Introduction to Medical Imaging

Signal Processing Basics

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Strange Effects

Ever tried to reduce the size of an image and you got this?

We call this effect ‘aliasing’

Better

But what you really wanted is this:

We call this ‘anti-aliasing’

Why Is This Happening?

The smaller image resolution cannot represent the image detail captured at the higher resolution

- skipping this small detail leads to these undesired artifacts

Overview

So how do we get the nice image?

For this you need to understand:

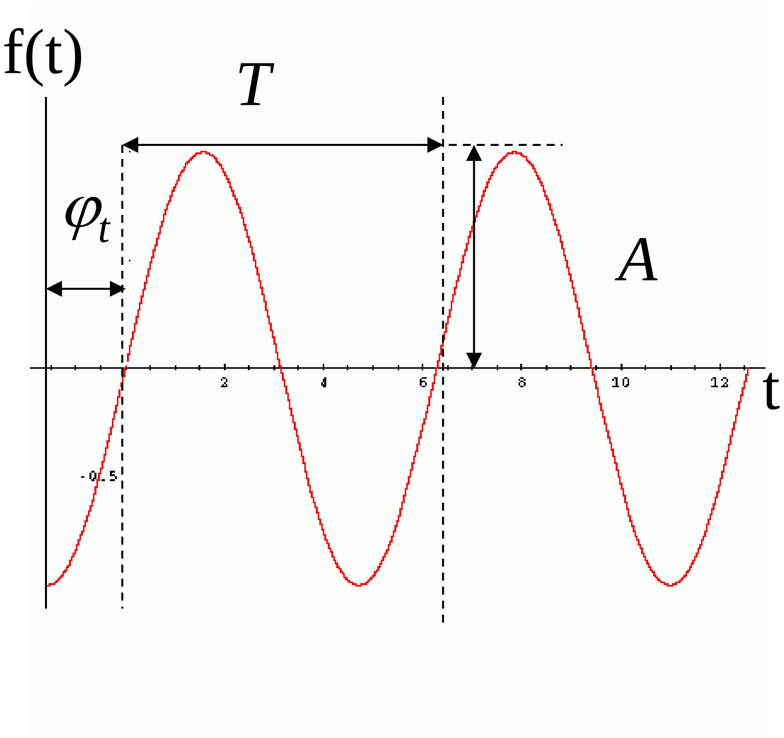
- Fourier theory
- Sampling theory
- Digital filters

Don’t be scared, we’ll cover these topics gently

Periodic Signals

A signal is periodic if $s(t+T) = s(t)$

- we call T the period of the signal
- if there is no such T then the signal is aperiodic



Sinusoids are periodic functions

- sinusoids play an important role

Write as:

$$A\sin(\frac{2\pi t}{T} + \varphi_t)$$

- where φ_t is the phase shift and A is the amplitude

Alternatively:

$$A\sin(2\pi ft + \varphi_t) = A\sin(\omega t + \varphi_t)$$

- where $f=1/T$ is the *frequency*
- we may write $\omega = 2\pi f$

Fourier Theory

Jean Baptiste Joseph Fourier (1768-1830)

His idea (1807):

- *Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.*

Don’t believe it?

- neither did Lagrange, Laplace, Poisson and other major mathematicians of his time
- in fact, the theory was not translated into English until 1878

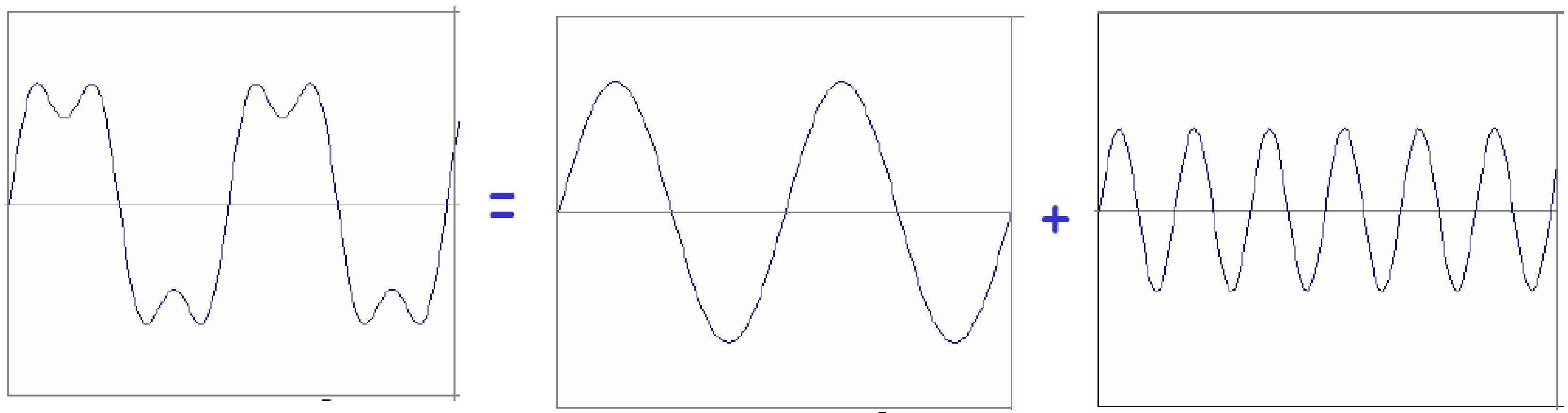
But it’s true!

- it is called the *Fourier Series*

Example

Consider the function:

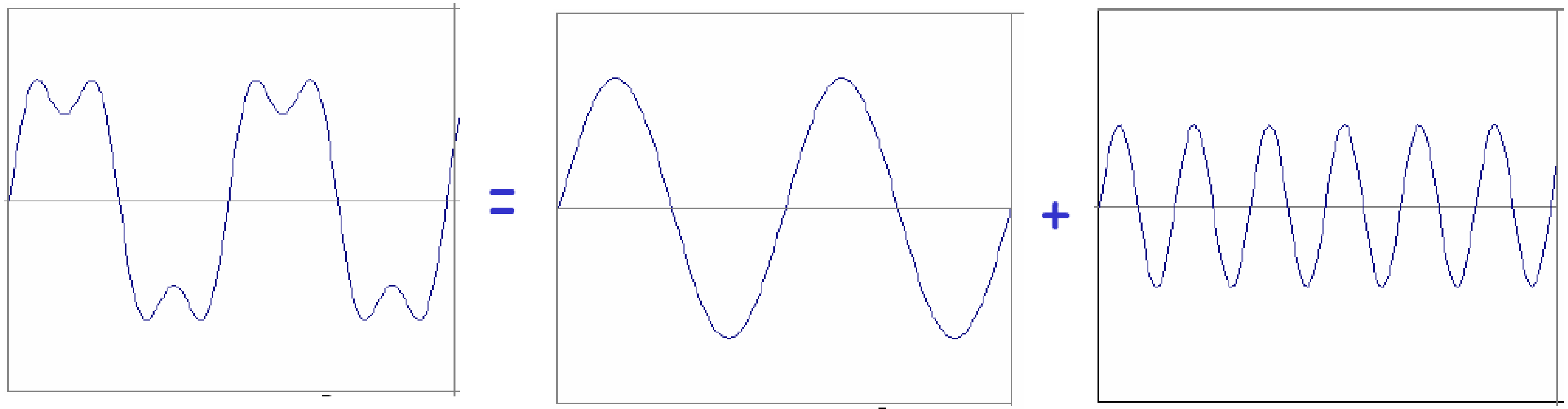
$$g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$$

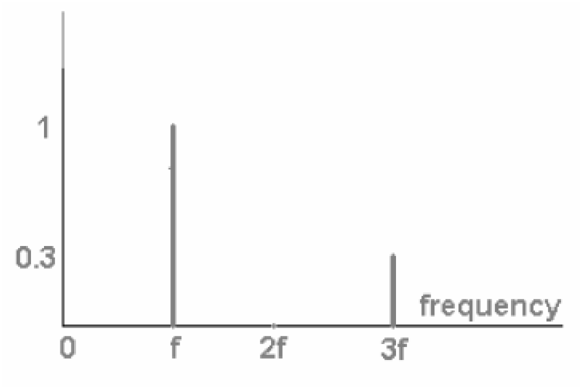


Frequency Spectrum

Consider the function:

$$g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$$





the function's frequency spectrum

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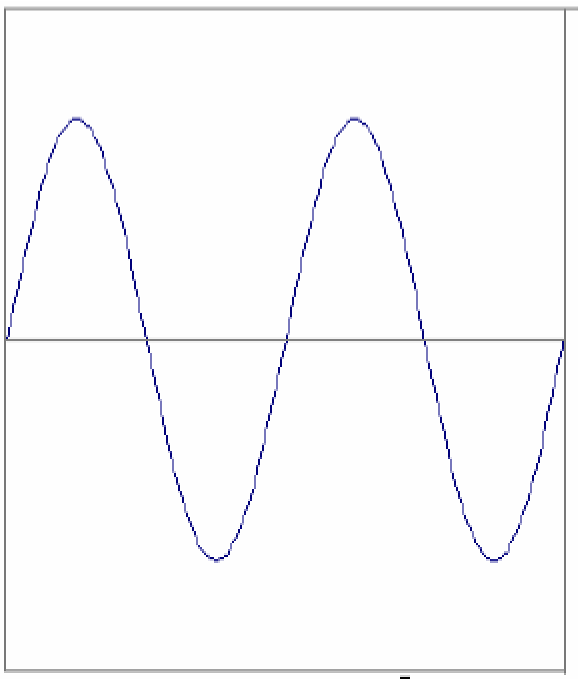
Further Example (1)



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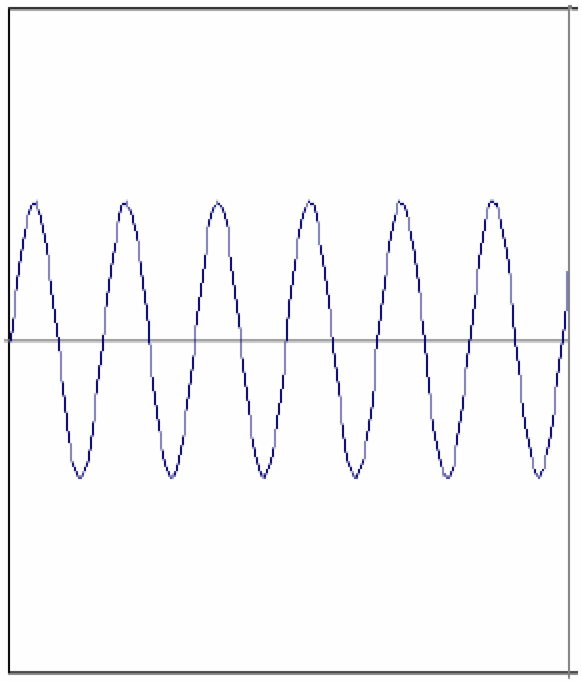

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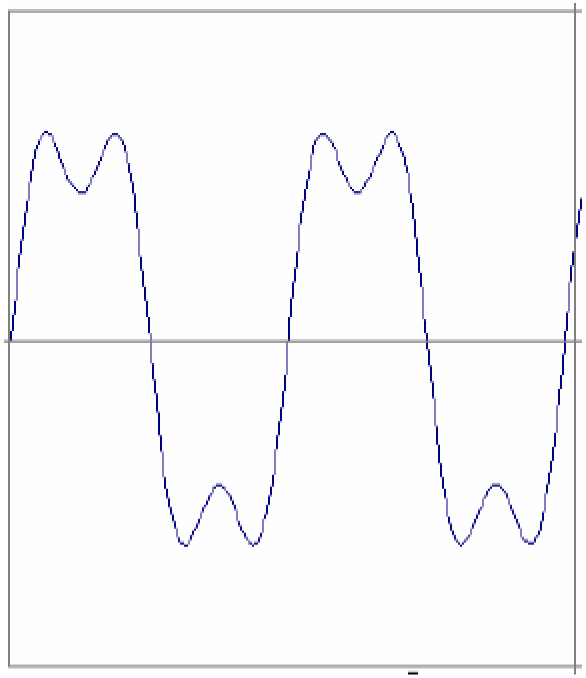
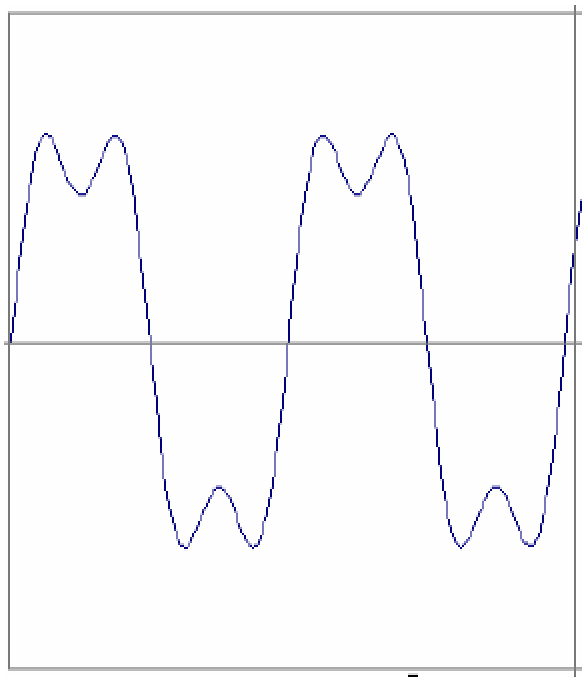


Further Example (2)



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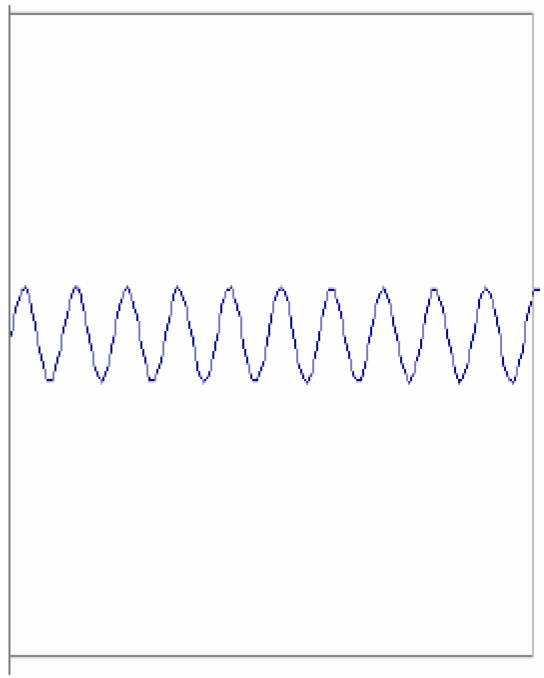

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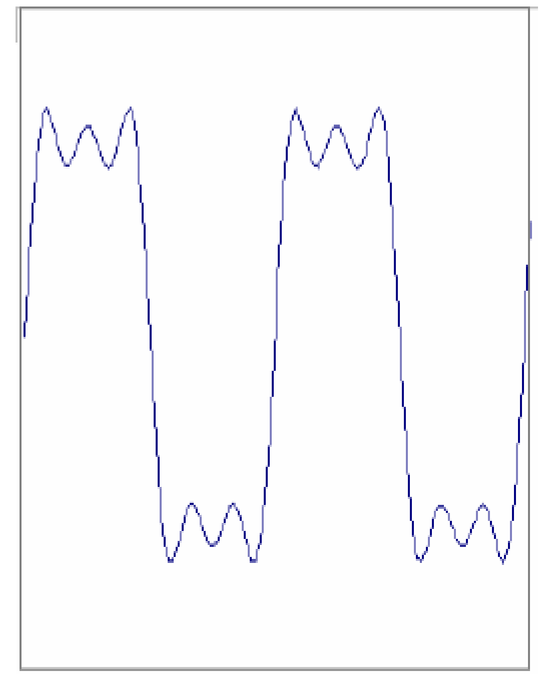
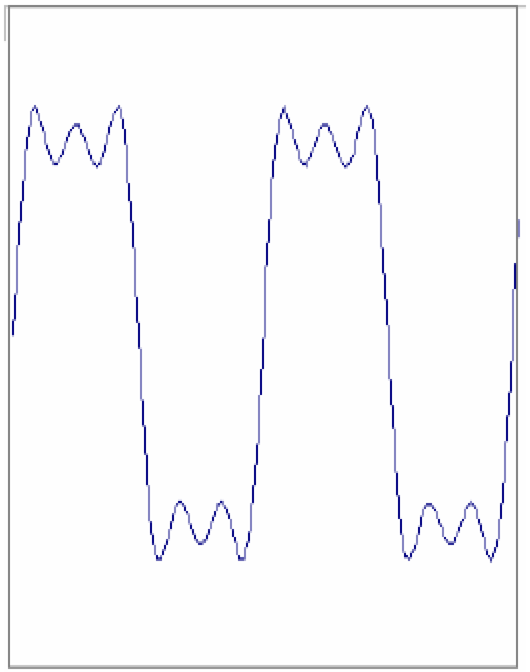


Further Example (3)



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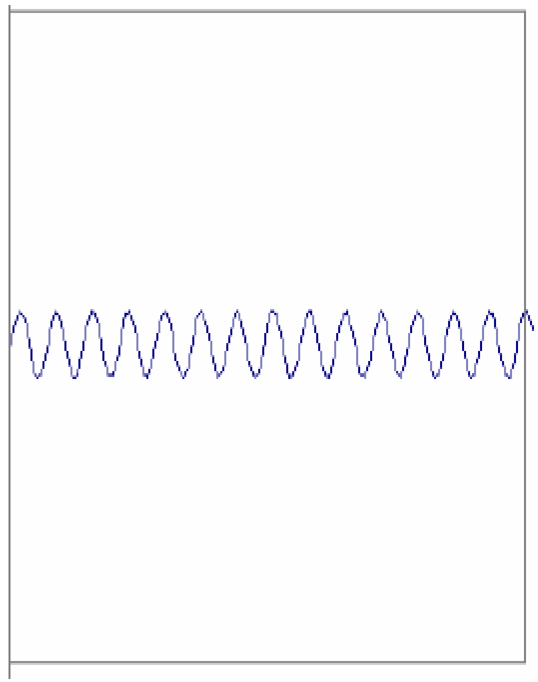

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Further Example (4)

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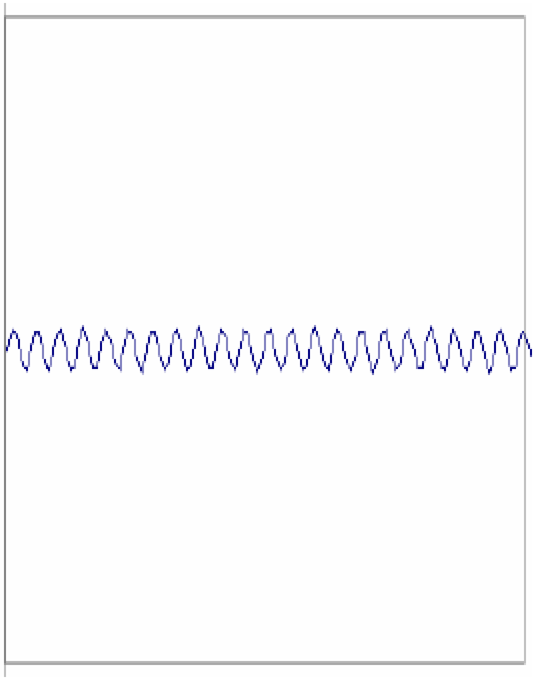

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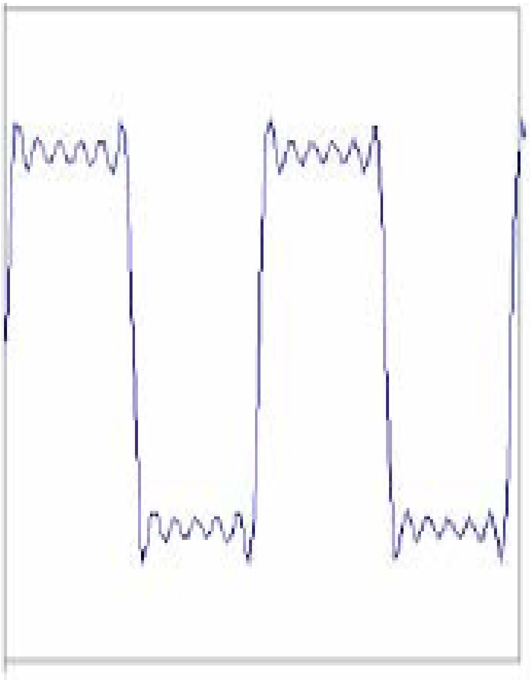
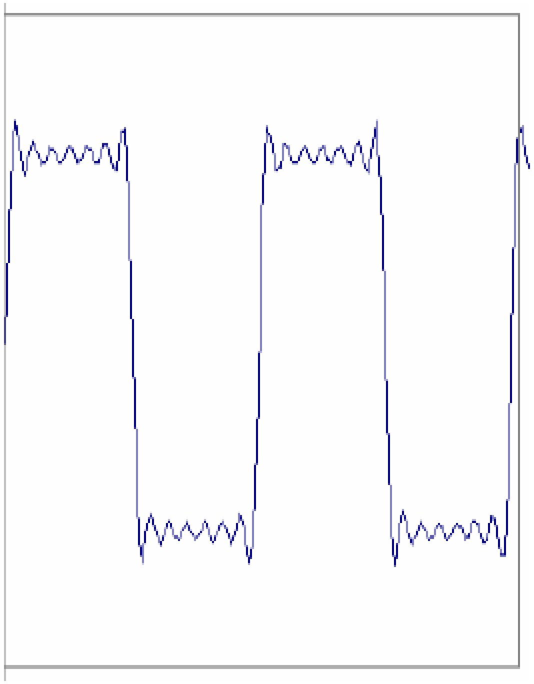


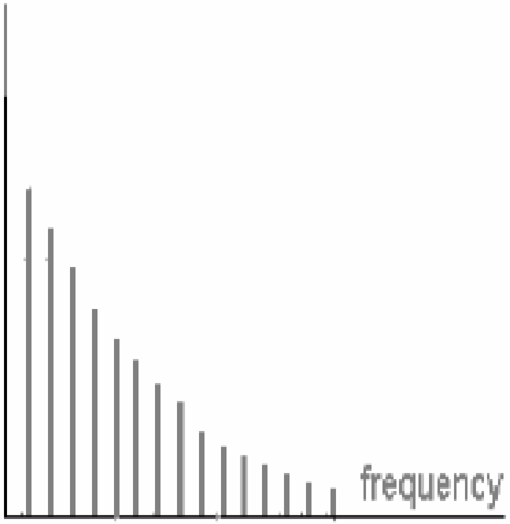
The Importance of the Frequency Spectrum

We observe:

- oscillations of different frequencies add to form the signal
- there is a characteristic frequency spectrum to any signal
- sharp edges can only be represented (generated) by high frequencies



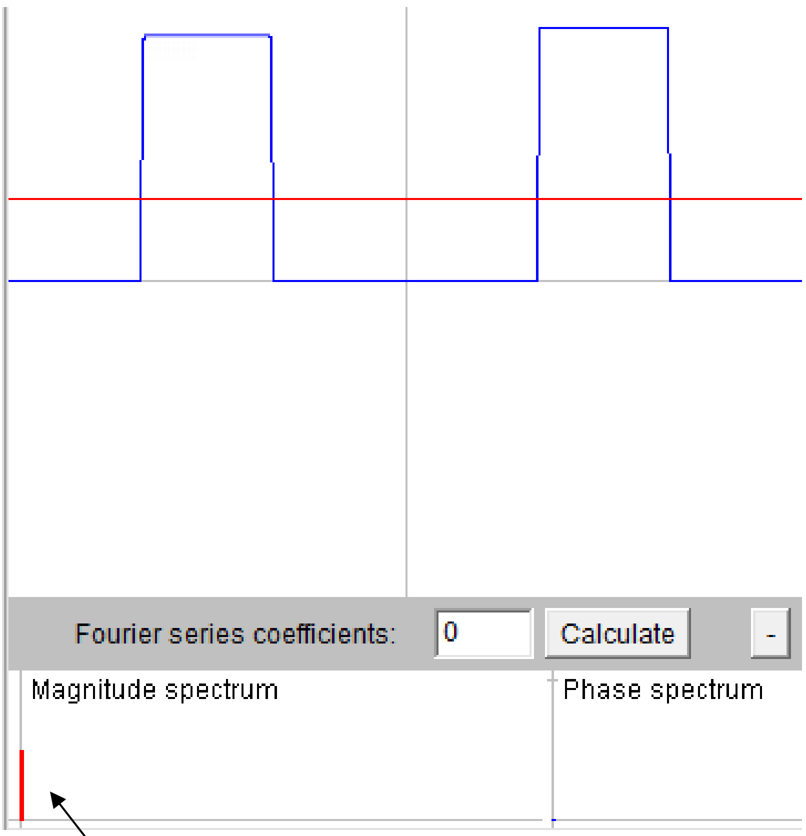
signal
(approximate square/box function)

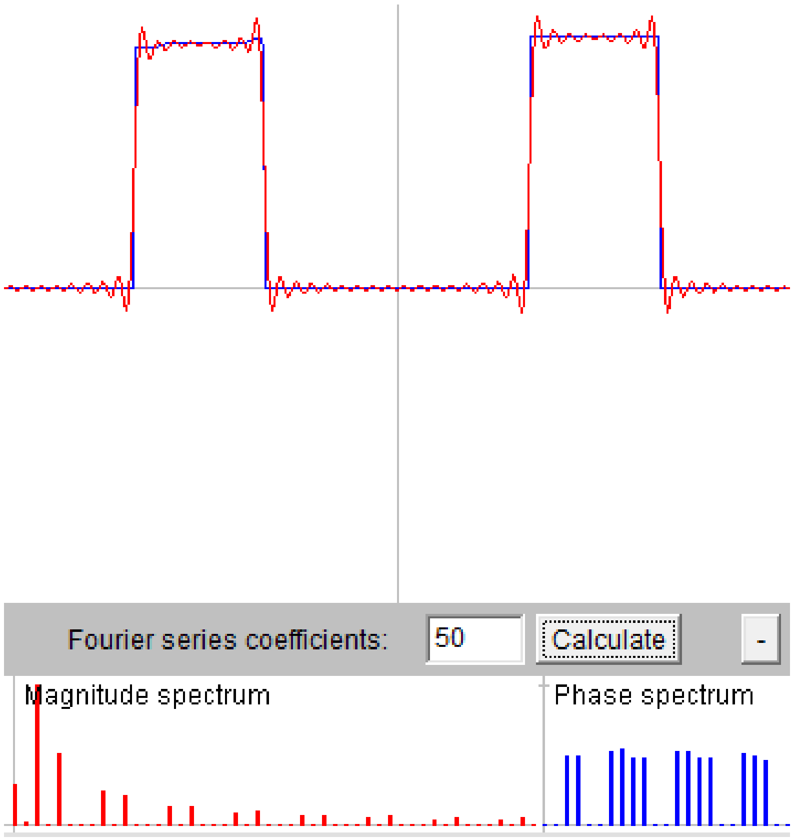


its frequency spectrum

The DC Component

The first component of the spectrum is the *signal average DC*





‘DC component’ = signal average

The Math…

The example just seen has the following Fourier Series:

$$s(t) = \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi k t)$$

- most of the time the phase is not interesting, so we shall omit it

In fact, this is an interesting series: the *sinc* function

- we shall see more of it soon

We can convert any discrete signal into its Fourier Series (and back)

- this is called the *Fourier Transform* (*Inverse Fourier Transform*)

$$\begin{array}{ccc} & \boxed{\text{Fourier Transform}} & \\ s(t) & \rightleftarrows & S(k) \\ & \boxed{\text{Inverse Fourier Transform}} & \end{array}$$

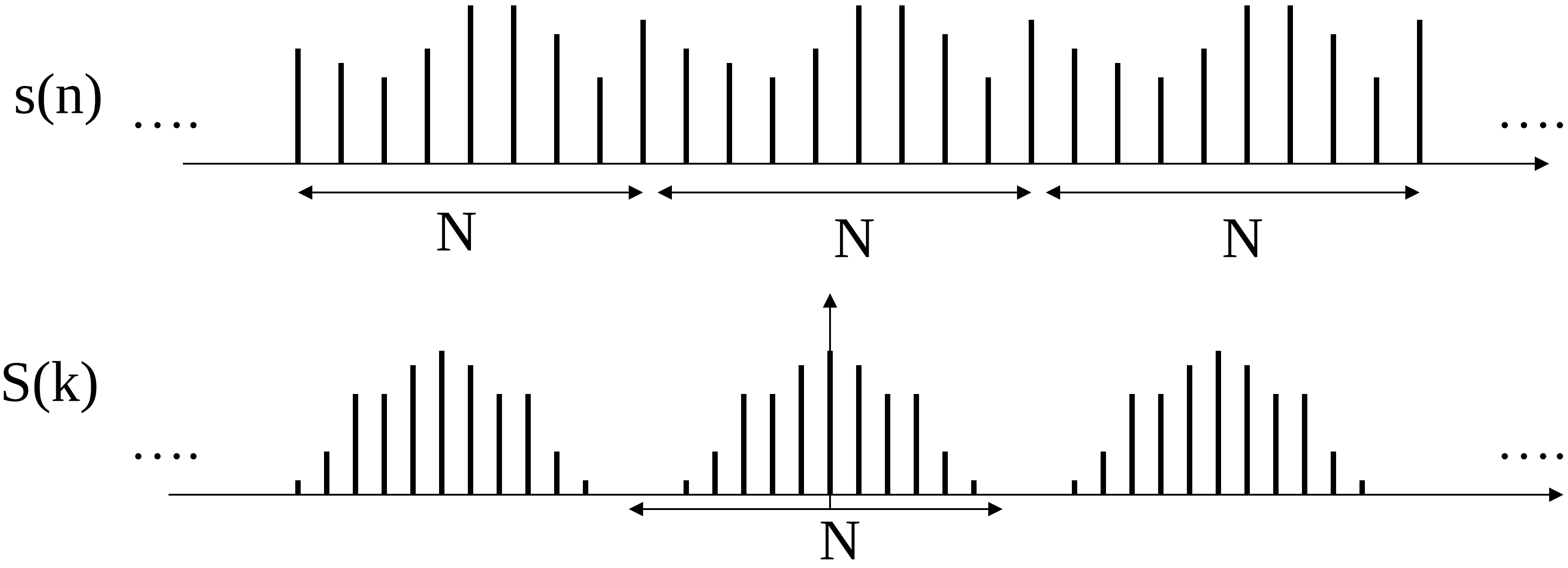
Fourier Transform of Discrete Signals: DFT

Discrete Fourier Transform (DFT)

- assumes that the signal is discrete and finite

$$S(k)=\sum_{n=0}^{N-1} s(n)e^{\frac{-i2\pi kn}{N}} \qquad s(n)=\frac{1}{N}\sum_{n=0}^{N-1} S(k)e^{\frac{i2\pi kn}{N}}$$

- we have *N* samples, from which we can calculate *N* frequencies
- the frequency spectrum is <u>discrete</u> and it is <u>periodic</u> in *N*



Periodicity

Images are discrete signals

- so their frequency spectra are finite and periodic (see last slide)
- and therefore they have an upper limit (a maximum frequency)

Images are also finite (in size)

- the DFT assumes that they are also periodic
- as odd as this may sound, this is the underlying assumption

Therefore:

- frequency spectra are finite and periodic
- images are also finite and periodic

Keep this in mind for now

- it will help explain the strange resizing effects presented before

Now, What About the Complex Exponential…

It is Fourier's way to encode phase and amplitude into one representation

- to understand it better, let's first review complex numbers
- and then see what it means in the Fourier context

Note, we only discuss this to illustrate the full picture

- essential for this class is only to know the concept of frequency spectrum discussed thus far

Recall: Complex Numbers

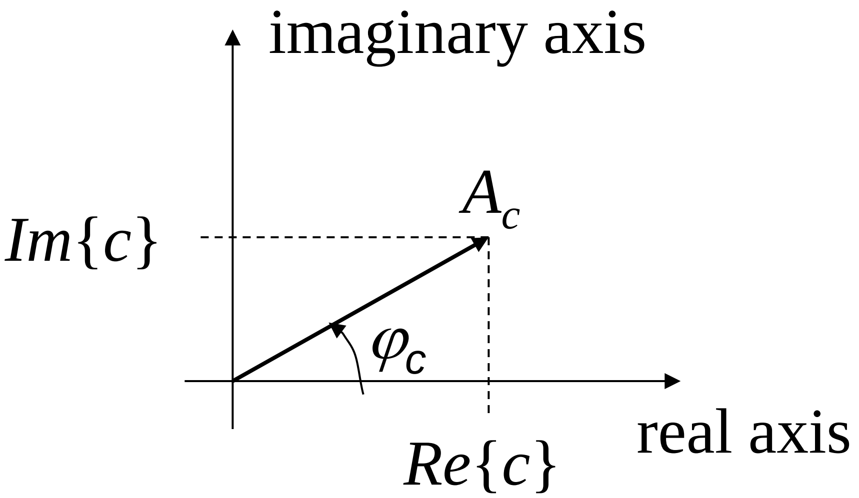
A complex number *c* has a real and an imaginary part:

- $c = Re\{c\} + i\,Im\{c\}$ (cartesian representation) $i = \sqrt{-1}$
- here, *i* always denotes the complex part

We can also use a polar representation:

$$A_c = \sqrt{\mathrm{Re}\{c\}^2 + \mathrm{Im}\{c\}^2}$$

$$\varphi_c = \tan^{-1}\left(\frac{\mathrm{Im}\{c\}}{\mathrm{Re}\{c\}}\right)$$

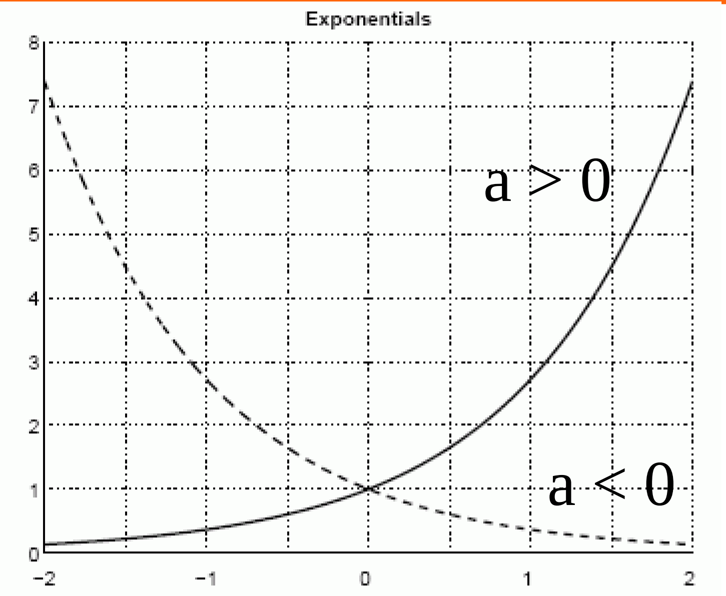


Application: Complex Sinusoids

Exponential *exp*

$$\exp(ax) = e^{ax}$$

- when *a* > 0 then *exp* increases with increasing *x*
- when *a* < 0 then *exp* approximates 0 with increasing *x*

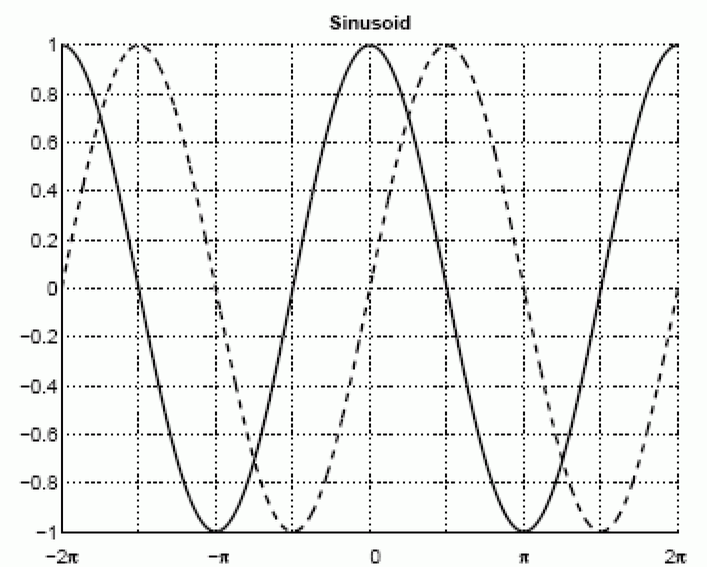


Complex exponential / sinusoid:

$$A_k e^{i(2\pi kt+\varphi)} = A_k(\cos(2\pi kt+\varphi)+i\sin(2\pi kt+\varphi))$$

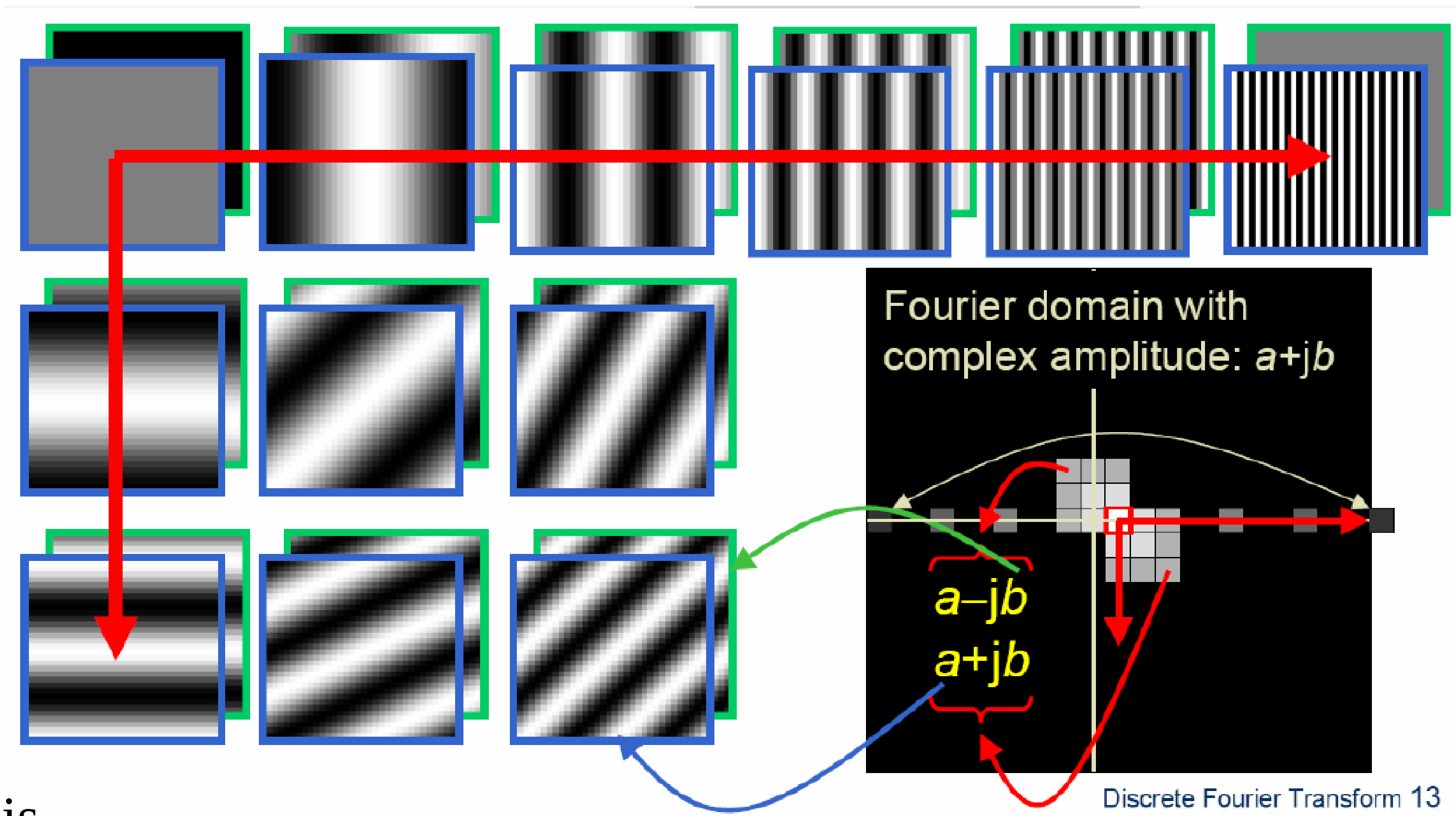
As before

- the *cos* term is the signal's real part
- the *sin* term is the signal's imaginary part
- *A* is the amplitude, φ the phase shift, *k* determines the frequency



Two-Dimensional Fourier Spectrum

u-axis



v-axis

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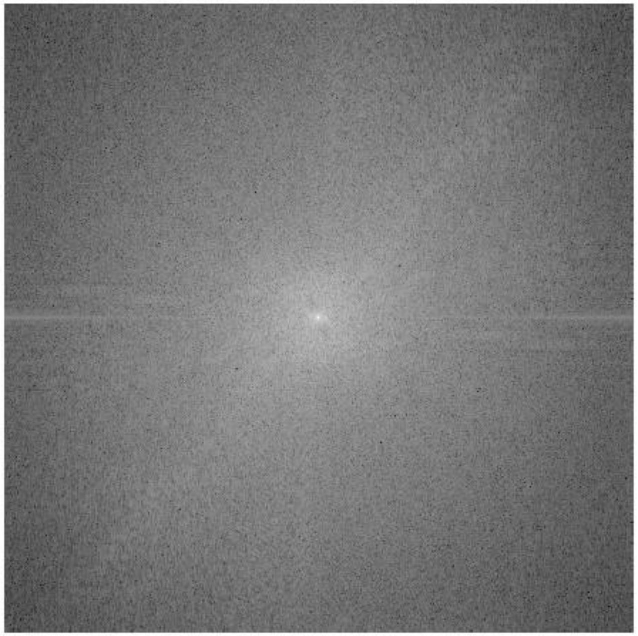
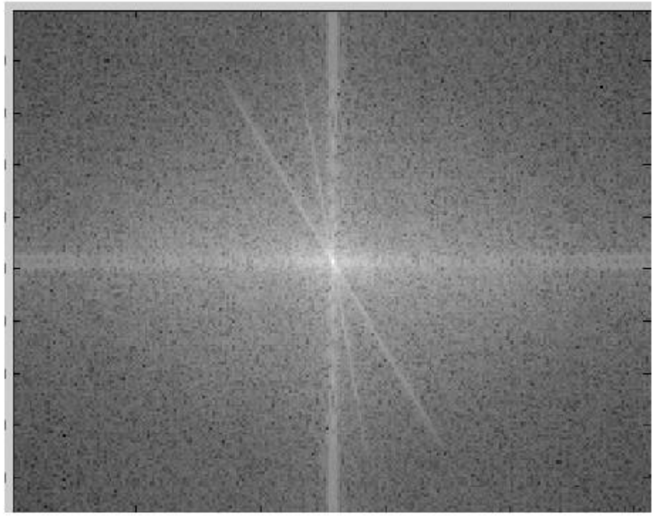
Some Example Spectra





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Effects of Missing Spectra Portions: Axial

(a) Spectrum along u determines detail along spatial x

(b) Spectrum along v determines detail along spatial y

(a)

(b)

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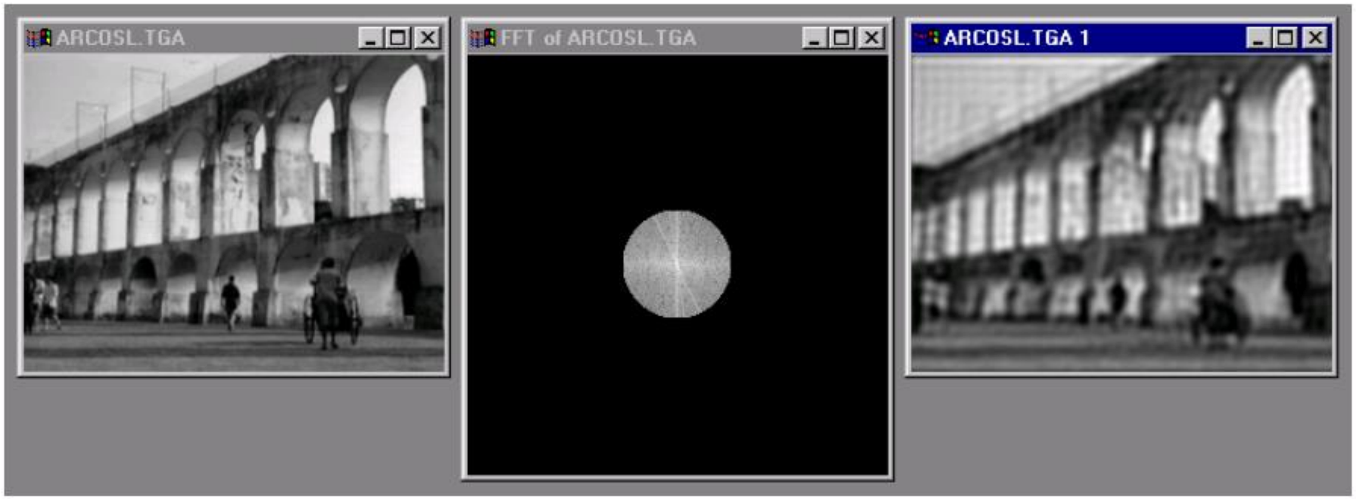
Effects of Missing Spectra Portions: Radial

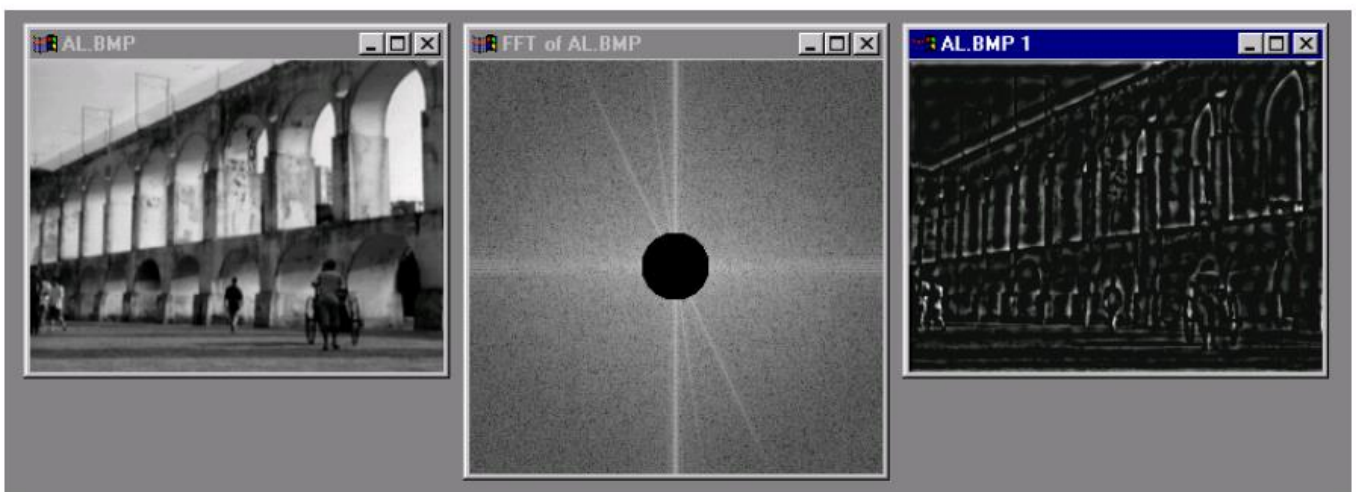
(a) Lower frequencies (close to origin) give overall structure

(b) Higher frequencies (periphery) give detail (sharp edges)

(a)



(b)



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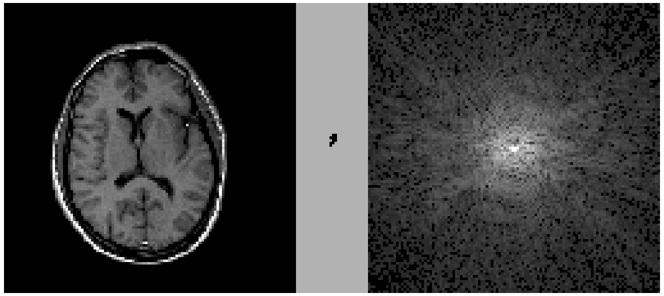
The Math… 2D DFT

The 2D transform:



$$S(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} s(n,m) e^{\frac{-i2\pi(kn+lm)}{NM}}$$

$$s(n,m) = \frac{1}{NM}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} S(k,l) e^{\frac{i2\pi(kn+lm)}{NM}}$$

Separability:

$$S(k,l) = \frac{1}{NM}\sum_{m=0}^{M-1} e^{\frac{-i2\pi lm}{M}} P(k,m) \quad \text{where } P(k,m) = \sum_{n=0}^{N-1} s(n,m) e^{\frac{-i2\pi kn}{N}}$$

$$s(n,m) = \frac{1}{NM}\sum_{l=0}^{M-1} e^{\frac{-i2\pi lm}{M}} p(n,l) \quad \text{where } \mathrm{p}(n,l) = \sum_{k=0}^{N-1} S(n,m) e^{\frac{-i2\pi kn}{N}}$$

- if M=N, complexity is $2\cdot O(2N^3)$

Fast Fourier Transform (FFT)

Recursively breaks up the FT sum into odd and even terms:

$$S(k) = \sum_{n=0}^{N-1} s(n) e^{\frac{-i2\pi kn}{N}} = \sum_{n=0}^{N/2-1} s(2n) e^{\frac{-i2\pi k2n}{N}} + \sum_{n=0}^{N/2-1} s(2n+1) e^{\frac{-i2\pi k(2n+1)}{N}}$$

$$= \sum_{n=0}^{N/2-1} s_{even}(n) e^{\frac{-i2\pi kn}{N/2}} + e^{\frac{-i2\pi k}{N}} \sum_{n=0}^{N/2-1} s_{odd}(n) e^{\frac{-i2\pi kn}{N/2}}$$

Results in an O(n·log(n)) algorithm (in 1D)

- O(n^2·log(n)) for 2D (and so on)

Fast Fourier Transform (FFT)

Gives rise to the well-known butterfly Divide + Conquer architecture

- invented by Cooley-Tuckey, 1965)