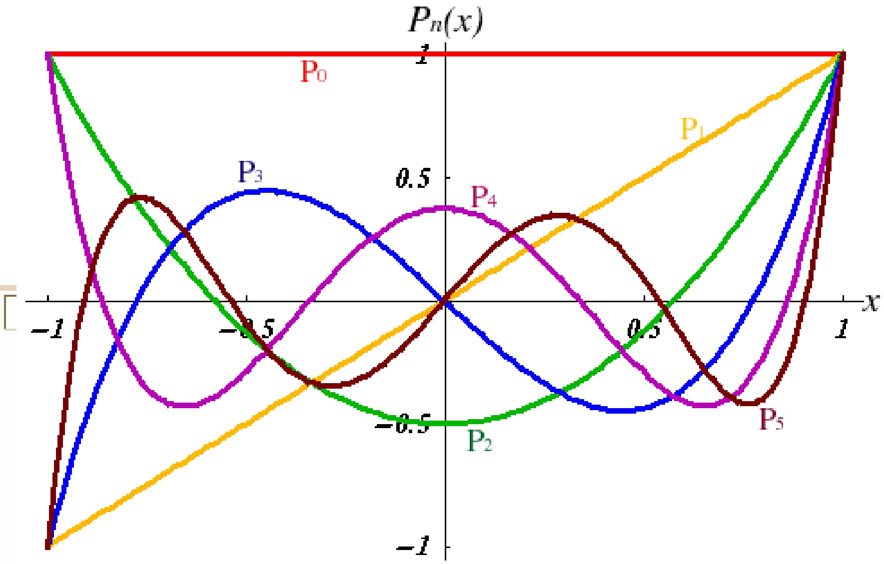


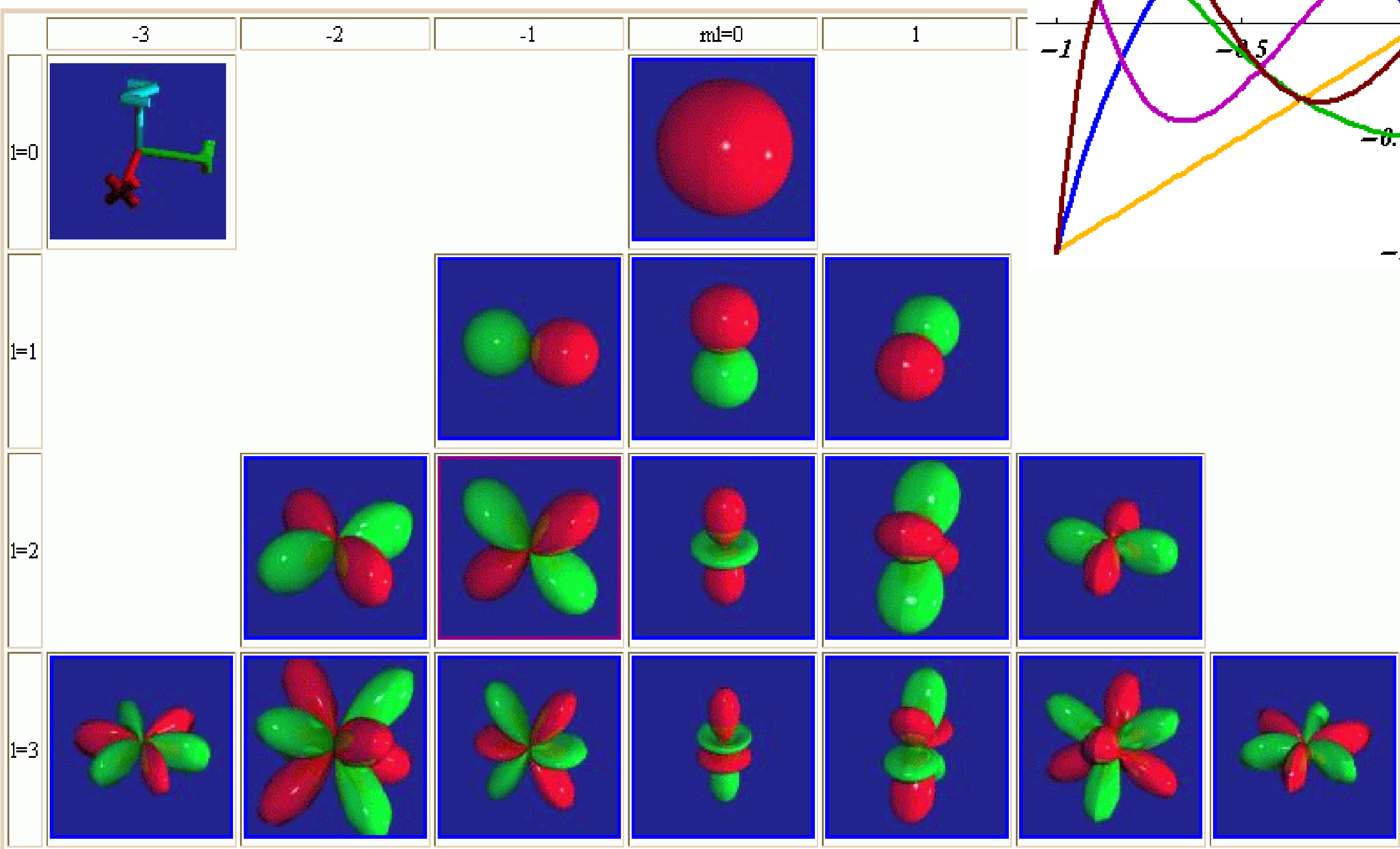
Spherical Harmonics - Plots

- Since a complex basis function would require 4D, let’s plot the real part:

$$S_{l,m}(\theta,\phi) = Re(Y_{l,m})^2$$

$$S_{l,0}(\theta,\phi) = Y_{l,0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

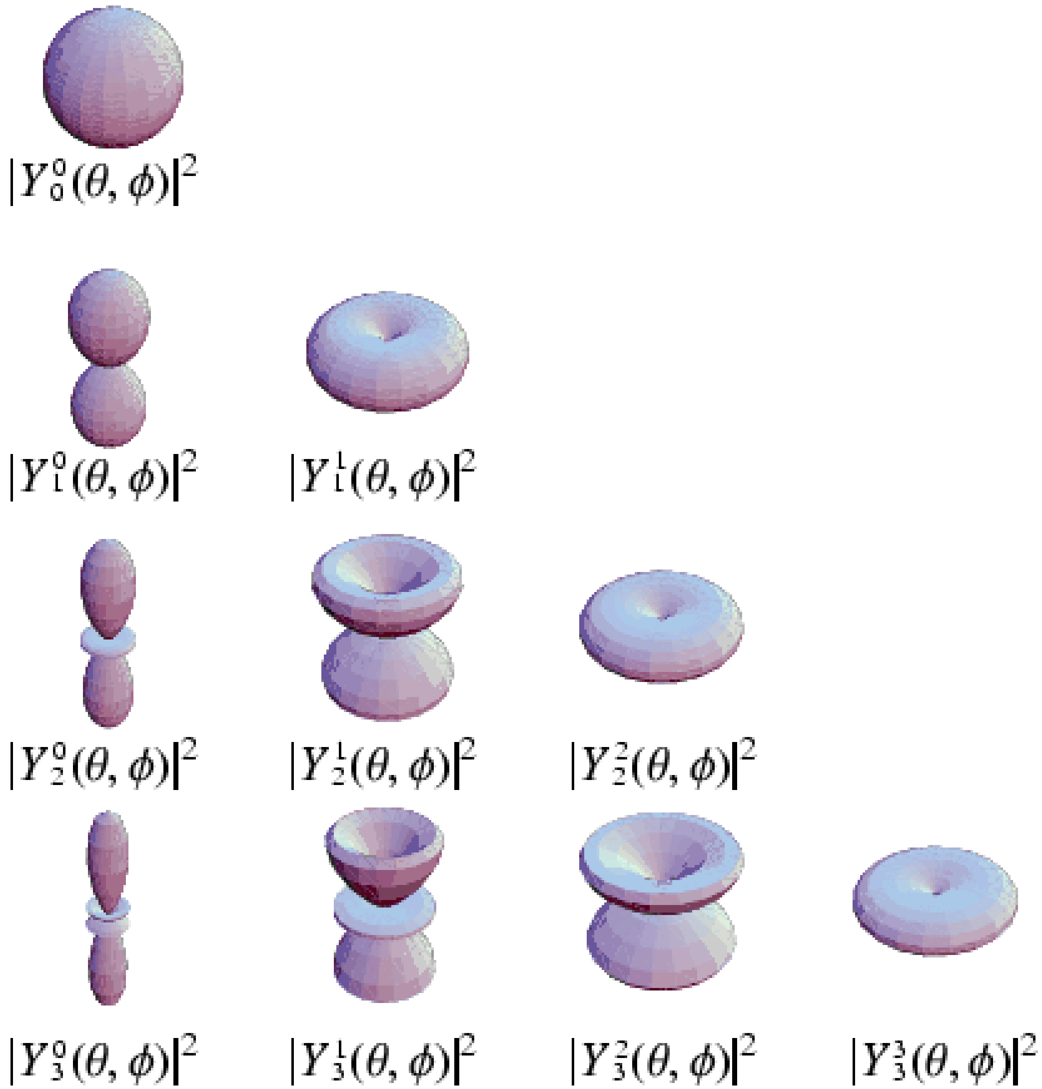




plot the radius at

Spherical Harmonics - Plots

- Alternatively, plot the magnitude:

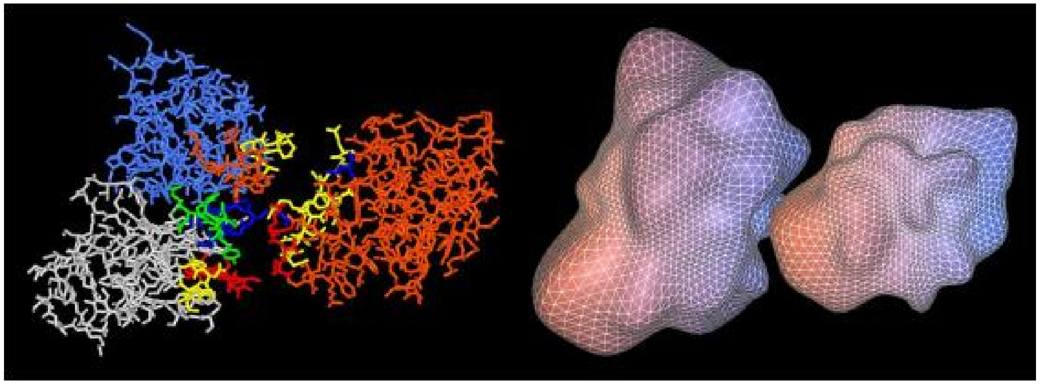


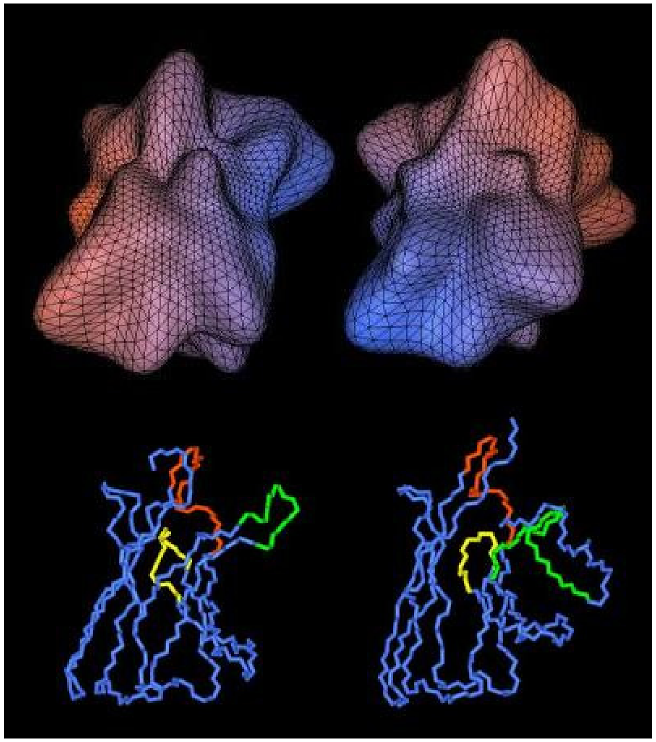
more info: mathworld.wolfram.com/SphericalHarmonic.html

Spherical Harmonics Applications

- Quantum mechanics
- Molecular modeling (need large L >14)

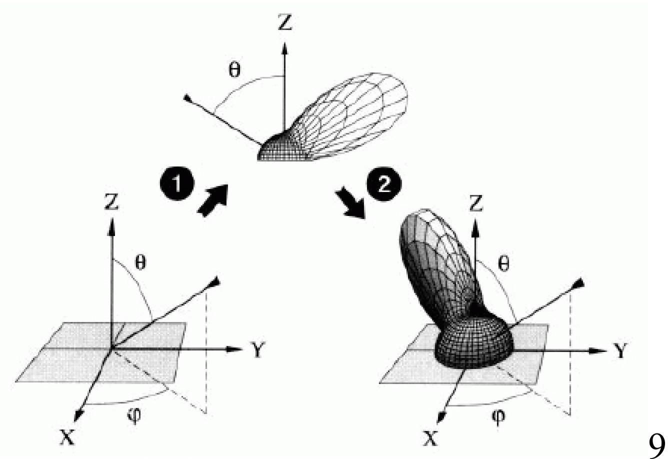
  - coefficent-wise comparisons, docking)





- BRDF (Bidirectional Reflection Distribution Function) modeling

  - gives sparse representation, but (like Fourier synthesis) will give somewhat smooth representation and does not offer much local control



Dave Ritchie, www.csd.abdn.ac.uk/~dritchie/graphics/

Spherical Harmonics Applications

- Modeling of shapes: