Introduction to the Discrete Fourier Transform

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Linear Shift Invariant System

A discrete image can be decomposed into a weighted field of equi-spaced impulses.

$$f(x) = \sum_{n=-\infty}^{+\infty} f(n)\,\delta(x-n)$$

$\delta(x-n)$: impulse at position *n*

$f(n)$: amplitude at position *n*

$f(x) \longrightarrow$ [LSI system $h(x)$] $\longrightarrow g(x)$

$\delta(x) \longrightarrow h(x)$ — *h*(*x*) = impulse response or Point Spread Function (PSF)

$a\,\delta(x) \xrightarrow{\text{linear}} a\,h(x)$

$\delta(x-n\Delta) \xrightarrow{\text{shift inv.}} h(x-n)$

$$f(x) = \sum_{n=-\infty}^{+\infty} f(n\Delta)\,\delta(x-n\Delta) \xrightarrow{\text{superposition}} g(x) = \sum_{n=-\infty}^{+\infty} f(n)\,h(x-n) \equiv f(x) * h(x)$$

Image *g* is the result of a ***convolution*** between image *f* and *h*

Convolution revisited

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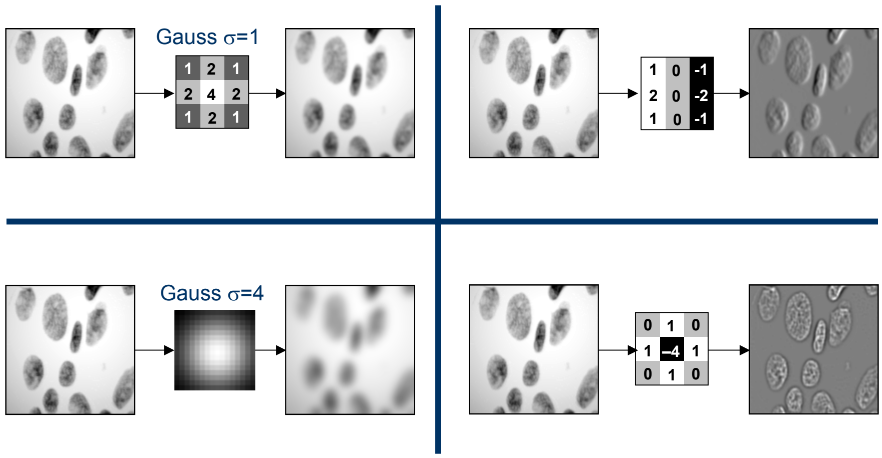
- Convolution: Replace the central pixel by a weighted sum of the gray-values inside an $n{\times}n$ neighborhood. Impulse response $h(x)$ is the filter.



Eigenfunction of LSI systems

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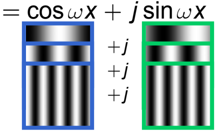
- Eigenfunctions of LSI systems are complex exponentials $\varphi(x)$.

$$\varphi_\omega(x) = e^{j\omega x} = \cos\omega x + j\sin\omega x$$

$$\varphi_{\omega=\omega 1}(x) = \quad +j$$
$$\varphi_{\omega=\omega 2}(x) = \quad +j$$
$$\varphi_{\omega=\omega 3}(x) = \quad +j$$



$\varphi_\omega(x)$ → LSI system $h(x)$ → $K\varphi_\omega(x) = \sum_{n=-\infty}^{+\infty} \varphi(n)h(x-n)$

$e^{j\omega x}$ →

$$g(x) = \sum_{n=-\infty}^{+\infty} e^{j\omega n}h(x-n) = e^{j\omega x}\underbrace{\sum_{n=-\infty}^{+\infty} h(m)e^{-j\omega m}}_{H(\omega)}$$

$$= e^{j\omega x} H(\omega)$$

- An LSI system multiplies each eigenfunction $\exp(j\omega x)$ by its corresponding eigenvalue $H(\omega)$.
- The Fourier transform decomposes an image into a weighted set of complex exponentials of varying frequency ω: “The Fourier spectrum”
- The system function $H(\omega)$ is the Fourier Transform of $h(x)$

Convolution property

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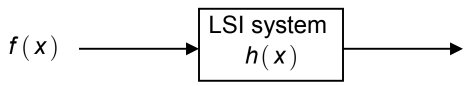
- A convolution between an image *f*(*x*) and an impulse response *h*(*x*) in space, corresponds to a multiplication of the Fourier spectra *F*(ω) and *H*(ω) in the Fourier domain.



$$f(x) * h(x) = \sum_{n=-\infty}^{+\infty} f(n\Delta) h(x - n\Delta)$$

$$\begin{aligned} F\{f(x) * h(x)\} &= \sum_{x=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} f(n) h(x-n) \right] e^{-j\omega x} \\ &= \underbrace{\sum_{n=-\infty}^{+\infty} f(n) e^{-j\omega n}}_{F(\omega)} \underbrace{\sum_{x=-\infty}^{+\infty} h(m) e^{-j\omega m}}_{H(\omega)} \\ &= F(\omega) H(\omega) \end{aligned}$$

Gaussian derivatives

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- How to compute a derivative in digital space?

$$f_x[x,y] \triangleq \frac{\partial}{\partial x} f[x,y] = ?$$

- Introduce (Gaussian) scale

$$\begin{aligned} f^{(\sigma)}[x,y] &= f^{(0)}[x,y] \otimes g^{(\sigma)}[x,y] \\ f^{(\sigma=5)}[x,y] &= f^{(\sigma=3)}[x,y] \otimes g^{(\sigma=4)}[x,y] \end{aligned}$$

- Derivative at scale σ is computed by convolution of the discrete image *f*[x,y] with discrete Gaussian derivative

$$f_x^{(\sigma)}[x,y] = f^{(0)}[x,y] \otimes g_x^{(\sigma)}[x,y]$$

- Note that the Gaussian function is known analytically, compute the derivative(s), and then sample to produce a discrete filter. The sampling of the Gaussian should be high enough, i.e. $\sigma > 0.9$ pixels

Fourier filters: Gaussian

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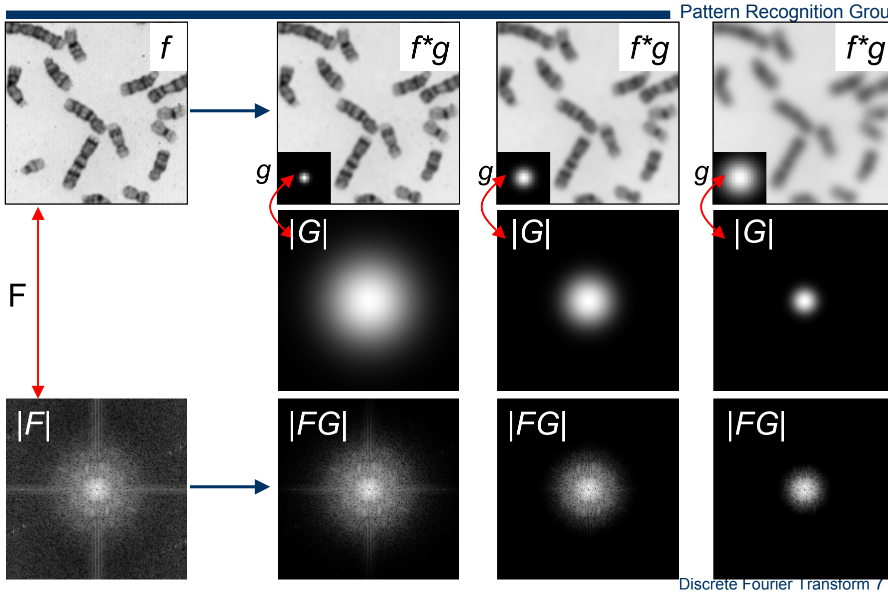

Gaussian derivative filters

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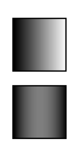
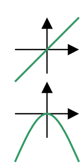
- In continuous space: the derivative operator corresponds to multiplication of the Fourier spectrum with $j\omega$

$$f(x) \stackrel{F}{\longleftrightarrow} F(\omega)$$

$$\frac{d}{dx} f(x) \stackrel{F}{\longleftrightarrow} j\omega F(\omega)$$

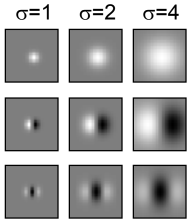
$$\frac{d^2}{dx^2} f(x) \stackrel{F}{\longleftrightarrow} -\omega^2 F(\omega)$$





- In discrete space: combine the derivative operator with Gaussian smoothing

σ=1 σ=2 σ=4

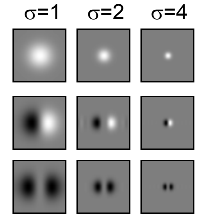


$$g^{(\sigma)}(x) * f(x) \stackrel{F}{\longleftrightarrow} G(\omega) F(\omega)$$

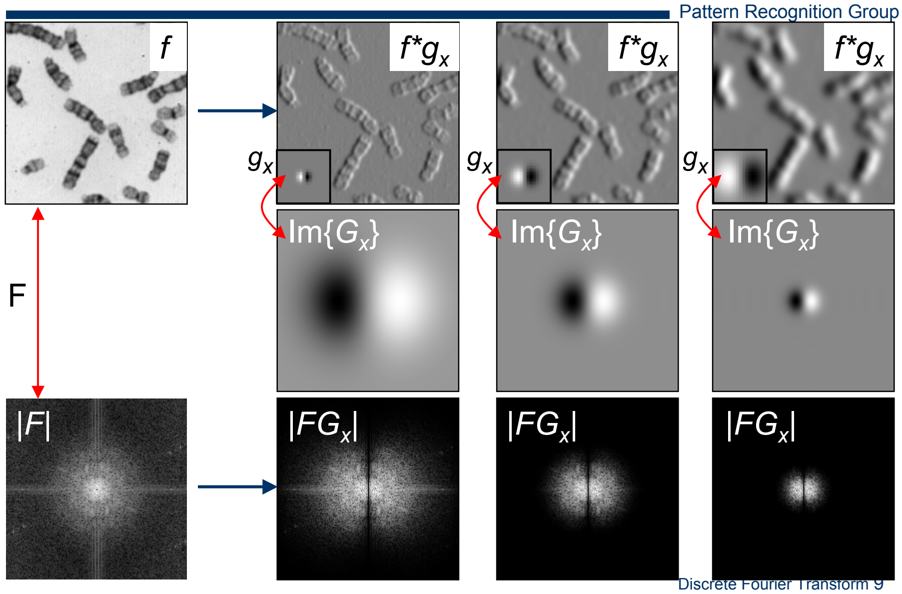
$$\frac{d}{dx} g^{(\sigma)}(x) * f(x) \stackrel{F}{\longleftrightarrow} j\omega\, G(\omega) F(\omega)$$

$$\frac{d^2}{dx^2} g^{(\sigma)}(x) * f(x) \stackrel{F}{\longleftrightarrow} -\omega^2 G(\omega) F(\omega)$$

σ=1 σ=2 σ=4



Fourier: Gaussian derivative



Fourier: Laplace & sharpening

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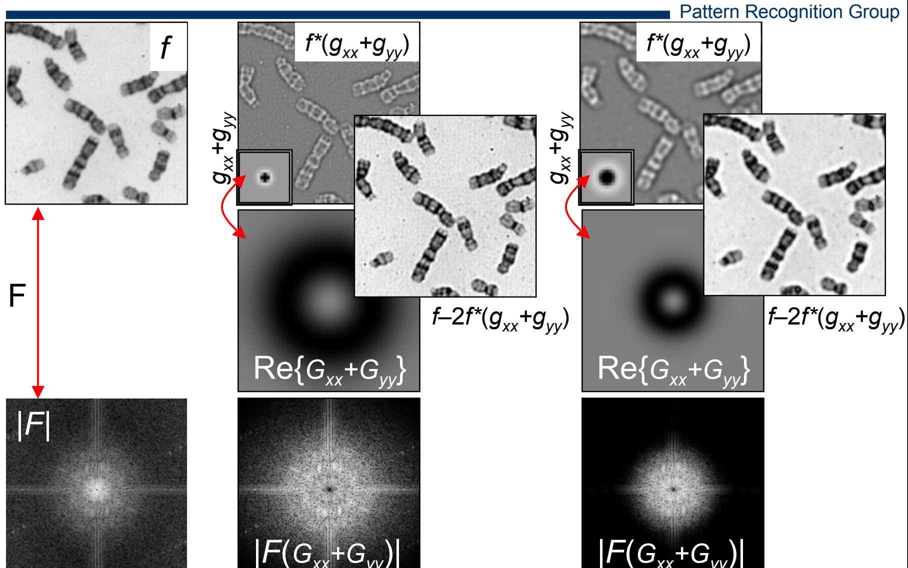


Discrete Fourier Transform

- Each image can be decomposed in weighted sum of complex exponentials (sines and cosines) of frequency *f* and angle ϕ. (or two frequency components *u* and *v*)

$$g(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} G(u,v)\, e^{j\frac{2\pi}{N}(ux+vy)}$$

image size *N*x*N*

$$G(u,v) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} g(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

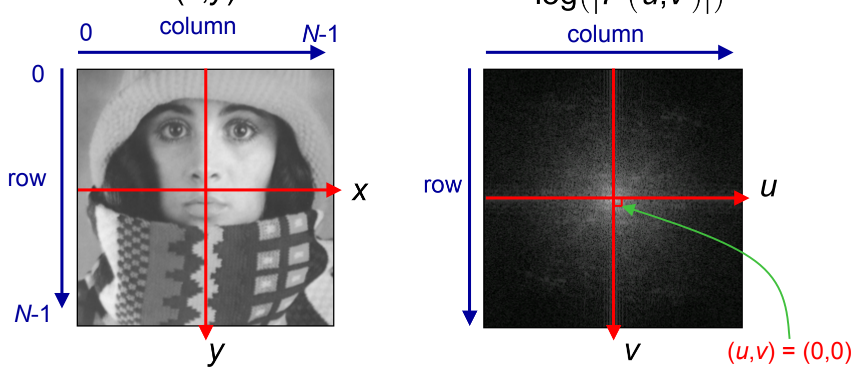
- For real-valued images: $\mathrm{Ev}\{g(x,y)\} \xleftrightarrow{F} \mathrm{Re}\{G(u,v)\}$

$$\mathrm{Od}\{g(x,y)\} \xleftrightarrow{F} j\,\mathrm{Im}\{G(u,v)\}$$

Fourier spectrum

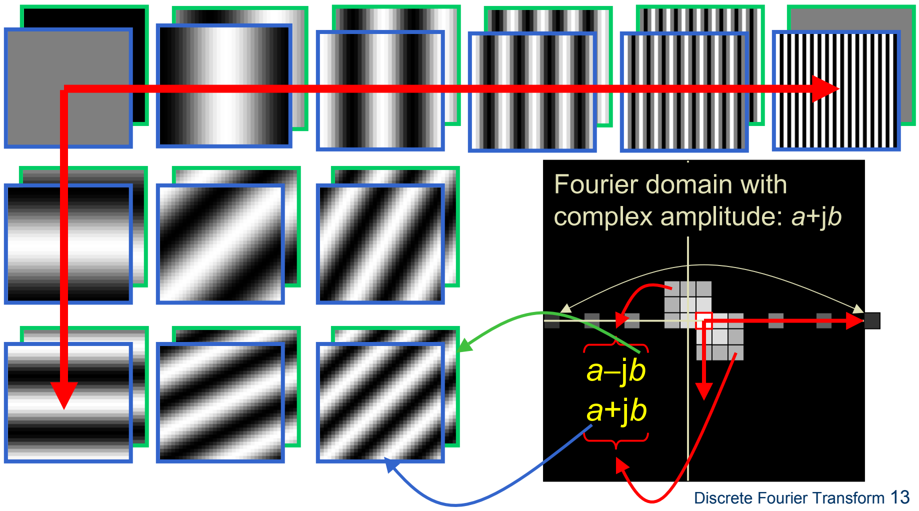
- $F(u,v)$ is the complex amplitude of the eigenfunction $\exp(j\,(2\pi/N)(ux+vy))$
  Note that $\mathbf{exp}(j\,(2\pi/N)(ux+vy)) = \mathbf{cos}((2\pi/N)(ux+vy)) + j\,\mathbf{sin}((2\pi/N)(ux+vy))$
- Standard display is the logarithm of the magnitude: $\log(|F(u,v|)$

$f(x,y)$ $\qquad$ ${}^{10}\log(|F(u,v)|)$

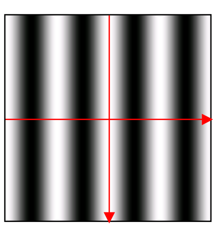


Getting used to Fourier (2)

An image is a weighted sum of cos (even) and sin (odd) images.



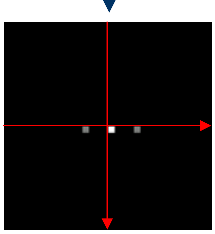
Eigenfunctions: Even & Odd



Real & even

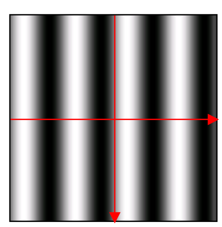
$$1+\cos\left(\frac{2\pi}{N}u_0x\right)=1+\frac{e^{j\frac{2\pi}{N}u_0x}+e^{j\frac{2\pi}{N}(-u_0)x}}{2}$$

$\mathcal{F}$



Real & even

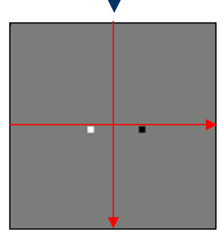
$$\frac{1}{N}\left(\begin{array}{l}\delta(u,v)+\\ \frac{1}{2}\delta(u-u_0,v)+\\ \frac{1}{2}\delta(u+u_0,v)\end{array}\right)$$



Real & odd

$$\sin\left(\frac{2\pi}{N}u_0x\right)=\frac{e^{j\frac{2\pi}{N}u_0x}-e^{j\frac{2\pi}{N}(-u_0)x}}{2j}$$

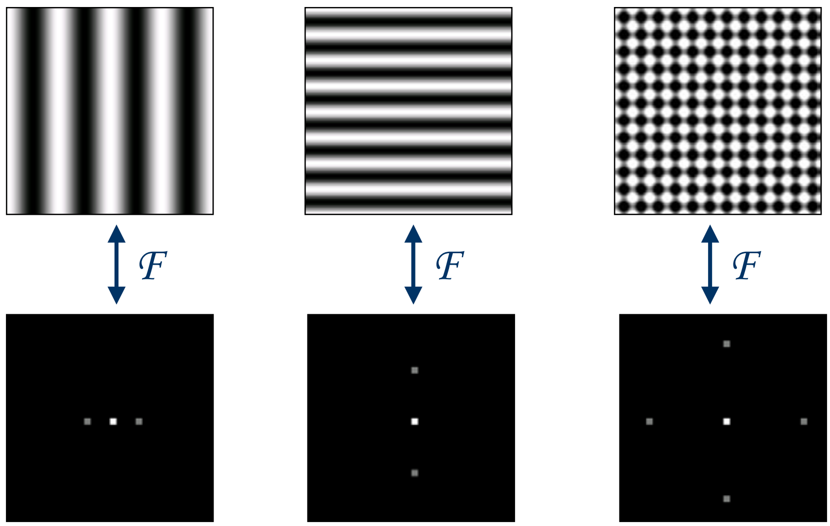
$\mathcal{F}$



Imag & odd

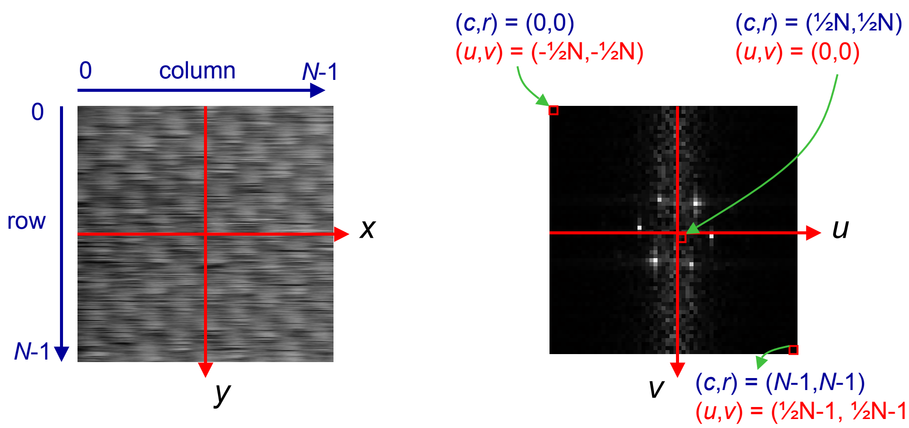
$$\frac{1}{N}j\left(\begin{array}{l}-\frac{1}{2}\delta(u-u_0,v)\\ +\frac{1}{2}\delta(u+u_0,v)\end{array}\right)$$

Orientation & frequency



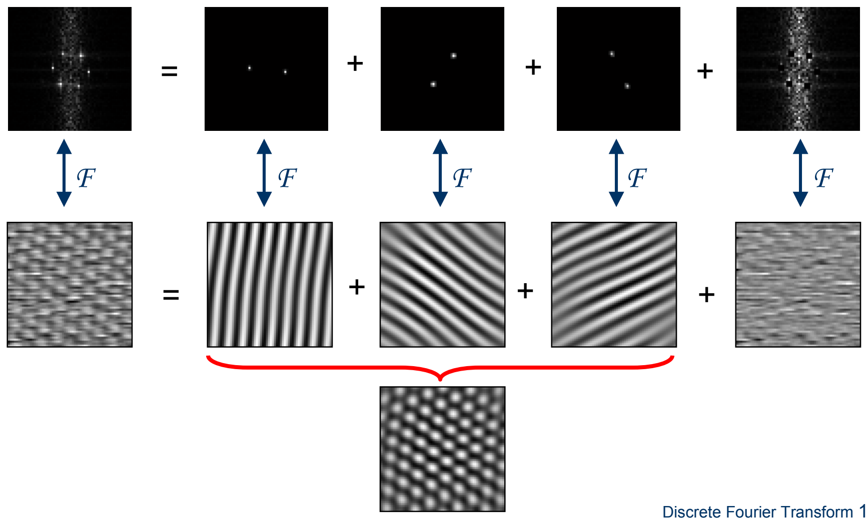
Getting used to Fourier (1)

- Graphite surface by Scanning Tunneling Microscopy
- Atomic structure of graphite shows a hexagonal surface

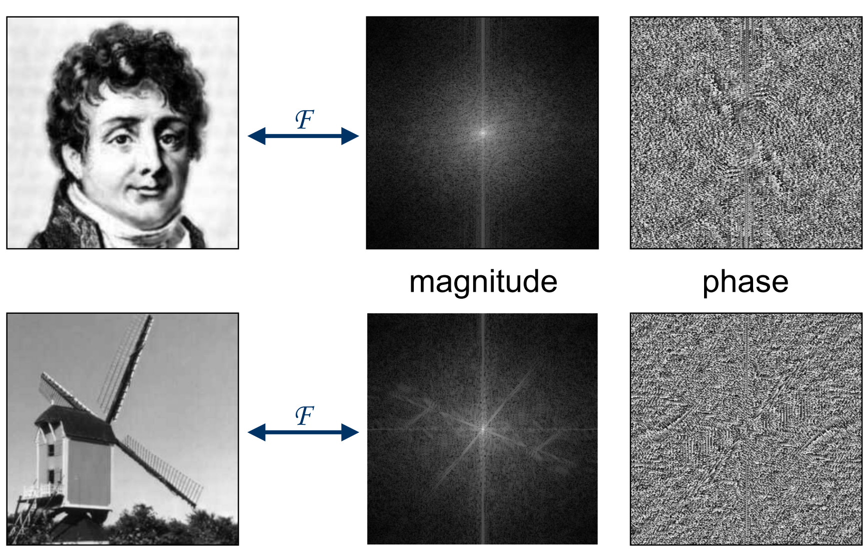


Superposition

- Fourier spectrum



Fourier transforms



Magnitude & phase

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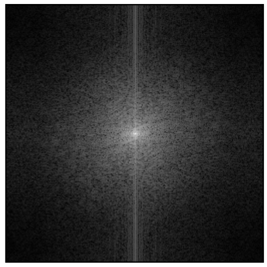
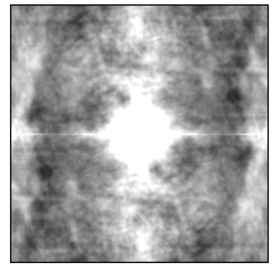
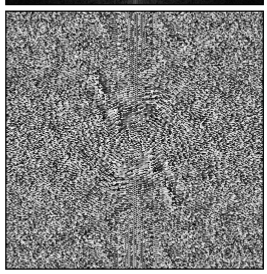
$|F(u,v)|$



$\mathcal{F}^{-1}$ ↔



$e^{j\sphericalangle F(u,v)} = \dfrac{F(u,v)}{|F(u,v)|}$



$\mathcal{F}^{-1}$ ↔

Local variance filter: power

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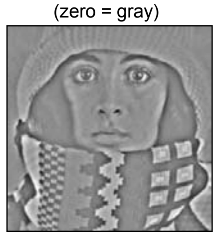
- *Recipe:* local variance filter (filter size = *n*)
  1. Compute the local mean (blurring filter of size *n*)
  2. Subtract the local mean.
  3. Compute the square of each pixel value
  4. Suppress the "double" response by local averaging (blurring filter of size *n*)
- Local variance is a measure for the local squared-contrast.

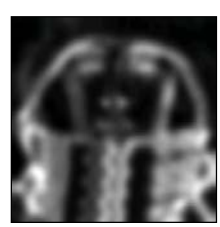
step 1

(zero = gray)



step 2

step 3



step 4

Scaling: local vs global

- Problem: Choosing the proper scale is an important, but tedious task.
  - Scale too small: Local characteristics are missed which yields an incomplete data description.
  - Scale too large: Confusion (mixing) of adjacent objects, lack of localization, and blindness for detail.
- Solution: Multi-scale analysis.
  - Analyze the image as function of scale: from fine detail to course “image-filling” objects.

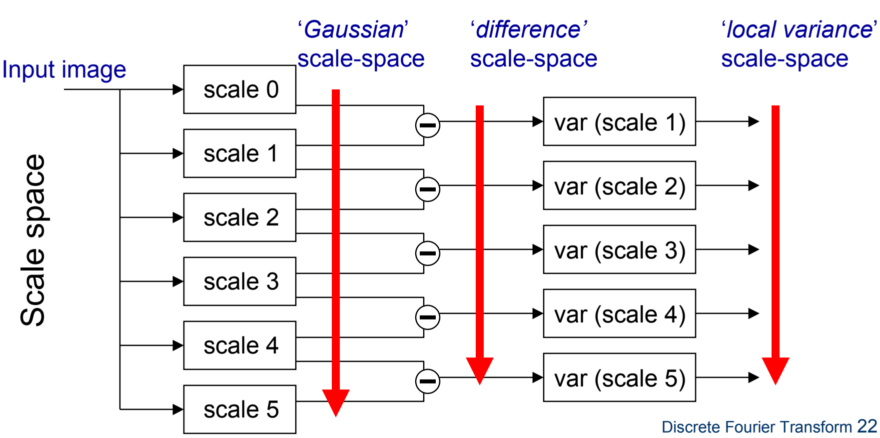
$$g(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \quad \overset{F}{\longleftrightarrow} \quad G(u,v) = e^{-\frac{\left(\left(\frac{2\pi}{N}u\right)^2 + \left(\frac{2\pi}{N}v\right)^2\right)\sigma^2}{2}}$$

Multi-Scale

Series of images of increasing scale: Scale-space

- Sample the scales logarithmically using filters of $size = base^{scale}$ yields n scales per octave

$$base \in \left\{2^1, 2^{1/2}, 2^{1/3}, \ldots, 2^{1/n}\right\}$$



Scale-spaces

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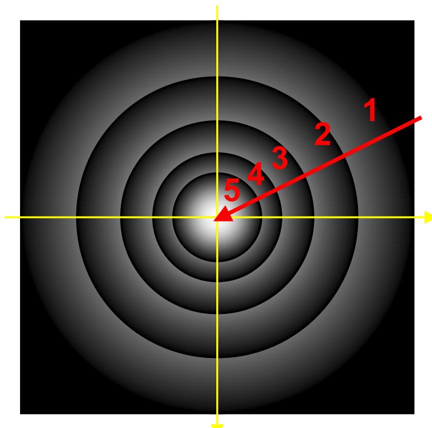
- Morphological scale-space: Use *openings* (*closings*)
- Gaussian scale-space: Use *Gaussian filters*



Increasing scales

Fourier domain with “footprints” of Gaussian filters of increasing scale

Filter size is inversely proportional to “footprint” in Fourier domain

Chirp example

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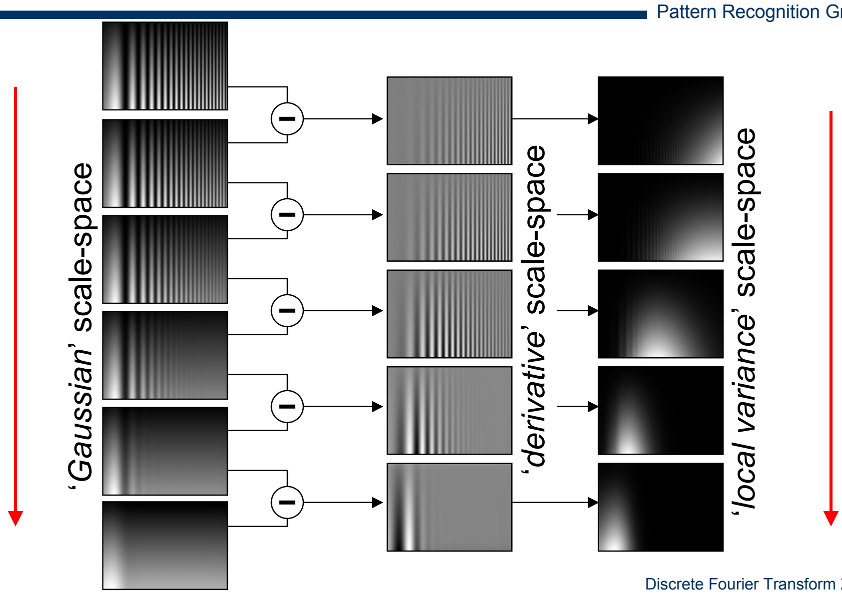

Gaussian derivatives

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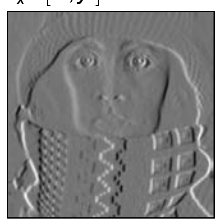
$f^{(0)}[x,y]$

$f^{(1)}[x,y]$

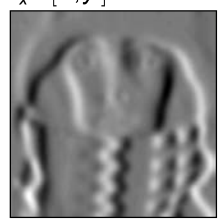
$f^{(5)}[x,y]$

$f_x^{(0)}[x,y] = ?$

$f_x^{(1)}[x,y]$



$f_x^{(5)}[x,y]$



Sampling

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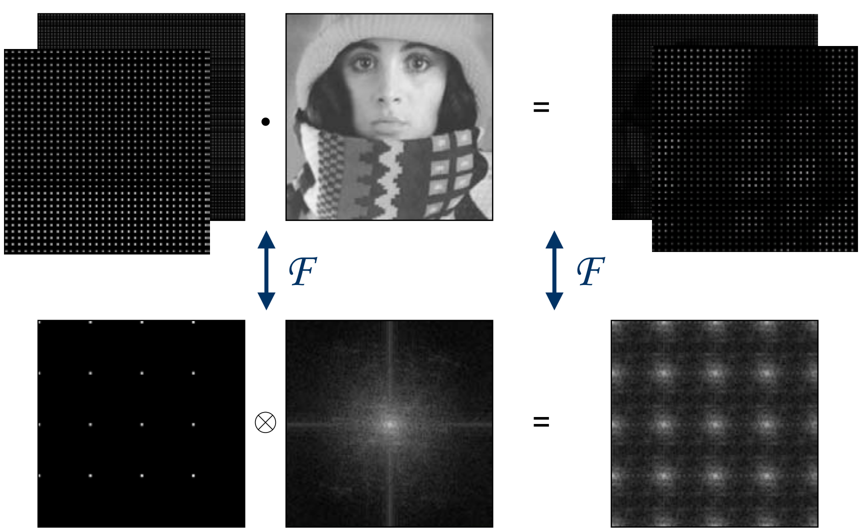
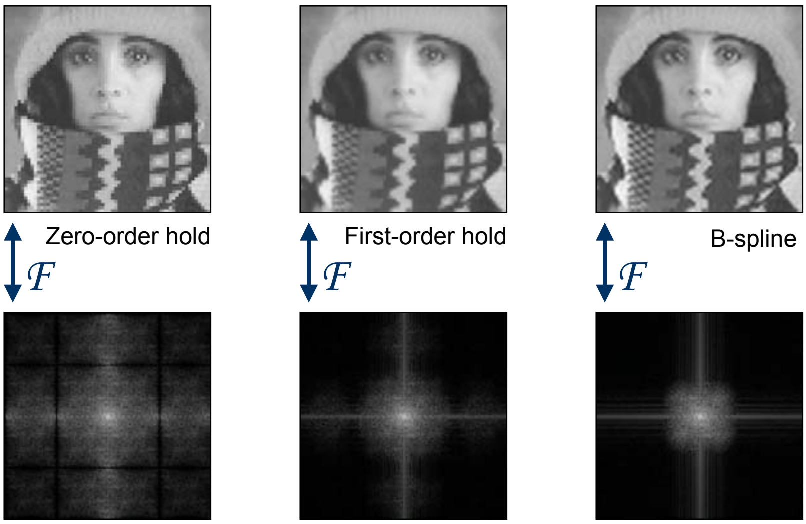
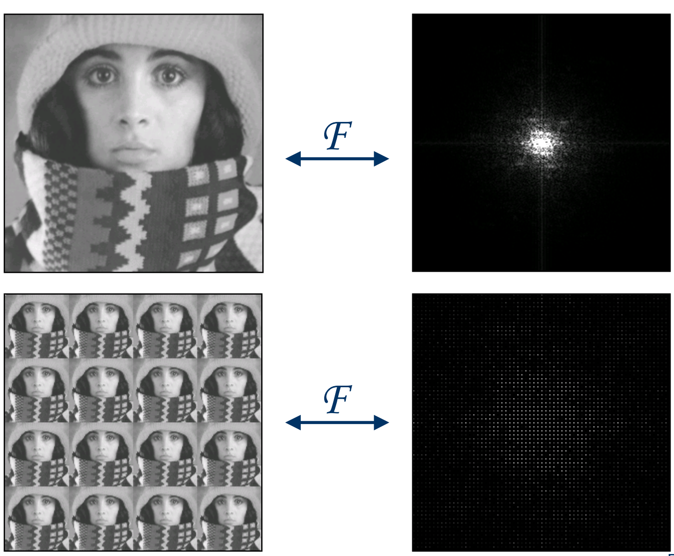


Interpolation



Periodic images



Periodic image

yields

Fourier spectrum
with impulses