Introduction to the Discrete Fourier Transform

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Linear Shift Invariant System

A discrete image can be decomposed into a weighted field of equi-spaced impulses.

\[ f(x) = \sum_{n=-\infty}^{+\infty} f(n)\delta(x-n) \]

\[ g(x) = \sum_{n=-\infty}^{+\infty} f(n)h(x-n) \equiv f(x) * h(x) \]

Image \( g \) is the result of a convolution between image \( f \) and \( h \)
Convolution revisited

- **Convolution**: Replace the central pixel by a weighted sum of the gray-values inside an $n \times n$ neighborhood. Impulse response $h(x)$ is the filter.

![Convolution Examples](image)

Eigenfunction of LSI systems

- **Eigenfunctions of LSI systems** are complex exponentials $\varphi(x)$.

  $$\varphi_\omega(x) = e^{j\omega x} = \cos \omega x + j \sin \omega x$$

  $$\varphi_{\omega1}(x) = +j$$  
  $$\varphi_{\omega2}(x) = +j$$  
  $$\varphi_{\omega3}(x) = +f$$

- LSI system $h(x)$

  $$K \varphi_\omega(x) = \sum_{n=-\infty}^{\infty} \varphi(n) h(x-n)$$

- $e^{j\omega x}$

  $$g(x) = \sum_{n=-\infty}^{\infty} e^{j\omega x} h(x-n) = e^{j\omega x} \sum_{n=-\infty}^{\infty} h(m) e^{-j\omega m}$$

  $$= e^{j\omega x} H(\omega)$$

- An LSI system multiplies each eigenfunction $\exp(j\omega x)$ by its corresponding eigenvalue $H(\omega)$.

- The Fourier transform decomposes an image into a weighted set of complex exponentials of varying frequency $\omega$. “The Fourier spectrum”

- The system function $H(\omega)$ is the Fourier Transform of $h(x)$.
Convolution property

A convolution between an image \( f(x) \) and an impulse response \( h(x) \) in space, corresponds to a multiplication of the Fourier spectra \( F(\omega) \) and \( H(\omega) \) in the Fourier domain.

\[
\begin{align*}
LSI \text{ system } & \quad f(x) \to h(x) \\
\longrightarrow & \quad f(x) * h(x) = \sum_{n=-\infty}^{\infty} f(n\Delta) h(x - n\Delta) \\
F \{ f(x) * h(x) \} & = \sum_{x=-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} f(n) h(x - n) \right] e^{-jx}
\end{align*}
\]

\[
= \sum_{n=-\infty}^{\infty} f(n) e^{-jn} \sum_{m=-\infty}^{\infty} h(m) e^{-jm}
\]

\[
= F(\omega) H(\omega)
\]

Gaussian derivatives

How to compute a derivative in digital space?

\[
f_x[x,y] \triangleq \frac{\partial}{\partial x} f[x,y] = ?
\]

Introduce (Gaussian) scale

\[
f^{(\sigma)}[x,y] = f^{(0)}[x,y] \otimes g^{(\sigma)}[x,y]
\]

\[
f^{(\sigma-5)}[x,y] = f^{(\sigma-3)}[x,y] \otimes g^{(\sigma-4)}[x,y]
\]

Derivative at scale \( \sigma \) is computed by convolution of the discrete image \( f[x,y] \) with discrete Gaussian derivative

\[
f_x^{(\sigma)}[x,y] = f^{(0)}[x,y] \otimes g_x^{(\sigma)}[x,y]
\]

Note that the Gaussian function is known analytically, compute the derivative(s), and then sample to produce a discrete filter. The sampling of the Gaussian should be high enough, i.e. \( \sigma > 0.9 \) pixels.
Fourier filters: Gaussian

Gaussian derivative filters

- In continuous space: the derivative operator corresponds to multiplication of the Fourier spectrum with $j\omega$

$$f(x) \xrightarrow{F} F(\omega)$$

$$\frac{d}{dx}f(x) \xrightarrow{F} j\omega F(\omega)$$

$$\frac{d^2}{dx^2}f(x) \xrightarrow{F} -\omega^2 F(\omega)$$

- In discrete space: combine the derivative operator with Gaussian smoothing
Fourier: Gaussian derivative

Fourier: Laplace & sharpening
Each image can be decomposed in weighted sum of complex exponentials (sines and cosines) of frequency \( f \) and angle \( \phi \). (or two frequency components \( u \) and \( v \))

\[
g(x,y) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} G(u,v) e^{j \frac{2\pi}{N} (ux + vy)}
\]

\[
G(u,v) = \frac{1}{N} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} g(x,y) e^{-j \frac{2\pi}{N} (ux + vy)}
\]

For real-valued images:

\[
\text{Ev}\{g(x,y)\} \xrightarrow{F} \text{Re}\{G(u,v)\}
\]

\[
\text{Od}\{g(x,y)\} \xrightarrow{F} j \text{Im}\{G(u,v)\}
\]

For real-valued images:

\[
F(u,v) \text{ is the complex amplitude of the eigenfunction } \exp(j \frac{2\pi}{N}(ux + vy))
\]

Note that \( \exp(j \frac{2\pi}{N}(ux + vy)) = \cos(\frac{2\pi}{N}(ux + vy)) + j \sin(\frac{2\pi}{N}(ux + vy)) \)

Standard display is the logarithm of the magnitude: \( \log(|F(u,v)|) \)
An image is a weighted sum of \( \cos \) (even) and \( \sin \) (odd) images.

Eigenfunctions: Even & Odd

Real & even
\[
1 + \cos \left( \frac{2\pi}{N} u_0 x \right) = \frac{1}{1 + \frac{2 e^{\frac{2\pi}{N} u_0 x} + e^{-\frac{2\pi}{N} u_0 x}}{2}}
\]

Real & odd
\[
\sin \left( \frac{2\pi}{N} u_0 x \right) = \frac{e^{\frac{2\pi}{N} u_0 x} - e^{-\frac{2\pi}{N} u_0 x}}{2j}
\]

Imag & odd
\[
\frac{1}{\pi} \begin{bmatrix} -\frac{1}{2} \delta(u-u_0,v) \\ \frac{1}{2} \delta(u-u_0,v) + \frac{1}{2} \delta(u+u_0,v) \end{bmatrix}
\]
Orientation & frequency

Getting used to Fourier (1)

- Graphite surface by Scanning Tunneling Microscopy
- Atomic structure of graphite shows a hexagonal surface

$$(c,r) = (0,0)$$
$$(u,v) = (-\frac{1}{2}N, -\frac{1}{2}N)$$
$$(c,r) = (\frac{1}{2}N, \frac{1}{2}N)$$
$$(u,v) = (0,0)$$
$$(c,r) = (N-1, N-1)$$
$$(u,v) = (\frac{1}{2}N-1, \frac{1}{2}N-1)$$
Superposition

Fourier spectrum

\[ F \]

\[ \begin{align*}
\mathcal{F} & = \mathcal{F} + \mathcal{F} + \mathcal{F} + \mathcal{F} \\
\mathcal{F} & = \mathcal{F} \end{align*} \]

Fourier transforms

\[ \mathcal{F} \]

magnitude

phase
Magnitude & phase

\[ |F(u,v)| \]

\[ e^{jF(u,v)} = \frac{F(u,v)}{|F(u,v)|} \]

Local variance filter: power

**Recipe:** local variance filter (filter size = \( n \))
1. Compute the local mean (blurring filter of size \( n \))
2. Subtract the local mean.
3. Compute the square of each pixel value
4. Suppress the "double" response by local averaging (blurring filter of size \( n \))

Local variance is a measure for the local squared-contrast.

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Scaling: local vs global

- Problem: Choosing the proper scale is an important, but tedious task.
  - Scale too small: Local characteristics are missed which yields an incomplete data description.
  - Scale too large: Confusion (mixing) of adjacent objects, lack of localization, and blindness for detail.

- Solution: Multi-scale analysis.
  - Analyze the image as function of scale: from fine detail to course “image-filling” objects.

\[
g(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2 + y^2}{2\sigma^2}} \xrightarrow{F} G(u,v) = e^{-\left[\left(\frac{x^2}{\sigma^2}\right)^2 + \left(\frac{y^2}{\sigma^2}\right)^2\right]/2}
\]

Multi-Scale

Series of images of increasing scale: Scale-space
- Sample the scales logarithmically using filters of size \( \text{base}^\text{scale} \) yields \( n \) scales per octave

\[
\text{size} = \text{base}^\text{scale}
\]

\[
\text{base} \in \{2^1, 2^\frac{3}{2}, 2^\frac{5}{2}, ..., 2^n\}
\]
Scale-spaces

- Morphological scale-space: Use openings (closings)
- Gaussian scale-space: Use Gaussian filters

Increasing scales

Fourier domain with “footprints” of Gaussian filters of increasing scale

Filter size is inversely proportional to “footprint” in Fourier domain

Chirp example
Gaussian derivatives

\[ f^{(0)}(x,y) \]

\[ f^{(1)}(x,y) \]

\[ f^{(5)}(x,y) \]

\[ f_x^{(0)}(x,y) = ? \]

\[ f_x^{(1)}(x,y) \]

\[ f_x^{(5)}(x,y) \]

Sampling

\[ f \]

\[ F \]

\[ F^* \]
Interpolation

- Zero-order hold
- First-order hold
- B-spline

Periodic images

- Periodic image yields Fourier spectrum with impulses