Polynomial-Time Computability in Analysis: A Survey

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Outline

Computational Theory of Real Analysis

Constructive Analysis Bishop, Bridges, Ishihara, $\cdots$

Intuitionistic Logic

Recursive Analysis (Computable Analysis)

Recursion Theory

Russian School Šanin, Moschovakis, Ceitin, $\cdots$

Polish School Grzegorczyk, Mostowski

Lacombe, Pour-El, Richards

Weihrauch, $\cdots$

Polynomial-Time Analysis

Complexity Theory

Turing machine model Ko, Friedman, Weihrauch, Müller
Rettinger, Zheng, Cook, Braverman, $\cdots$

Real-valued circuit model Hoover

Algebraic model Blum, Shub, Smale, Cucker, $\cdots$

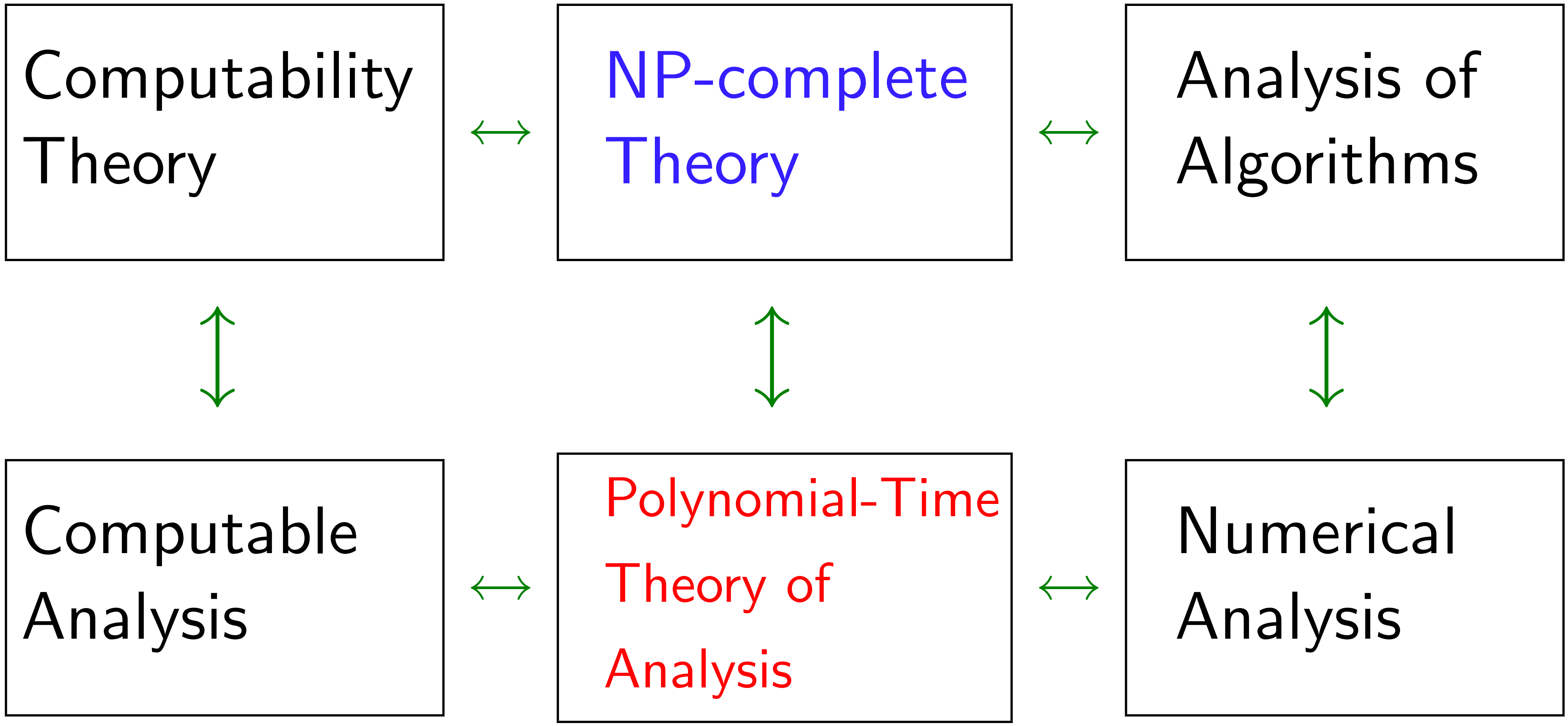
Information-based complexity theory
Traub, Wozniakowski, $\cdots$

Numerical Analysis

Classical analysis, Arithmetic complexity theory

Interval analysis, Scientific computing

Relationship between these theories



Example: Roots of Polynomials

Bishop: Fundamental Theorem of Algebra has a constructive proof.

Specker: All roots of a computable polynomial function are computable.

The mapping from coefficients to roots is computable.*

Ko-Friedman: All roots of a polynomial-time computable polynomial function are polynomial-time computable.

Neff: The mapping from coefficients to roots is in NC.

Schönhage: The mapping from coefficients to roots is computable in time $O(n^3\phi(n))$.

Smale: Newton's method runs in polynomial time on average.

Warning They may use different models.
$\Longrightarrow$ There is **no Church's Thesis** in computational analysis.

The models of the following theories are consistent:

Recursive analysis (Polish school)

Polynomimal-time analysis (Turing machine model)

Discrete NP-completeness theory

Classical numerical analysis (e.g., interval analysis)

Real Numbers

A real number is an infinite object, and has no finite representations.

Basic representation: Cauchy functions with a fixed converging rate

$$\varphi_x : \mathbb{N} \longrightarrow \mathbb{D} \text{ with } |\varphi_x(n) - x| \le \frac{1}{2^n}. \quad \left(\text{Why } \frac{1}{2^n}?\right)$$

$\mathbb{D}$: dyadic rationals

x is computable if $\exists$ a computable φ_x.

x is P-time computable if $\exists$ a P-time computable φ_x.

Other Representations?

Dedekind cuts: $L_x = \{d \in \mathbb{D} : d < x\}$
Binary expansions: $b_x : \mathbb{N}^+ \to \{0,1\}$ and $b_x(0) \in \mathbb{Z}$, with
$x = \sum_{n=0}^{\infty} b_x(n) \cdot 2^{-n}$.
Continued fractions: $c_x : \mathbb{N} \to \mathbb{N}^+$ with

$$x = c_x(0) + \frac{1}{c_x(1) + \frac{1}{c_x(2) + \frac{1}{\cdots}}}$$

For computable real numbers, these representations are equivalent
to Cauchy function representation.

For P-time computable real numbers, they are not equivalent.

Real Numbers as Discrete Objects

$P_{\mathbb{R}}$: Set of P-time computable real numbers
$NP_{\mathbb{R}}$: Set of NP-time computable real numbers
$\#P_R$, $PSPACE_{\mathbb{R}}, \ldots$

What are the relations between these complexity classes?

General Observation

Representations of real numbers behave like selective sets or sparse sets.

$P_{\mathbb{R}} = NP_{\mathbb{R}} \Longleftrightarrow P_1 = NP_1$

$\#P_{\mathbb{R}} =? \#NP_{\mathbb{R}}$ (YES if $NP = UP$)

Real Functions

Representation of $f : \mathbb{R} \to \mathbb{R}$:

Type-2 function with a fixed converging rate

$\Phi_f : \Psi \times \mathbb{N} \to \mathbb{D}$, with $|\Phi_f(\varphi_x, n) - f(x)| \leq \dfrac{1}{2^n}$

Ψ: set of Cauchy functions φ_x

Computational Model for type-2 functions:

Oracle Turing machine

f is computable if Φ_f is computable by an oracle TM M

$|M^{\varphi_x}(n) - f(x)| \leq 2^{-n}$

$f : [0,1] \to \mathbb{R}$ is P-time computable if $M^{\varphi_x}(n)$ halts in time $n^{O(1)}$ for every oracle φ_x with $x \in [0,1]$.

Compute $f(x) = x^2$:

Input n (the output precision)
Oracle φ_x (representation of a real x)
Algorithm
(1) Compute required input precision m from n

($n \mapsto m$ is called modulus function);

(2) Ask oracle to get a rational r with $|r - x| \leq 2^{-m}$;
(3) Compute $s \leftarrow r^2$;
(4) Output first n bits of s.

Note: Modulus function may also depend on x. So, Steps (1) and (2) may be repeated to find the right m.

An Alternative type-1 representation

(with an additional continuity requirement)

(φ_f, m_f) where $\varphi_f : \mathbb{D} \times \mathbb{N} \longrightarrow \mathbb{D}$, $m_f : \mathbb{N} \longrightarrow \mathbb{N}$,

with $|\varphi_f(d, n) - f(d)| \leq 2^{-n}$, and

$$|x - y| \leq 2^{-m_f(n)} \Rightarrow |f(x) - f(y)| \leq 2^{-n}$$

f is computable iff φ_f, m_f are computable

f is P-time computable iff φ_f is P-time computable, and m_f is a polynomial function.

Warning

In this model, comparison of two real numbers is noncomputable.

- $\exists$ oracle TM M such that $M^{\varphi_x,\varphi_y}(0) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{if } x > y, \\ \uparrow & \text{if } x = y. \end{cases}$

- No oracle TM: $M^{\varphi_x,\varphi_y}(0) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$

- The problem of determining whether a given polynomial function (represented by its coefficients) has multiple roots is undecidable.

Numerical Operators

$F : C[0,1] \to \mathbb{R}$ is a type-3 function.

We can use Oracle TM as a computational model.

F is computable if $\exists$ oracle TM M such that $|M^{\Phi_f}(n) - F(f)| \leq 2^{-n}$.

(In the computation, M may ask the oracle to find an approximate value of $f(x)$ by asking the oracle for the value of $\Phi_f^d(n)$, where $d \approx x$.)

P-Time Computable Operators?

Weak form: Consider only P-time invariance

If f is P-time computable, what is the complexity of $F(f)$?

Strong form [Kawamura-Cook, 2010]

Use regular functions as representations of f, a more general notion of P-time computable operator can be defined.

Many known results about P-time computability of numerical operators in the weak form can be extended to the strong form.

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1 Computational Models

Church's thesis in computational analysis?

2 Complexity Hierarchy of Numerical Operations

Applying NP-theory to analysis

3 Applications to Computational Geometry

P-time computable Jordan domains

4 Applications in Complex Analysis

Julia sets, conformal mappings

A **Complexity Hierarchy** of Numerical Operations

Differentiation	Noncomputable
Integral Eq (with local Lipschitz cond)	EXPSPACE-Complete
Ordinary Diff Eq (with Lipschitz cond)	PSPACE-complete
Integration	#P-complete
Minimax	$\mathrm{NP}^{\mathrm{NP}}$-complete
Maximization	NP-complete
Roots (of 2-dim. functions)	between UP and NP
Fixed Points (of 2-dim. functions)	PPAD-complete
Roots (of 1-1 functions)	P-complete
Differentiation (f' has poly. modulus)	P

Maximization:

What is the complexity of finding

$$\max\{x_1, x_2, \ldots, x_K\}?$$

Depending on the representation of $x_1, x_2, \ldots, x_K$.

(1) Explicit Representation:

$x_1, x_2, \ldots, x_K$ are given as input (input size $n \approx K$):

Input: $\underbrace{38, 25, 19, 55, \ldots, 49}_{\text{find max}}$

Complexity: In P (needs $K - 1$ comparisons)

(2) Oracle Representation:

$x_1, x_2, \ldots, x_K$ are given by an oracle function Φ $(\Phi(i) = x_i)$

Oracle:

38	25	19	55	$\cdots$	49

Input: K (input size $n = \lceil logK \rceil$)

Complexity: Exponential time

(must ask the oracle Φ for $K \approx 2^n$ times)

(3) Machine Representation:

$x_1, x_2, \ldots, x_K$ are presented by a polynomial-time algorithm A that computes the function Φ

Input: $\boxed{A}$ $\quad (n = size(A) \approx \lceil \log K \rceil^{O(1)})$

$\downarrow\downarrow \cdots \downarrow$

$38, 25, 19, 55, \cdots, 49$ (hidden input, size $= K$)

Complexity: In NP; NP-complete for some A
(Actually, the following variation is in NP: Given A and an integer M, determine whether $M < \max\{\Phi(1), \ldots, \Phi(K)\}$.)

- Most NP-ccomplete optimization problems can be viewed in this form.

Traveling Salesman:

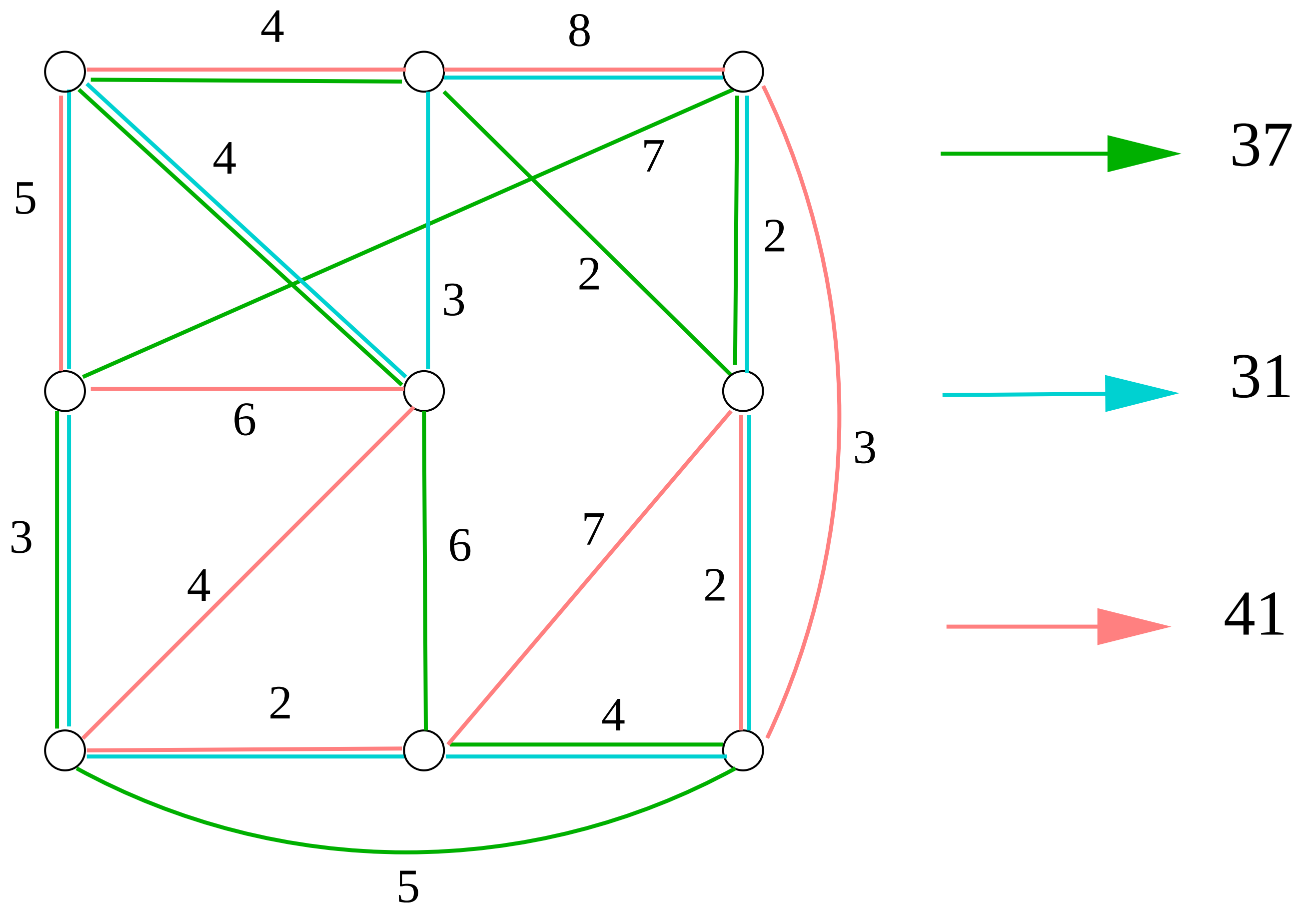
Input: Graph G with n vertices; weight $w : E \longrightarrow \mathbb{N}^+$
Question: Find the min-weight Hamiltonian tour of G

- There are $K = (n-1)!$ different Hamiltonian tours of G, and they can be enumerated as $H_1, H_2, \ldots, H_K$.

- Now, Traveling Salesman can be restated as follows:

  Find the minimum of the output from A_G:

A_G: For $i = 1, 2, \ldots, K$, identify i with a Hamiltonian tour H_i and output $\Phi(i) =$ total weight of H_i.



Numerical Maximization

Given $f : [0,1] \rightarrow] R$ (as an oracle), find $\max_{0 \leq x \leq 1} f(x)$.

Discretize this problem:

Assumption: Function f has a polynomial modulus:

$$|x - y| \leq 2^{-n^c} \Rightarrow |f(x) - f(y)| \leq 2^{-n}$$

With this assumption, the discretized problem becomes

Find the maximum value of

$$f\left(\frac{1}{2^{n^c}}\right), f\left(\frac{2}{2^{n^c}}\right), \ldots, f\left(\frac{2^{n^c}}{2^{n^c}}\right)$$

(For convenience, we use $c = 1$ in the following discussion.)

Representation of f:

(1) Explicit representation

Function values $f\left(\frac{1}{2^{n^c}}\right), f\left(\frac{2}{2^{n^c}}\right), \ldots, f\left(\frac{2^{n^c}}{2^{n^c}}\right)$ are given as input.

Complexity: Polynomial in input size, exponential in output precision n

- This is the common practice of Computational Geometry (with n input points, instead of 2^n points).

(2) Oracle representation

Function f is given by an oracle. The maximization algorithm may ask for $f(r)$ for any rational number r.

Complexity: Exponential in the output precision n.

- This is used in some theoretical study of numerical analysis (e.g., Information-Based Complexity Theory of [Traub et al.]).

(3) Machine representation

Function f is assumed to be computable by a machine M_f in polynomial time (polynomial in output precision n), and the maximization algorithm may simulate M_f on any input r.

Complexity: NP-complete.

Note: The Machine representation approach is equivalent to the model in the Turing Machine-Based P-Time Theory of Analysis.

Theorem [Ko, Friedman]

P $=$ NP $\Longleftrightarrow$ For every polynomial-time computable function $f : [0,1] \to \mathbb{R}$, $\max f \in$ P.

Ordinary Differential Equations (IVP)

$$y'(x) = f(x, y(x)),\ 0 \leq x \leq 1,$$
$$y(0) = 0.$$

- $\exists$ computable f: all solutions y are not computable on $[0, \delta]$ for all $\delta > 0$. Pour-El, Richards

- f computable, solution y unique $\Longrightarrow$ y computable.

- $\exists$ P-time computable f: solution y is unique, but complexity of y is arbitrary high. Miller

Lipschitz Condition

$f \in Lip(\alpha)$: $(\forall x \in [0,1])\ (\forall y_1, y_2 \in [-1,1])$
$$|f(x,y_1) - f(x,y_2)| \leq \alpha \cdot |y_1 - y_2|.$$

- f P-time computable, $f \in Lip(\alpha) \Longrightarrow y$ P-space computable. Ko
- $(\exists$ P-time computable $f)$: $f \in Lip(\alpha)$, y is P-space complete. Ko, Kawamura
- The mapping $f \mapsto y$ is P-space complete. Kawamura, Cook

Volterra Integral Equations (of the 2nd kind)

$$y(x) = f(x) + \int_0^x K(x, s, y(s))ds, \ 0 \le x \le 1,$$

with $K \in Lip_3(\alpha)$: $|K(x, s, y_1) - K(x, s, y_2)| \le \alpha \cdot |y_1 - y_2|$

- If α is independent of x, then this problem is P-space complete. Ko, Kawamura
- If $\alpha \le 2^{n^{O(1)}}$ for $x \le 1 - 2^{-n}$, then y is EXP-space computable. Ko
- Under the above local Lipschitz condition, this problem is EXP-space complete. Kawamura

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Subsets of $\mathbb{R}^2$

Computable sets of real numbers?

Again, there does not seem to be a Church's Thesis.

For discrete $A \subseteq \{0,1\}^*$, A is computable if

$$\chi_A(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{array} \right\} \text{ is computable.}$$

Try: For $S \subseteq \mathbb{R}^2$, S is computable if

$$\chi_S(\mathbf{z}) = \left\{ \begin{array}{ll} 1 & \text{if } \mathbf{z} \in S \\ 0 & \text{if } \mathbf{z} \notin S \end{array} \right\} \text{ is computable.}$$

??

Warning The function χ_S is not computable for nontrivial S (i.e., $S \neq \emptyset$, $S \neq \mathbb{R}^2$).

For an oracle TM, let

$$Err_n(M) = \{\mathbf{z} : M^{\mathbf{z}}(n) \neq \chi_S(\mathbf{z})\}.$$

P-time Approximable (Measurable) Sets

$\exists$ P-time oracle TM M: $\mu(Err_n(M)) \leq 2^{-n}$.

P-time Recognizable Sets

$\exists$ P-time oracle TM M:

$$\mathbf{z} \in Err_n(M) \Rightarrow \delta(\mathbf{z}, \partial S) \leq 2^{-n}.$$

Strongly P-time Recognizable Sets

$\exists$ P-time oracle TM M:

$\mathbf{z} \in Err_n(M) \Rightarrow \delta(\mathbf{z}, \partial S) \leq 2^{-n}$ and $\mathbf{z} \notin S$.

P-time Computable Sets [Weihrauch, $\cdots$]

$\exists$ P-time oracle TM M:

$\mathbf{z} \in Err_n(M) \Rightarrow 2^{-n} < \delta(\mathbf{z}, S) \leq 2 \cdot 2^{-n}$.

P $=$ NP $\Longleftrightarrow$ the above two classes are equivalent.

P-time Computable Sets wrt Hausdorff Distance

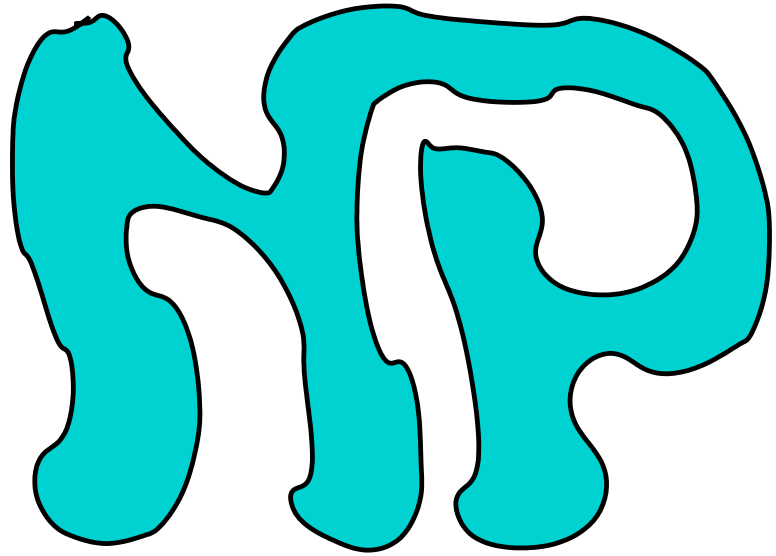
$\exists$ P-time oracle TM M: [Braverman, Yampolsky]

$\delta_{\mathrm{HAUS}}(S, \{\mathbf{z}\ M^{\mathbf{z}}(n) = 1\}) \leq 2^{-n}$

All of the above definitions are not equivalent.

Jordan Domains

A Jordan domain is a singly-connected set whose boundary is a Jordan curve Γ (the image of a mapping $f : [0,1] \rightarrow \mathbb{R}^2$).



Computable Curves — still no unique definition

Monotonically Computable: f is one-to-one

Retraceably Computable: f is not necessarily one-to-one

Gu, Lutz, Mayordomo

Normalizably Computable: Length of $f[0,t]$ is proportional to t, for $0 < t < 1$ (if $leng(\Gamma)$ is finite). Rettinger, Zheng

Continuous Computational Geometry

Goals: Resolve the numerical non-robustness problem

Deal with more general geometric objects

Allow efficient implementation of traditional algorithms

E.g. Exact Geometry Computation (EGC)

Yap, Melhorn, ...

Jordan Domain-Based Approach

General Question

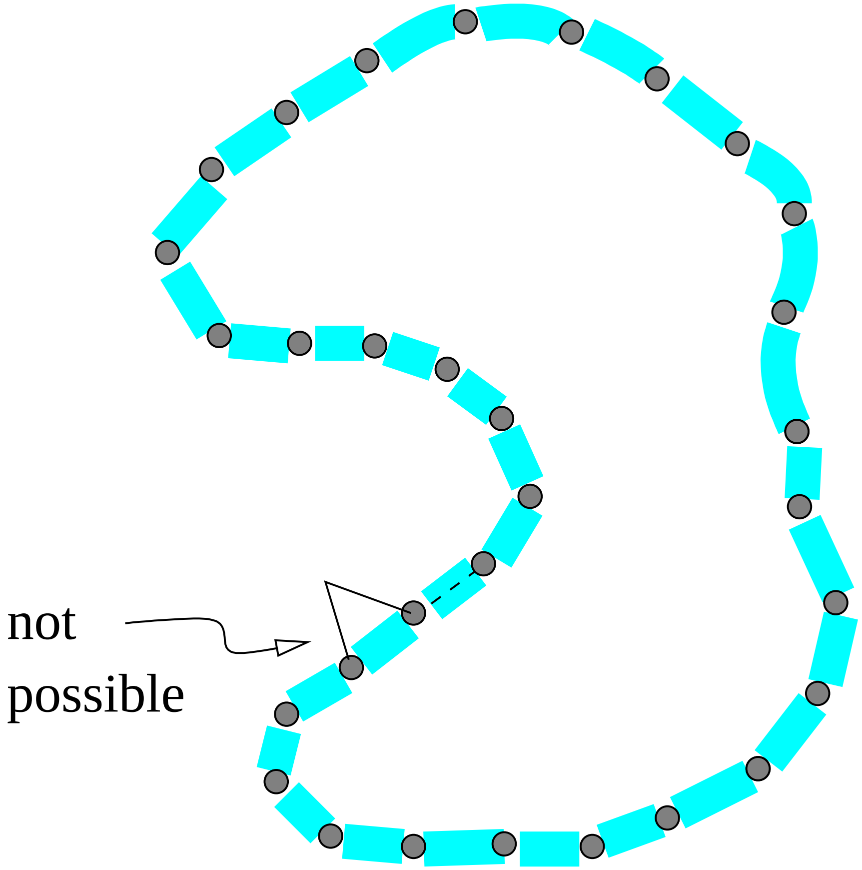
Given a two-dimensional domain S whose boundary is a P-time computable Jordan curve, what is the complexity of the related problems?

P-Time Computable Jordan Domains

as an extension of Polygon Representation

If ∂S is P-computable, then it has polynomial modulus.

So, ∂S is represented by an implicit polygon of $2^{p}(n)$ vertices.



Complexity of Jordan Domains S

Area	Noncomputable (fractal)
Length of ∂S	Noncomputable (fractal)
Shortest Paths in S	between #P and PSPACE
Pancake Cutting	#P-complete
Membership ($x \in S$?)	between UP and #P
Circumscribed Rectangle	$\mathrm{NP}^{\mathrm{NP}}$-complete
Distance of x from S	NP-complete
Convex Hull	NP-complete

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Analytic Functions

If f is real analytic and P-time computable, then integral $\int_0^x f$, derivative $f'(x)$, maximum value $\max f(x)$, and roots $\{x : f(x) = 0\}$ are all P-time computable.

Parallel Complexity

If f is analytic and is NC (or LOG-space) computable, then integral, derivative, maximum value and roots of f are all NC (or LOG-space, resp.) computable. Yu

Zeroes of an Analytic Fuction f

on a Jordan domain S

Assumptions

- f is analytic on $S \cup \partial S$
- $f(\mathbf{z}) > 0$ on ∂S
- f and ∂S are NC computable

Quadrature Method

(1) Compute the number of zeroes

$$n = \frac{1}{2\pi i} \int_{\partial S} \frac{f'(\mathbf{z})}{f(\mathbf{z})} d\mathbf{z}$$ (by principle of argument)

(2) Compute the Newton sums

$$s_p = \sum_{i=1}^{n} \mathbf{z}_i^p = \frac{1}{2\pi i} \int_{\partial S} \mathbf{z}^p \frac{f'(\mathbf{z})}{f(\mathbf{z})} d\mathbf{z}$$

(3) Compute the associated polynomial

$$p_n(\mathbf{z}) = \prod_{i=1}^{n} (\mathbf{z} - \mathbf{z}_i)$$ (by Newton's identity and $s_p, \; p = 1, \ldots, n$)

(4) Solve the associated polynomial equation [Neff]

All the above calculations can be parallelized.

Some problems related to Membership Problem

- Computing Winding Number of a closed curve
- Computing Single-Valued Analytic Branch of a multi-valued function

Square Root Problem

On a complex domain, $\sqrt{\mathbf{z}} = \sqrt{|\mathbf{z}|} \cdot e^{iarg(\mathbf{z})/2}$ has 2 single-valued, analytic branches:

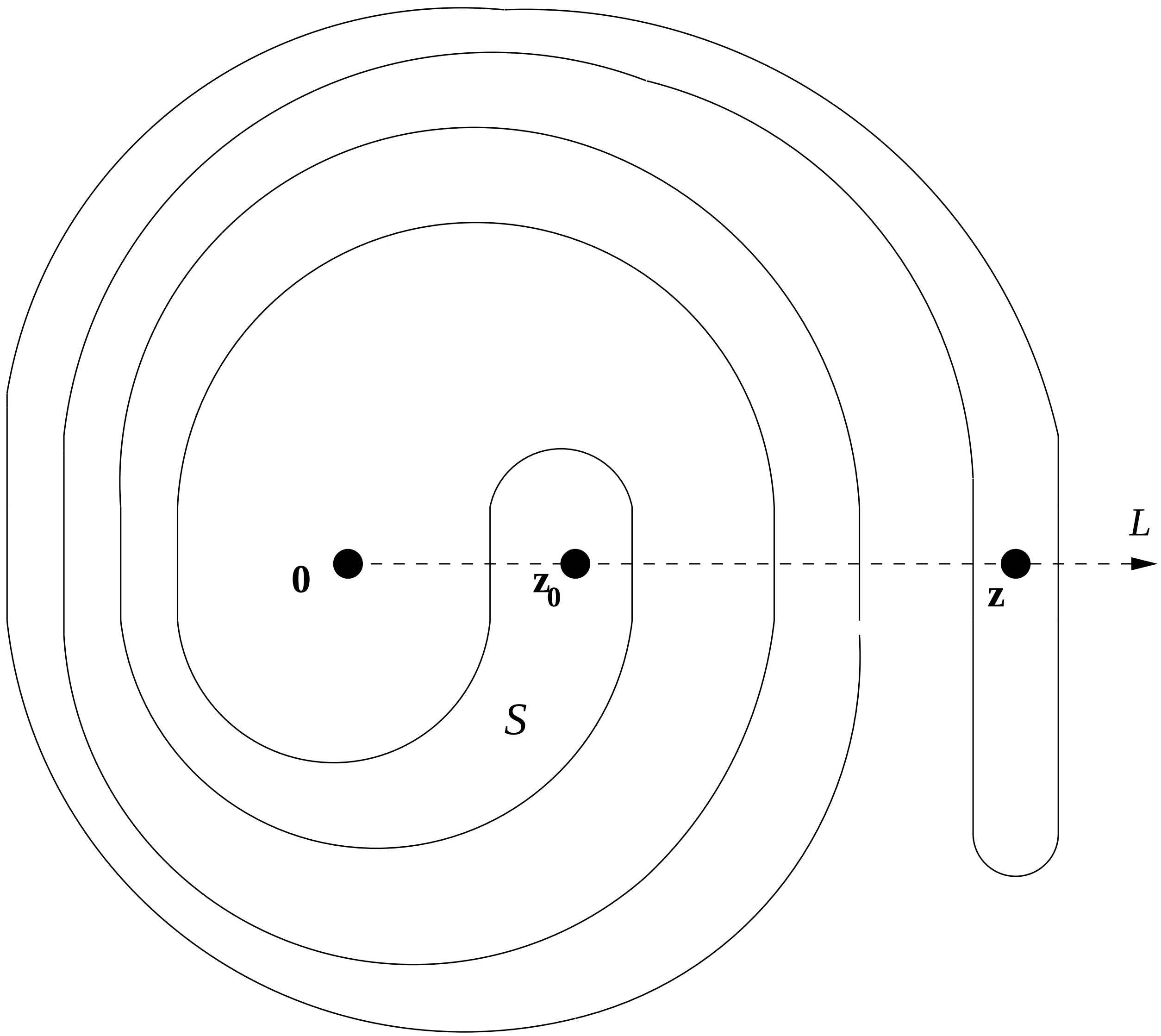
$$\sqrt{\mathbf{z}} = \sqrt{|\mathbf{z}|} \text{ or } \sqrt{|\mathbf{z}|} \cdot e^{i\pi}$$

Logarithm Problem

On a complex domain,
$\log \mathbf{z} = \log |\mathbf{z}| + i\, arg(\mathbf{z})$
has ∞ single-valued analytic branches:

$arg(\mathbf{z}) = \cdots, -4\pi,$
$-2\pi, 0, 2\pi, 4\pi, \cdots$

corresponding to
$arg(\mathbf{z}_0) = \cdots, 0,$
$2\pi, 4\pi, 6\pi, 8\pi, \cdots$



Analytic Branch Problem

Given a P-time computable closed Jordan curve Γ, what is the complexity of finding a single-valued analytic branch of $\log \mathbf{z}$ or $\sqrt{\mathbf{z}}$ on $S = \mathrm{Int}(\Gamma)$?

Equivalent Problem: Given Γ, what is the complexity of computing a continuous argument function $h(\mathbf{z}) \in arg(\mathbf{z})$ on S?

$$\log \mathbf{z} \equiv h(\mathbf{z}) - h(\mathbf{z}_0) \qquad \sqrt{\mathbf{z}}, \equiv \frac{h(\mathbf{z}) - h(\mathbf{z}_0)}{2\pi} \mathrm{mod} 2$$

If $\mathbf{z}$ and $\mathbf{z}_0$ are on the boundary of S,

$h(\mathbf{z}) \approx$ winding number $\mathrm{about}\ \mathbf{z}$.

Complexity

Problem	Lower bound	Upper bound
Winding Number	#P	#P
Logarithm	#P	#P
Square root	$\oplus$P	MP
Membership	UP	MP

NP: $\{x \mid (\exists^p y)\, R(x,y)\}$, where $R \in P$

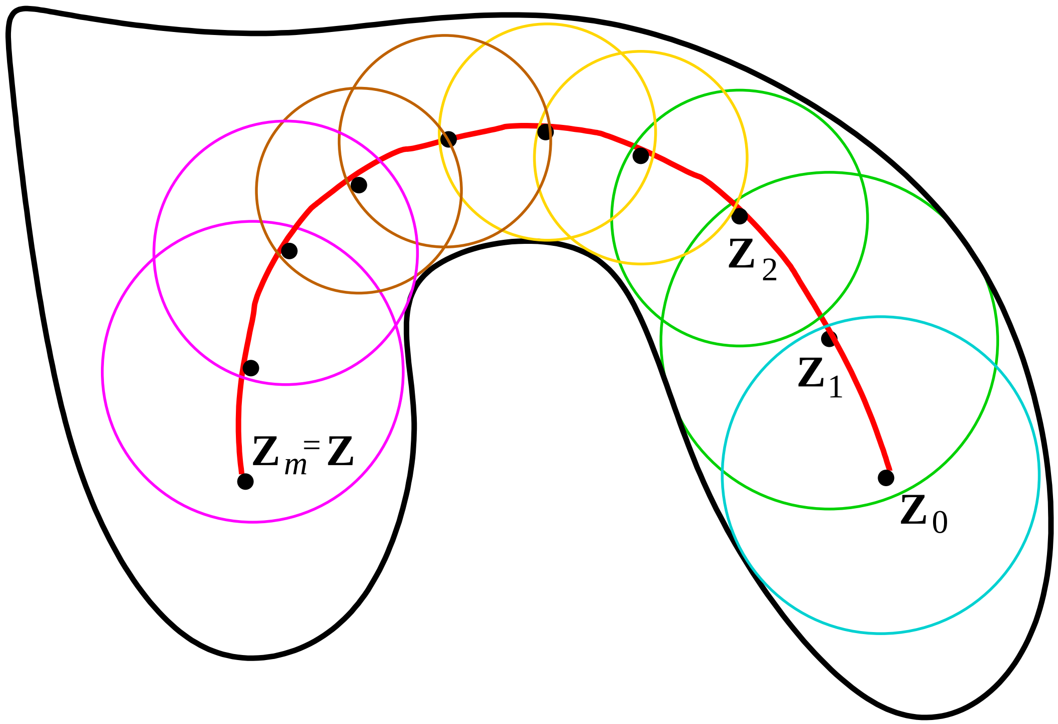
#P: $f(x) =$ number of y such that $R(x,y)$

$\oplus P$ (Parity P): $f(x)$ is odd

MP (Midbit P): the middle bit of $f(x) = 1$.

Analytic Continuation

Assume that f is an analytic function defined on a domain S. Then, the power series of f at any $\mathbf{z} \in S$ can be computed from that of f at a starting point $\mathbf{z}_0$.



Complexity?

Depends on geometric properties of ∂S?

Julia Sets

For a function $f : \mathbb{C} \rightarrow \mathbb{C}$, define

$$K(f) = \{\mathbf{z} \in \mathbb{C} \mid (\exists C > 0)(\forall n)\ |f^n(\mathbf{z})| \leq C\},$$

$$J(f) = \text{boundary of } K(f).$$

- $\exists$ P-time computable $f : \mathbb{C} \rightarrow \mathbb{C}$ such that $J(f)$ encodes the halting problem of the universal Turing machine.
- Membership problem of J_f for a hyperbolic polynomial f is P-time computable [Rettinger, Weihrauch, Braverman, Yampolsky]
- A special group of functions: $f_c(\mathbf{z}) = \mathbf{z}^2 + \mathbf{c}$, $\mathbf{z}, \mathbf{c} \in \mathbb{C}$. For most $\mathbf{c}$ (including all $\mathbf{c}$ outside the Mandelbrot set), f_c is hyperbolic.

Conformal Mappings

Given a Jordan domain S, what is the complexity of the Riemann mapping from S to the unit disk (relative to the complexity of S)?

- Under some restrictions on the boundary of S, the complexity is $\#P$-complete (if S is P-time computable). [Braverman, Yampolsky, Rettinger]

- **Open Question**:
  In the general case, when it is only known that ∂S is P-time computable, is the complexity still $\#P$?

Thank You