Differentiation, Finite Differencing, Incrementalization: From Clarity To Efficiency

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At the center of computer science

two major concerns of study:

- what to compute
- how to compute efficiently

problem solving:

- from clear specifications for "what"
- to efficient implementations for "how"

From clarity to efficiency

conflict between clarity and efficiency:

clear specifications usually correspond to straightforward implementations, not at all efficient.

efficient implementations are usually difficult to understand, not at all clear.

challenge:

develop a method that is both general and systematic

A general and systematic method

iterate: determine a minimum increment to take repeatedly, iteratively, to arrive at the desired program output

incrementalize: make expensive operations incremental in each iteration by using and maintaining useful additional values

implement: design appropriate data structures for efficiently storing and accessing the values maintained

applies to different programming paradigms		abstraction
loops:	**incrementalize**	none
sets:	**incrementalize**, implement	data
recursion:	iterate, **incrementalize**	control
rules:	iterate, **incrementalize**, implement	both
objects:	**incrementalize** across modules	module

iterate and incrementalize $\rightarrow$ integration by differentiation

Loops — a simple example

eliminating multiplications:

```
i:=1                    -- in grid with a columns and b rows
while i <= b:
  :
  ...a*i...             -- access last element of each row
  :
  i:=i+1
```

strength reduction: an oldest optimization, for array access.

Difference Engine, ENIAC: tabulating polynomials.

need to use language semantics and cost model

exploit algebraic properties: `a*(i+1)` = `a*i+a`

store, update, initialize value of `a*i`: where? how?

Loops — incrementalize

incrementalize

maintain invariant: `c` = `a*i`, use and update

```
i:=1                    → i:=1; c:=a;
while i <= b:
  :
  ...a*i...             → ...c...
  :
  i:=i+1                → i:=i+1; c:=c+a;
```

exploit algebraic properties

maintain additional information

iterate and implement: too little or too much to do

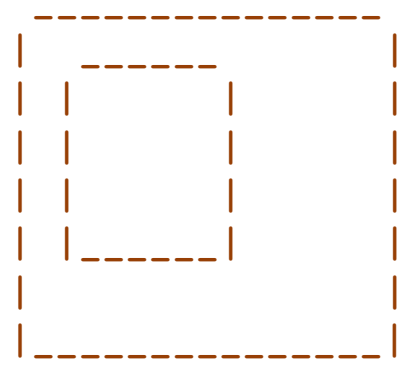
Loops — more examples

hardware design: non-restoring binary integer square root

```
n := input()
m := 2^(l-1)
for i := l-2 downto 0:
  p := n - m^2
  if p > 0:
    m := m + 2^i
  else if p < 0:
    m := m - 2^i
output(m)
```

goal: a few +- and shifts per bit

image processing: blurring



goal: a few operations per pixel

need higher-level abstraction

Sets — a simple example

graph reachability: edges, source vertices → reachable vertices

```
r := s
while exists x in e[r] - r:        -- e[r] = {y: (x,y) in e, x in r}
  r := r + {x}
```

need to

- handle composite set expressions: `x[y]`, `x - y`
- design representations of interrelated sets: `e`, `s`, `r`

Sets — incrementalize and implement

incrementalize: retrieve/add/delete element, test membership

two invariants for `e[r]-r`: `t = e[r]`, `w = t-r`

chain rule: maintain `t` and then `w`.

derive rules for maintaining simple and complex invariants.

implement: `s`, domain of `e`, range of `e`, `r`, `t`, `w`

based representations: records for all elements of related sets;

a set retrieved from is a linked list of pointers to the records;

a set tested against is a field in the records.

iterate: directly from `min r: s subset r, r + e[r] = r`

Sets — more examples

query processing: join optimization

```
r := {[x,y]: x in s, y in t | f(x) = g(y)}
```

iterate:

```
r := {}
for x in s:
  r := r + {[x,y]: y in t | f(x)=g(y)}
```

incrementalize: maintain

```
ginverse = {[g(y),y]: y in t}
```

derived:

```
ginverse := {}
for y in t:
  ginverse = ginverse + {[g(y),y]}
r := {}
for x in s:
  for y in ginverse{f(x)}
    r := r + {[x,y]}
```

previous algorithm:

```
finverse := {}
for x in s:
  finverse := finverse + {[f(x),x]}
ginverse := {}
for y in t:
  if g(y) in domain(finverse):
    ginverse := ginverse + {[g(y),y]}
r := {}
for z in domain(ginverse):
  for x in finverse{z}:
    for y in ginverse{z}:
      r := r + {[x,y]}
```

compare:

same asymptotic time: $O(\#s + \#t + \#r)$; fewer loops and ops;
less space: $O(\#t)$ or $O(\min(\#s, \#t))$, not $O(\#s + \#t)$; simpler and shorter; derived!

role-based access control (RBAC)

core RBAC: 16 expensive queries, 9 kinds, updated in many places.
125 lines python $\rightarrow$ hundreds of lines. CheckAccess: constant time.

Recursion — a simple example

longest common subsequence: sequences `x` and `y` $\rightarrow$ length

```
lcs(i,j)
= if i=0 or j=0: 0
  else if x[i]=y[j]: lcs(i-1,j-1)+1
  else: max(lcs(i,j-1),lcs(i-1,j))
```

need to

- determine how to iterate: recursion to iteration
- determine what and how to cache: dynamic programming

Recursion — iterate and incrementalize

```
lcs(i,j)
= if i=0 or j=0: 0
  else if x[i]=y[j]: lcs(i-1,j-1)+1
  else: max(lcs(i,j-1),lcs(i-1,j))
```

iterate: minimum increment from arguments of recursive calls

`i,j` $\rightarrow$ `i+1,j`

incrementalize: cache and use

```
lcs(i+1,j)   use r = lcs(i,j)          → lcs'(i,j,r)
= if i+1=0 lor j=0: 0
  else if x[i+1]=y[j]: lcs(i,j-1)+1    use lcs(i,j-1), cache
  else: max(lcs(i+1,j-1),lcs(i,j))     use lcs(i,j-1)
                                       → lcs'(i,j-1,lcs(i,j-1))
                                       recursively
```

implement: directly map to recursive or indexed data structures

Recursion — more examples

sequence processing: editing distance, paragraph formatting, matrix chain multiplications, ...

math puzzles: Hanoi tower, find solution in linear time

```
h(n,a,b,c)                -- move n disks from a to b using c
= if n<=0: skip
  else: h(n-1,a,c,b)::move(a,b)::h(n-1,c,b,a)
```

iterate: `n,a,b,c` $\rightarrow$ `n+1,a,c,b`

cache: `hExt(n,a,b,c) = <h(n,a,b,c), h(n,b,c,a), h(n,c,a,b)>`

```
hExt(n+1,a,c,b)   use rExt=hExt(n,a,b,c)  → hExt'(n,a,b,c,
= if n+1<=0: <skip,skip,skip>                      rExt)
  else: 1st(rExt)::move(a,c)::2nd(rExt),
        3rd(rExt)::move(c,b)::1st(rExt),
        2nd(rExt)::move(b,a)::3rd(rExt)>
```

simpler than others: maintain 2 additional values, not 5

Rules — a simple example

transitive closure:

```
edge(u,v) -> path(u,v)
edge(u,w), path(w,v) -> path(u,v)
```

need to

- find a way to proceed
- determine what and how to maintain
- design representations of different kinds of facts

additional question

- can we give time and space complexity guarantees?

Rules — iterate, incrementalize, implement

iterate: add one fact at a time until fixed point is reached

incrementalize: maintain maps indexed by shared arguments

implement: design nested linked lists and arrays of records

time and space guarantees:

```
edge(u,v) -> path(u,v)
edge(u,w), path(w,v) -> path(u,v)
```

time: # of combinations of hypotheses — optimal

$O(\min(\#\texttt{edge} \times \#\texttt{path}.2/1,\ \#\texttt{path} \times \#\texttt{edge}.1/2))$

edges vertices output indegree

space: $O(\#\texttt{edge})$, for storing inverse map of `edge`

Rules — more examples

program analysis: dependence analysis, pointer analysis, information flow analysis, ...

trust management: SPKI/SDSI authorization

```
auth(k1,[k2],TRUE,a1,v1), auth(k2,s2,d2,a2,v2)
        -> auth(k1,s2,d2,PInt(a1,a2),VInt(v1,v2))

auth(k1,[k2 [n2 ns3]],d,a,v1), name(k2,n2,[k3],v2)
        -> auth(k1,[k3 ns3],d,a,VInt(v1,v2))

name(k1,n1,[k2 [n2 ns3]],v1), name(k2,n2,[k3],v2)
        -> name(k1,n1,[k3 ns3],VInt(v1,v2))
```

find authorized keys: $O(in * kp * kn)$, better than $O(in * k * k)$.

Objects — a simple example

the "what" of a software component:

queries: compute information using data w/o changing data.

updates: change data.

example:

class `LinkedList` in Java has many methods:

`size()`, 18 add or remove, several other queries.

Objects — incrementalize

how to implement the queries and updates: varies significantly

straightforward:

queries compute requested information.

updates change base data.

example: `size()` contains a loop that computes the size.

observe:

queries are often repeated, many are easily expensive;

updates can be frequent, they are usually small.

sophisticated — incrementalized:

store derived information; queries return stored information.

updates also update stored information.

example: maintain size in a field, and update it in 18 places.

Objects — more examples

examples: wireless protocols, electronic health records, virtual reality, games, ...

```
findStrongSignals(): return {s in signals | s.getStrength() > threshold}
```

```
  class Protocol
    signals: set(Signal)
    threshold: float
+   strongSignals: set(Signal)
    ...
    addSignal(signal): signals.add(signal)
+     signal.takeProtocol(this)
+     if signal.getStrength() > threshold
+       strongSignals.add(signal)
*   findStrongSignals(): return strongSignals
+   updateSignal(signal):
+     if signals.contains(signal)
+       if strongSignals.contains(signal)
+         if not signal.getStrength()>threshold
+           strongSingals.remove(signal)
+       else
+         if signal.getStrength()>threshold
+           strongSingals.add(signal)
    ...
```

```
  class Signal
    strength: float
+   protocols: set(Protocol)
    ...
+   takeProtocol(protocol):
+     protocols.add(protocol)
    setStrength(v):
      strength = v
+     for protocol in protocols
+       protocol.updateSignal(this)
    getStrength(): return strength
    ...
  ...
```

	original lines
*	changed lines
+	added lines

findStrongSignal: $O(\#\text{signals}) \to O(1)$. setStrength: $O(1) \to O(\#\text{protocols})$.

Iterate, Incrementalize, Implement

iterate at a minimum increment step; incrementalize expensive computations; implement on efficient data structures.

loops — iter, **inc**, impl
maintaining invariants, algebraic properties, additional values

sets — iter, **inc**, **impl**
chain rule, deriving maintenance rules; based representations

recursion — **iter**, **inc**, impl
recursion to iteration; dynamic programming

rules — **iter**, **inc**, **impl**
all, giving time and space complexity guarantees

objects
all, across modules

connect theory w/ practice. like differentiation & integration.

References

loops [Allen69..., Liu97, LS98a/LSLR05...]

sets [Earley76..., PK82, Willard96, Willard02, LWGRCZZ06...]

recursion [BD77..., Smith90, LS99/03, LS00, LS02/09...]

rules [Forgy82, Vardi82..., McAllester99, LS03/09...]

objects [..., LSGRL05, RL08...]

more: *Systematic Program Design: From Clarity to Efficiency*

Beyond — far and near, new and old

distributed: synchronous and asynchronous communications
sets (of procs) and sequences (of msgs), quantifications

secure: cryptographic primitives
synchronous and asynchronous comm, declarative policies

probabilistic: probability distribution functions
sets and aggregations, continuous mathematics

game theoretical: conflict and cooperation, equilibrium
fixed points, logic semantics and constraints

In DistAlgo: Lamport's distributed mutex

```
 1   def setup(s):
 2     self.s = s                              # set of all other processes
 3     self.q = {}                             # set of pending requests with logical clock

 4   def mutex(task):                          # for doing task() in critical section
 5     -- request
 6     self.c = logical_clock()                                          # rule 1
 7     send ('request', c, self) to s                                    #
 8     q.add(('request', c, self))                                       #
 9     await each ('request',c2,p2) in q | (c2,p2) != (c,self) implies (c,self) < (c2,p2)
10           and each p2 in s | some received('ack',c2,=p2) | c2 > c     # rule 5
11     task()                                  # critical section
12     -- release
13     q.del(('request', c, self))                                       # rule 3
14     send ('release', logical_clock(), self) to s                      #

15   receive ('request', c2, p2):                                        # rule 2
16     q.add(('request', c2, p2))                                        #
17     send ('ack', logical_clock(), self) to p2                         #

18   receive ('release', _, p2):                                         # rule 4
19     q.del(('request', _, =p2))                                        #
```

Complete program in DistAlgo

```
 0 class P extends Process:

... # content of the previous slide

20    def run():
        ...
21      def task(): ...
22      mutex(task)
        ...

23 def main():
     ...
24   configure channel = reliable, fifo
25   configure clock = Lamport
26   ps = 50 new P
27   for p in ps: p.setup(ps-{p})
28   ps.start()
     ...
```

some syntax in Python:

```
class P( process )
send( m, to= ps )
some( elem in s, has= bexp )
config( channel= {'reliable','fifo'} )
new( P, num= 50 )
```

Simplified spec by un-incrementalization

```
 0 class P extends Process:
 1   def setup(s):
 2     self.s = s

 3   def mutex(task):
 4     -- request
 5     self.c = logical_clock()
 6     send ('request', c, self) to s
 7     await each received('request',c2,p2) |
 8             not (some received('release',c3,=p2) | c3 > c2) implies (c,self) < (c2,p2)
             and each p2 in s | some received('ack',c2,=p2) | c2 > c
 9     task()
10     -- release
11     send ('release', logical_clock(), self) to s

12   receive ('request', _, p2):
13     send ('ack', logical_clock(), self) to p2
```

Multi-Paxos for distributed consensus

in C++/Java/Python by students: thousands of lines

in DistAlgo by students: hundreds of lines

in DistAlgo better spec: tens of lines

Some more examples of incrementalization

birthday conflict probability: from Russel's BLOG

```
some x in P, some y in P: x != y, x.bday = y.bday
```

efficient MCMC sampling: O(1) after each update

win-not-win game:

```
win(x) if move(x,y), not win(y)
```

efficient WFS computation: O(#`move`) total

Some ongoing projects and results

objects and nested sets: generating incremental implementations of queries with complexity guarantees [arxiv/PPDP 2016]

datalog and extensions: generating efficient implementations for demand-driven queries [PPDP 10, SIGMOD 11],
applications to many pointer analyses [arxiv/ICLP/TPLP 2016]

quantifications and arbitrary negation: generating optimized implementations by using aggregations [OOPSLA 12],
founded semantics and constraint semantics [arxiv 16]

distributed algorithms: from high-level specifications to efficient implementations [OOPSLA 12, SSS 12, arxiv16/TOPLAS to appear]