Computational Conformal Geometry Lecture Notes

Topology, Differential Geometry, Complex Analysis

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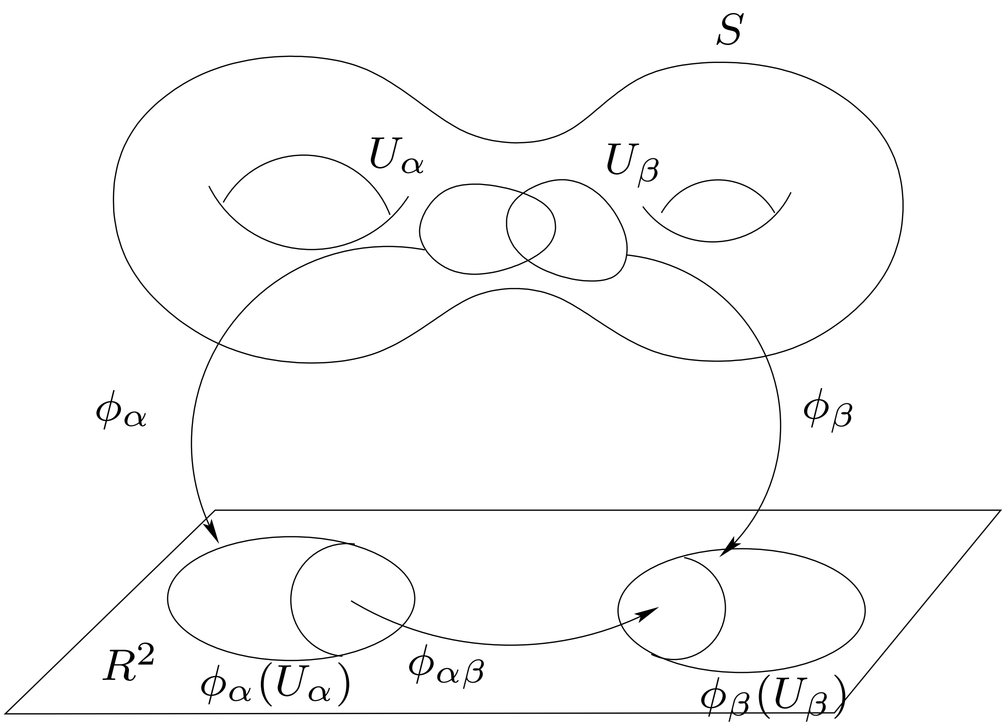
Definition of Manifold

A *manifold* of dimension n is a connected Hausdorfff space M for which every point has a neighbourhood U that is homeomorphic to an open subset V of R^n. Such a hemeomorphism

$$\phi : U \rightarrow V$$

is called a coordinate chart. An *atlas* is a family of charts $\{U_\alpha, \phi_\alpha\}$ for which U_α constitute an open covering of M.



Differential Manifold

- *Transition function*: Suppose $\{U_\alpha, \phi_\alpha\}$ and $\{U_\beta, \phi_\beta\}$ are two charts on a manifold S, $U_\alpha \cap U_\beta \neq$, the chart transition is

$$\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

- *Differentiable Atlas*: An atlas $\{U_\alpha, \phi_\alpha\}$ on a manifold is called differentiable if all charts transitions are differentiable of class C^∞.
- *Differential Structure*: A chart is called *compatible* with a differentiable atlas if adding this chart to the atlas yields again a differentiable atlas. Taking all charts compatible with a given differentiable atlas yieds a differentiable structure.
- *differentiable manifold*: A differentiable manifold of dimension n is a manifold of dimension n together with a differentiable structure.

Differential Map

- *Differential Map*:A continuous map $h : M \to M'$ between differential manifolds M and M' with charts $\{U_\alpha, \phi_\alpha\}$ and $\{U'_\alpha, \phi'_\alpha\}$ is said to be differentiable if all the maps $\phi'_\beta \circ h\phi_\alpha^{-1}$ are differentiable of class C^∞ wherever they are defined.
- *Diffeomorphism*: If h is a homeomorphism and if both h and h^{-1} are differentiable, then h is called a diffeomorphism.

Regular Surface Patch

Suppose $D = \{(u,v)\}$ is a planar domain, a map $\mathbf{r} : D \rightarrow R^3$,

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)),$$

satisfying

1. $x(u,v), y(u,v), z(u,v)$ are differentiable of class C^∞.
2. $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent, namely

$$\mathbf{r}_u = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}), \mathbf{r}_v = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}), \frac{\partial z}{\partial u}), \mathbf{r}_u \times \mathbf{r}_v \neq 0,$$

is a surface patch in R^3, (u,v) are the coordinates parameters of the surface $\mathbf{r}$.

Regular Surface Patch

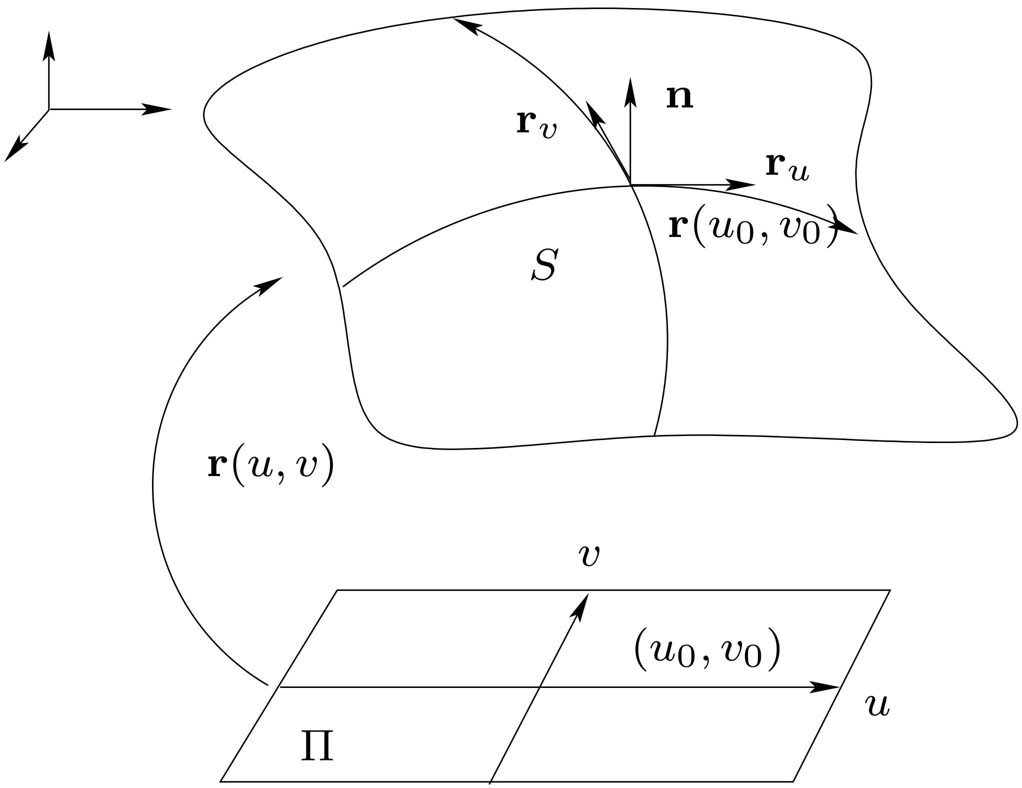


Figure 2: Surface patch.

Different parameterizations

Surface $\mathbf{r}$ can have different parameterizations. Consider a surface

$$\mathbf{r}(u,v) : D \rightarrow R^3,$$

and parametric transformation

$$\sigma : (\bar{u},\bar{v}) \in \bar{D} \rightarrow (u,v) \in D,$$

namely $\sigma : \bar{D} \rightarrow D$ is bijective and the $Jacobin$

$$\frac{\partial(u,v)}{\partial(\bar{u},\bar{v})} = \begin{vmatrix} \frac{\partial u(\bar{u},\bar{v})}{\partial \bar{u}} & \frac{\partial v(\bar{u},\bar{v})}{\partial \bar{u}} \\ \frac{\partial u(\bar{u},\bar{v})}{\partial \bar{v}} & \frac{\partial v(\bar{u},\bar{v})}{\partial \bar{v}} \end{vmatrix} \neq 0.$$

then we have new parametric representation of the surface $\mathbf{r}$,

$$\mathbf{r}(\bar{u},\bar{v}) = \mathbf{r} \circ \sigma(\bar{u},\bar{v}) = \mathbf{r}(u(\bar{u},\bar{v}), v(\bar{u},\bar{v})) : \bar{D} \rightarrow R^3.$$

First fundamental form

Given a surface S in R^3, $\mathbf{r} = \mathbf{r}(u, v)$ is its parametric representation, denote

$$E =< \mathbf{r}_u, \mathbf{r}_u >, F =< \mathbf{r}_u, \mathbf{r}_v >, G =< \mathbf{r}_v, \mathbf{r}_v >,$$

the quadratic differential form

$$I = ds^2 = Edu \cdot du + 2Fdu \cdot dv + Gdv \cdot dv,$$

is called the *first fundamental form* of S.

Invariant property of the first fundamental form

- The first fundamental form of a surface S is invariant under parametric transformation,

$$Edu^2 + 2Fdudv + Gdv^2 = \bar{E}d\bar{u}^2 + 2\bar{F}d\bar{u}d\bar{v} + \bar{G}d\bar{v}^2.$$

- The first fundamental form of a surface S is invariant under the rigid motion of S.

Second fundamental form

Suppose a surface S has parametric representation $\mathbf{r} = \mathbf{r}(u, v)$, $\mathbf{r}_u, \mathbf{r}_v$ are coordinate tangent vectors of S, then the unit normal vector of S is

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|},$$

the second fundamental form of S is defined as

$$II = - < d\mathbf{r}, d\mathbf{n} > .$$

Define functions

$$\begin{aligned}
L &= < \mathbf{r}_{uu}, \mathbf{n} > = - < \mathbf{r}_u, \mathbf{n}_u > && (1)\\
M &= < \mathbf{r}_{uv}, \mathbf{n} > = - < \mathbf{r}_u, \mathbf{n}_v > = - < \mathbf{r}_v, \mathbf{n}_u > && (2)\\
N &= < \mathbf{r}_{vv}, \mathbf{n} > = - < \mathbf{r}_v, \mathbf{n}_u > && (3)
\end{aligned}$$

then the second fundamental form is represented as

$$II = Ldu^2 + 2Mdudv + 2Ndv^2.$$

normal curvature

Suppose $\mathbf{w} = \epsilon \mathbf{r}_u + \eta \mathbf{r}_v$ is a tangent vector at point $S = \mathbf{r}(u, v)$, a plane Π through normal $\mathbf{n}$ and $\mathbf{w}$, the planar curve $\Gamma = S \cap \Pi$ has curvature k_n at point $\mathbf{r}(u, v)$, which is called the normal curvature of S along the tangent vector $\mathbf{w}$.

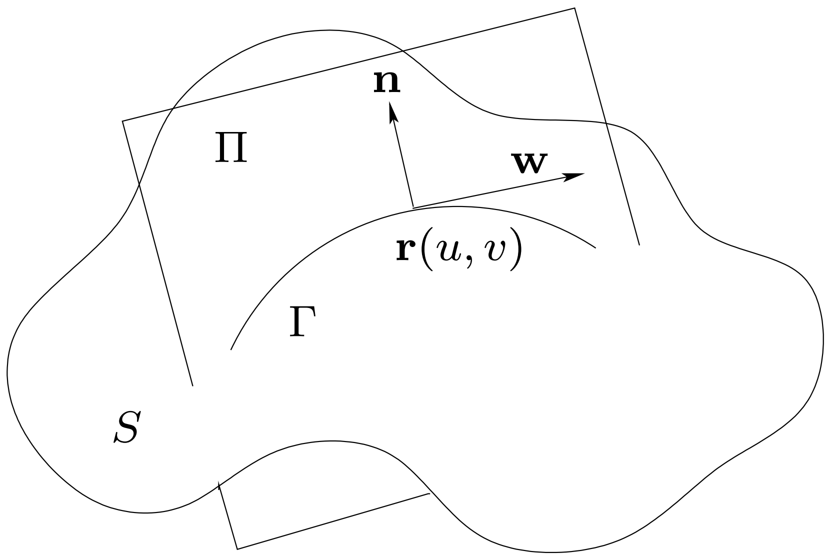


Figure 3: normal curvature.

Normal curvature

Suppose a surface S, a tangent vector $\mathbf{w} = \epsilon \mathbf{r}_u + \eta \mathbf{r}_v$, the normal curvature along $\mathbf{w}$ is

$$k_n(\mathbf{w}) = \frac{II(\mathbf{w},\mathbf{w})}{I(\mathbf{w},\mathbf{w})} = \frac{L\epsilon^2 + 2M\epsilon\eta + N\eta^2}{E\epsilon^2 + 2F\epsilon\eta + G\eta^2}$$

On convex surface patch, the normal curvature along any directions are positive. On saddle surface patch, the normal curvatures may be positive and negative, or zero.

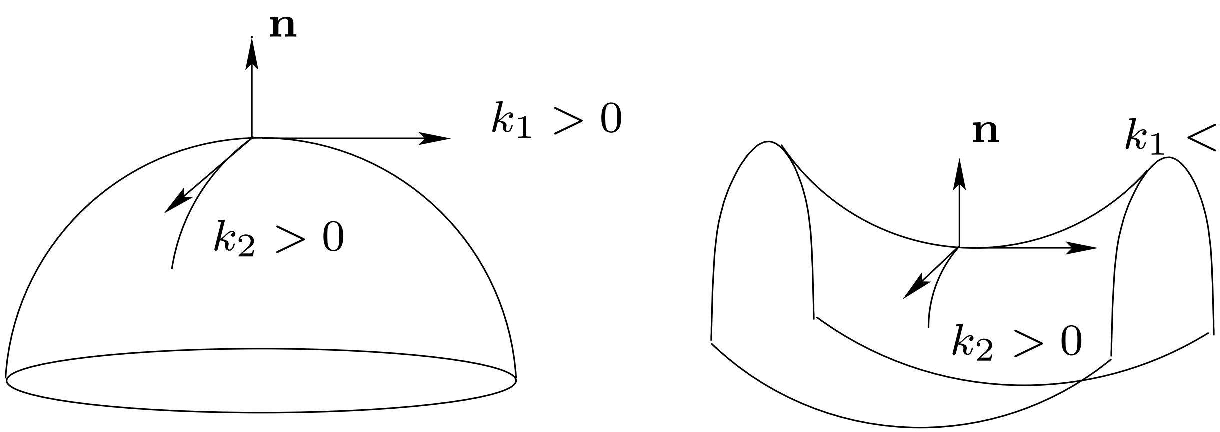


Figure 4: Convex surface patch and saddle surface patch.

Gauss Map

Suppose S is a surface with parametric representation $\mathbf{r}(u, v)$, the normal vector at point (u, v) is $\mathbf{n}(u, v)$, the mapping

$$\mathbf{g} : S \to S^2, \mathbf{r}(u, v) \to \mathbf{n}(u, v),$$

is called the *Gauss map* of S.

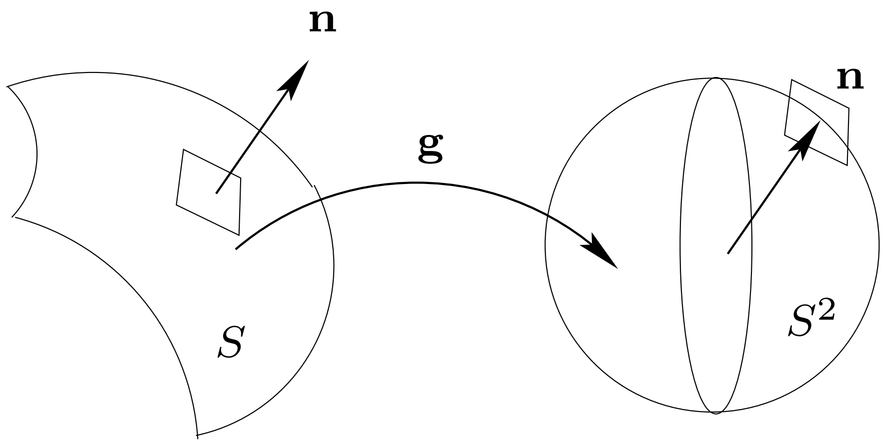


Figure 5: Gauss map.

Weingarten Transform

The differential map $\mathbf{W}$ of Gauss map $\mathbf{g}$ is called the *Weingarten transform*. $\mathbf{W}$ is a linear map from the tangent space of S to the tangent space of S^2,

$$\mathbf{W}: T_pS \to T_pS^2$$

$$\mathbf{v} = \lambda \mathbf{r}_u + \mu \mathbf{r}_v \to \mathbf{W}(\mathbf{v}) = -(\lambda \mathbf{n}_u + \mu \mathbf{n}_v).$$

The properties of Weingarten transform

- Weingarten transform is independent of the choice of the parameters.
- Suppose $\mathbf{v}$ is a unit tangent vector of S, the normal curvature

$$k_n(\mathbf{v}) = < \mathbf{W}(\mathbf{v}), \mathbf{v} > .$$

- Weingarten transform is a self-conjugate transform from the tangent plane to itself.

$$< \mathbf{W}(\mathbf{v}), \mathbf{w} > = < \mathbf{v}, \mathbf{W}(\mathbf{w}) > .$$

Principle Curvature

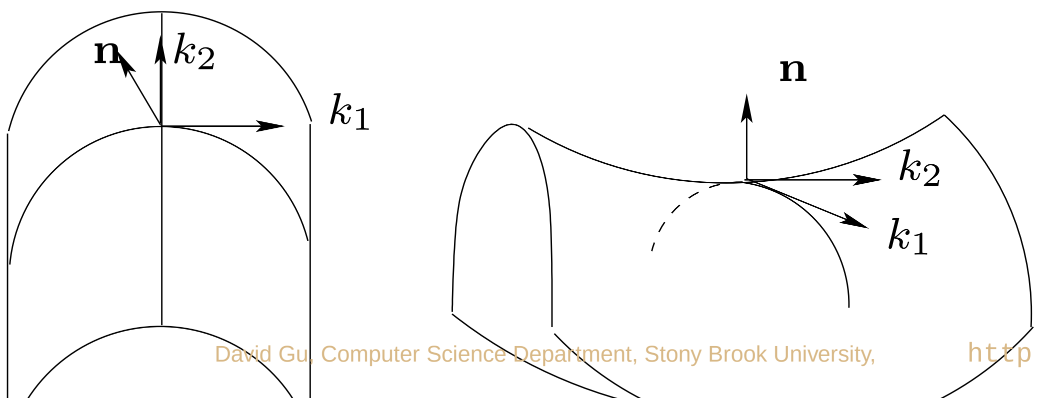
The eigen values of Weingarten transformation are called *principle curvatures*. The eigen directions are called principle directions, namely

$$\mathbf{W}(\mathbf{e}_1) = k_1\mathbf{e}_1, \mathbf{W}(\mathbf{e}_2) = k_2\mathbf{e}_2,$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are unit vectors. Because Weingarten map is self conjugate, it is symmetric. Therefore, the principle directions are orthogonal.
Suppose an arbitrary unit tangent vector $\mathbf{v} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, then the normal curvature along $\mathbf{v}$ is

$$k_n(\mathbf{v}) =< \mathbf{W}(\mathbf{v}), v >= \cos^2\theta k_1 + \sin^2\theta k_2,$$

therefore, normal curvature reaches its maximum and minimum at the principle curvatures.



Weingarten Transformation

Weingarten transformation coefficients matrix is

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} LG - MF & ME - LF \\ MG - NF & NE - MF \end{pmatrix}$$

Principle curvatures satisfy the quadratic equation

$$k^2 - \frac{LG - 2MF + NE}{EG - F^2} k + \frac{LN - M^2}{EG - F^2} = 0.$$

Locally, a surface can be approximated by a quadratic surface

$$\begin{cases} x = u, \\ y = v, \\ z = \frac{1}{2}(k_1 u^2 + k_2 v^2). \end{cases}$$

Mean curvature and Gaussian curvature

The *mean curvature* is defined as

$$H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2},$$

the *Gaussian curvature* is defined as

$$K = \frac{LN - M^2}{EG - F^2}$$

Mean curvature is related the area variation of the surface.
Suppose D is a region on S including point P, $\mathbf{g}(D)$ is the image of D under Gauss map $\mathbf{g}$. The Gaussian curvature is the limit of the area ratio between D and $\mathbf{g}(D)$,

$$K(p) = \lim_{D \to p} \frac{Area(\mathbf{g}(D))}{Area(D)}.$$

Gauss Equation

The first fundamental form E, F, G and the second fundamental form are not independent, they satisfy the following Gauss equation

$$-\frac{1}{\sqrt{EG}}\{(\frac{(\sqrt{E})_v}{\sqrt{G}})_v + (\frac{(\sqrt{G})_u}{\sqrt{E}})_u\}$$

Codazzi equations are

$$\left\{\begin{array}{l} (\frac{L}{\sqrt{E}})_v - (\frac{M}{\sqrt{E}})_u - N\frac{(\sqrt{E})_v}{G} - M\frac{(\sqrt{G})_u}{\sqrt{EG}} = 0 \\ (\frac{N}{\sqrt{G}})_u - (\frac{M}{\sqrt{G}})_v - L\frac{(\sqrt{G})_u}{E} - M\frac{(\sqrt{E})_v}{\sqrt{EG}} = 0 \end{array}\right.$$

Discrete interpretation.

Fundamental theorem in differential geometry

Suppose D is a planar domain, given functions $E(u,v), F(u,v), G(u,v)$ and $L(u,v), M(u,v), N(u,v)$ satisfying the Gauss equation and Codazzi equations, then for any $(u,v) \in D$, there exists a neighborhood $U \subset D$, and a surface $\mathbf{R}(u,v) : U \longrightarrow R^3$, such that E, F, G and L, M, N are the first and the second fundamental forms of $\mathbf{r}$.

Fundamental Theorem

Suppose $D = \{(u^1, u^2\}$ is a planar region, $\phi = g_{\alpha\beta} du^\alpha du^\beta$ and $\psi = b_{\alpha\beta} du^\alpha du^\beta$ are differential forms defined on D, $(g_{\alpha\beta})$ and $(b_{\alpha\beta})$ are symmetric, $(g_{\alpha\beta}$ is positive definite. Denote $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$, $b_\alpha^\beta = g^{\beta\gamma} b_{\gamma\alpha}$, construct Christoffel symbols

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\eta} \{ \frac{\partial g_{\alpha\eta}}{\partial u^\beta} + \frac{\partial g_{\beta\eta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\eta} \}$$

Then consider the first order partial differential equation with $\mathbf{r}, \mathbf{r}_1, \mathbf{r}_2, \mathbf{n}$ as unknowns,

$$\begin{cases} \frac{\partial \mathbf{r}}{\partial u^\alpha} = \mathbf{r}_\alpha, \\ \frac{\partial \mathbf{r}_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^\gamma \mathbf{r}_\gamma + b_{\alpha\beta} \mathbf{n}, \\ \frac{\partial \mathbf{n}}{\partial u^\beta} = -b_\beta^\gamma \mathbf{r}_\gamma, \alpha, \beta = 1, 2. \end{cases}$$

This partial differential equation is to solve the motion equations of the natural frame of the surface. The sufficient and necessary conditions for the equation group to be solvable (equivalent to Gauss Codazzi equations)are

$$\begin{aligned} \frac{\partial}{\partial u^\beta}\left(\frac{\partial \mathbf{r}}{\partial u^\alpha}\right) &= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial \mathbf{r}}{\partial u^\beta}\right), \\ \frac{\partial}{\partial u^\gamma}\left(\frac{\partial \mathbf{r}_\alpha}{\partial u^\beta}\right) &= \frac{\partial}{\partial u^\beta}\left(\frac{\partial \mathbf{r}_\alpha}{\partial u^\gamma}\right), \\ \frac{\partial}{\partial u^\beta}\left(\frac{\partial \mathbf{n}}{\partial u^\alpha}\right) &= \frac{\partial}{\partial u^\alpha}\left(\frac{\partial \mathbf{n}}{\partial u^\beta}\right) \end{aligned}$$

Isometry

- Isometry: Suppose S and $\tilde{S}$ are two surfaces in R^3, σ is a bijection from S to $\tilde{S}$. An arbitrary curve C on S is mapped to curve $\tilde{C}$ on $\tilde{S}$, $\tilde{C} = \sigma(C)$. If C and $\tilde{C}$ have the same length, then σ is an *isometry*.
- Suppose the parametric representations of S and $\tilde{S}$ are $\mathbf{r} = \mathbf{r}(u, v), (u, v) \in D$ and $\mathbf{r} = \tilde{\mathbf{r}}(\tilde{u}, \tilde{v}), (\tilde{u}, \tilde{v}) \in \tilde{D}$, their first fundamental forms are $I(u, v) = Edu^2 + 2Fdudv + Gdv^2$ and $\tilde{I}(\tilde{u}, \tilde{v}) = \tilde{E}d\tilde{u}^2 + 2\tilde{F}d\tilde{u}d\tilde{v} + \tilde{G}d\tilde{v}^2$. Suppose the parametric representation of the isometry σ is

$$\begin{cases} \tilde{u} = \tilde{u}(u, v) \\ \tilde{u} = \tilde{u}(u, v) \end{cases}$$

then

$$ds^2(u, v) = d\tilde{s}^2(\tilde{u}, \tilde{v}).$$

namely,

$$\begin{matrix} E & F \\ F & G \end{matrix} = \mathbf{J}_\sigma \begin{matrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{matrix} \mathbf{J}_\sigma^T, where\ \mathbf{J}_\sigma = \begin{matrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{matrix}$$

Tangent Map

Suppose $\mathbf{v} = a\mathbf{r}_u + b\mathbf{r}_v \in T_pS$ is a tangent vector at point p on S, take a curve $\gamma(t) = \mathbf{r}(u(t), v(t))$ on S such that

$$\gamma(0) = p, \frac{\gamma}{dt}|_{t=0} = \mathbf{r}_u \frac{du}{dt}(0) + \mathbf{r}_v \frac{dv}{dt}(0) = a\mathbf{r}_u + b\mathbf{r}_v,$$

then $\tilde{\gamma}(t)$ is a curve on $\tilde{S}$, $\tilde{\gamma}(0) = \sigma(p)$, the tangent vector at $t = 0$ is

$$\begin{array}{rcl} \tilde{\mathbf{v}} & = & \frac{d\tilde{\gamma}}{dt}(0) = \tilde{\gamma}_{\tilde{u}} \frac{d\tilde{u}}{dt}(0) + \tilde{\gamma}_{\tilde{v}} \frac{d\tilde{v}}{dt}(0) \\ & = & \tilde{\gamma}_{\tilde{u}}(a\frac{\partial \tilde{u}}{\partial u} + b\frac{\partial \tilde{u}}{\partial v})|_{t=0} + \tilde{\gamma}_{\tilde{v}}(a\frac{\partial \tilde{v}}{\partial u} + b\frac{\partial \tilde{v}}{\partial v})|_{t=0} \end{array}.$$

Tangent vector $\mathbf{tildev}$ only depends on σ and $\mathbf{v}$, and is independent of the choice of curve γ.

This induces a map between the tangent spaces on S and $\tilde{S}$,

$$\begin{array}{rcl} \sigma_* : T_pS & \rightarrow & T_{\sigma(p)}\tilde{S} \\ \mathbf{v} & \rightarrow & \tilde{\mathbf{v}} = \sigma_*(\mathbf{v}) \end{array}$$

σ_* is called the tangent map of σ.

Tangent Map

Under natural frame, the tangent map is represented as

$$\begin{matrix} \sigma_*(\mathbf{r}_u) \\ \sigma_*(\mathbf{r}_v) \end{matrix} = \begin{matrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{matrix} \quad \begin{matrix} \tilde{\mathbf{r}}_{\tilde{u}} \\ \tilde{\mathbf{r}}_{\tilde{v}} \end{matrix} = \mathbf{J}_\sigma \quad \begin{matrix} \tilde{\mathbf{r}}_{\tilde{u}} \\ \tilde{\mathbf{r}}_{\tilde{v}} \end{matrix}$$

A bijection σ between surfaces S and $\tilde{S}$ is an isometry if and only if for any two tangent vectors $\mathbf{v}, \mathbf{w}$,

$$< \sigma_*(\mathbf{v}), \sigma_*(\mathbf{w}) >=< \mathbf{v}, \mathbf{w} > .$$

Conformal Map

A bijection $\sigma : S \to \tilde{S}$ is a *conformal map*, if it preserves the angles between arbitrary two intersecting curves.
The sufficient and necessary condition of σ to be conformal is there exists a positive function λ, such that the first fundamental forms of S and $\tilde{S}$ satisfy

$$\tilde{I} = \lambda^2 \cdot I.$$

Isothermal Coordinates

- (S S Chern): For an arbitrary point p on a surface S, there exists a neighborhood U_p, such that it can be conformally mapped to a planar region.
- Under conformal parameterization, the first fundamental form is represented as

$$I = \lambda^2(u,v)(du^2 + dv^2), \lambda > 0,$$

then (u,v) is called the *isothermal coordinates*.

using isothermal coordinates

Isothermal coordinates is useful to simplify computations.

- Gaussian curvature is
$$K = -\frac{1}{\lambda^2}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) \ln \lambda$$
- Mean curvature
$$2H\mathbf{n} = \frac{1}{\lambda^2}(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2})\mathbf{r}$$

Complex Representation

For convenience, we introduce the complex coordinates $z = u + iv$, let

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$$

then

$$K = -\frac{4}{\lambda^2}\frac{\partial^2}{\partial z \partial \bar{z}} \ln \lambda.$$

Laplace Operator

The *Laplace* operator on surface

$$\Delta_S f = \frac{1}{\lambda^2}(f_{uu} + f_{vv}).$$

The Green formula is

$$\int\int_U f\Delta_S g dA + \int\int_U < \nabla f, \nabla g > dA = \int_C f\frac{\partial g}{\partial v},$$

where C is the boundary of U, $\partial U = C$, v is outward normal of U.

λ, H *representation*

Suppose (u,v) is the isothermal coordinates, then

$$\begin{aligned} <\mathbf{r}_z, \mathbf{r}_z> &= \frac{1}{4} <\mathbf{r}_u - i\mathbf{r}_v, \mathbf{r}_u - i\mathbf{r}_v> \\ &= \frac{(}{<}\mathbf{r}_u, \mathbf{r}_u> - <\mathbf{r}_v, <r>_v> -2i <\mathbf{r}_u, \mathbf{r}_v>) \end{aligned}$$

because (u,v) is isothermal,

$$<\mathbf{r}_u, \mathbf{r}_u> = <\mathbf{r}_v, \mathbf{r}_v>, <\mathbf{r}_u, \mathbf{r}_v> = 0$$

we can get

$$<\mathbf{r}_z, \mathbf{r}_z> = 0, <\mathbf{r}_{\bar{z}}, \mathbf{r}_{\bar{z}}> = 0, <\mathbf{r}_z, \mathbf{r}_{\bar{z}}> = \frac{\lambda^2}{2}, <\mathbf{n}, \mathbf{n}> = 1, <\mathbf{r}_z, \mathbf{n}> = <\mathbf{r}_{\bar{z}}, \mathbf{n}> = 0.$$

therefore

$$\mathbf{r}_{z\bar{z}} = \frac{\lambda^2}{2} H\mathbf{n}.$$

Let $Q = <\mathbf{r}_{zz}, \mathbf{n}>$, then Q is a locally defined function on S.

λ, H *representation*

Differentiate above equations, we get

$$< \mathbf{r}_z, \mathbf{r}_{zz} >=< \mathbf{r}_z, \mathbf{r}_{z\bar{z}} >= 0, < \mathbf{r}_{zz}, \mathbf{r}_{\bar{z}} >= \lambda\lambda_z, < \mathbf{n}_z, \mathbf{r}_z >= - < \mathbf{r}_{zz}, \mathbf{n} >= Q, < \mathbf{n}_z, \mathbf{r}_{\bar{z}} =<$$

Therefore, the motion equation for the frame $\{\mathbf{r}_z, \mathbf{r}_{\bar{z}}, \mathbf{n}\}$ is

$$\begin{cases} \mathbf{r}_{zz} & = & \frac{2}{\lambda}\lambda_z \mathbf{r}_z + Q\mathbf{n} \\ \mathbf{r}_{z\bar{z}} & = & \frac{\lambda^2}{2} H\mathbf{n} \\ \mathbf{n}_z & = & -H\mathbf{r}_z - 2\lambda^{-2} Q\mathbf{r}_{\bar{z}} \end{cases}$$

From $\mathbf{r}_{z\bar{z}z} = \mathbf{r}_{zz\bar{z}}$, we get the Gauss-Codazzi equation in complex form

$$\begin{array}{rcll} (\ln\lambda)_{z\bar{z}} & = & \frac{|Q|^2}{\lambda^2} - \frac{\lambda^2}{4} H^2 & (Gauss\ equation) \\ Q_{\bar{z}} & = & \frac{\lambda^2}{2} H_z & (Codazzi\ equation) \end{array}$$

λ, H *representation*

Given a planar domain $D \subset R^2$, (u, v) are parameters, and 2 functions $\lambda(u, v)$ and $H(u, v)$ satisfying Gauss-Codazzi equations, with appropriate boundary condition, then there exists a unique surface S, such that, (u, v) is its isothermal parameter, $H(u, v)$ is its mean curvature function, and the surface first fundamental form is

$$ds^2 = \lambda(u, v)^2 (du^2 + dv^2).$$

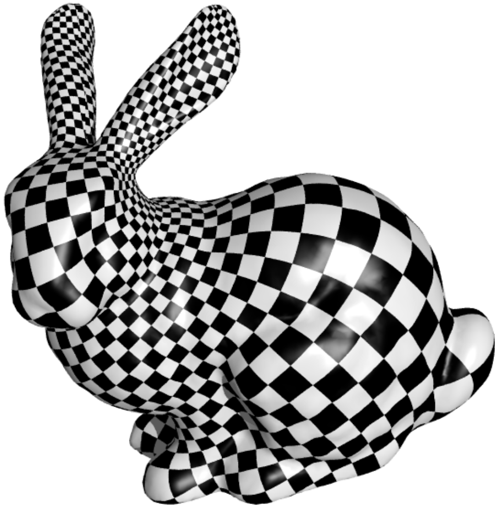
From Codazzi equation, Q can be reconstructed, then the motion equation of the natural frame $\{\mathbf{r}_z, \mathbf{r}_{\bar{z}}, \mathbf{n}\}$ can be solved out.
The quadratic differential form

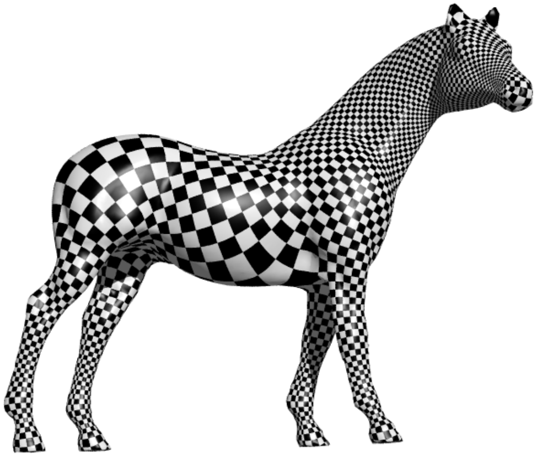
$$\Psi = - < \mathbf{r}_z, \mathbf{n}_z > dz^2 = Q dz^2$$

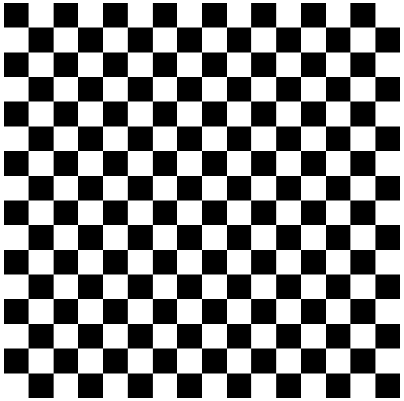
is called the *Hopf* differential. It has the following speical properties

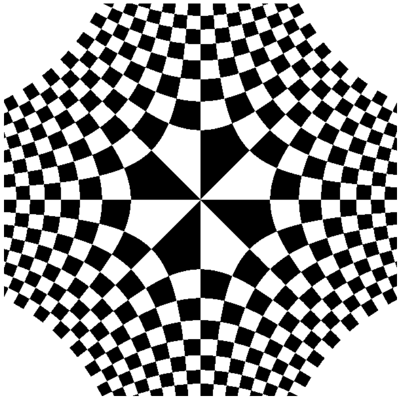
- If all points on a surface S are umbilical points, then Hopf differential is zero.
- Surface S has constant mean curvature if and only if Hopf differential is holomorphic quadratic differentials.

Isothermal Coordinates









Fundamental Group

Two continuous maps $f_1, f_2 : S \to M$ between manifolds S and M are homotopic, if there exists a continuous map

$$F : S \times [0,1] \to M$$

with

$$\begin{aligned} F|_{S\times 0} &= f_1, \\ F|_{S\times 1} &= f_2. \end{aligned}$$

we write $f_1 \sim f_2$.

fundamental group

Let $\gamma_i : [0,1] \to S, i = 1,2$ be curves with

$$\begin{aligned}\gamma_1(0) &= \gamma_2(0) = p_0\\ \gamma_1(1) &= \gamma_2(1) = p_1\end{aligned}$$

we say γ_1 and γ_2 are homotopic, if there exists a continuous map

$$G : [0,1] \times [0,1] \to S,$$

such that

$$\begin{aligned}G|_{\{0\}\times[0,1]} &= p_0 \quad & G|_{\{1\}\times[0,1]} &= p_1,\\ G|_{[0,1]\times\{0\}} &= \gamma_1 \quad & G|_{[0,1]\times\{1\}} &= \gamma_2.\end{aligned}$$

we write $\gamma_1 \sim \gamma_2$.

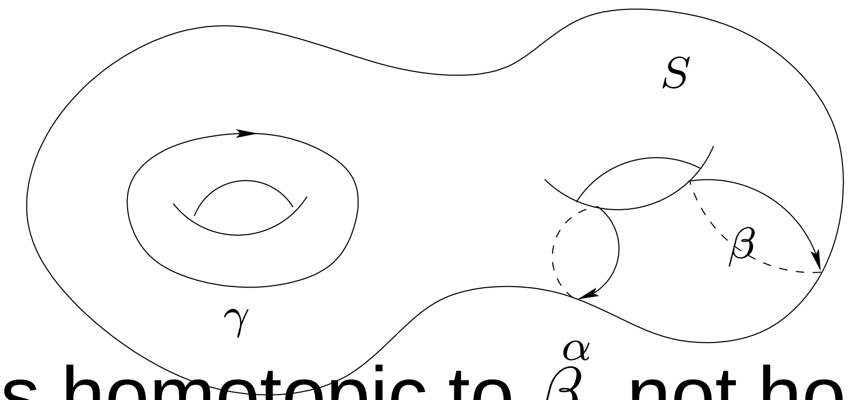


Figure 8: α is homotopic to β, not homotopic to γ.

fundamental group

Let $\gamma_1, \gamma_2 : [0,1] \to M$ be curves with

$$\gamma_1(1) = \gamma_2(0),$$

the product of $\gamma_1\gamma_2 := \gamma$ is defined by

$$\gamma(t) := \begin{cases} \gamma_1(2t) & for\ t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & for\ t \in [\frac{1}{2}, 1]. \end{cases}$$

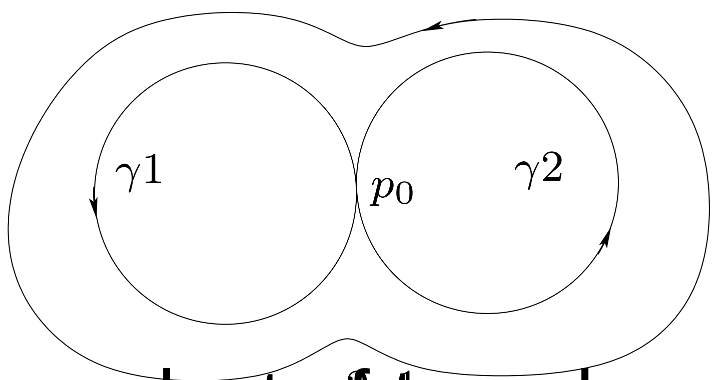


Figure 9: product of two closed curves.

Fundamental Group

For any $p_0 \in M$, the fundamental group $\pi_1(M, \_0)$ is the group of homotopy classes of paths $\gamma : [0,1] \to M$ with $\gamma(0) = \gamma(1) = p_0$, i.e. closed paths with p_0 as initial and terminal point.

$\pi_1(M, p_0)$ is a group with respect to the operation of multiplication of homotopy classes.

The identity element is the class of the constant path $\gamma_0 \equiv p_0$.

For any $p_0, p_1 \in M$, the groups $\pi_1(M, p_0)$ and $\pi_1(M, p_1)$ are isomorphic.

If $f : M \to N$ be a continuous map, and $q_0 := f(p_0)$, then f induces a homomorphism

$$f_* : \pi_1(M, p_0) \to \pi_1(N, q_0)$$

of fundamental groups.

Need more contents for topology

Canonical Fundamental Group Basis

For genus g closed surface, there exist canonical basis for $\pi_1(M, p_0)$, we write the basis as $\{a_1, b_1, a_2, b_2, \cdots, a_g, b_g\}$, such that

$$a_i \cdot a_j = 0, a_i \cdot b_j = \delta_i^j, b_i \cdot b_j = 0,$$

where $\cdot$ represents the algebraic intersection number. Especially, through any point $p \in M$, we can find a set of canonical basis for $\pi_1(M)$, the surface can be sliced open along them and form a canonical fundamental polygon

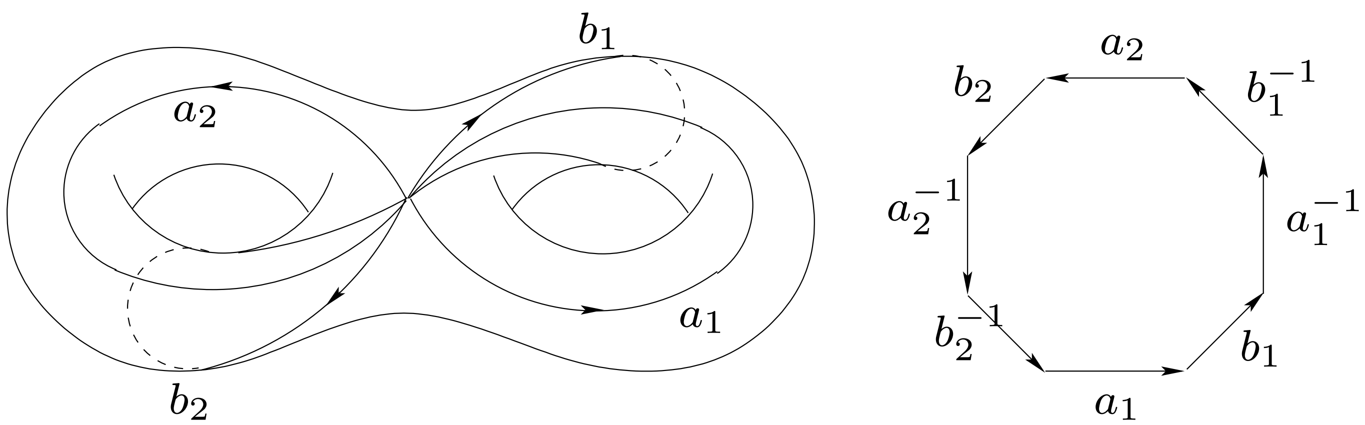


Figure 10: canonical basis of fundamental group $\pi_1(M, p_0)$.

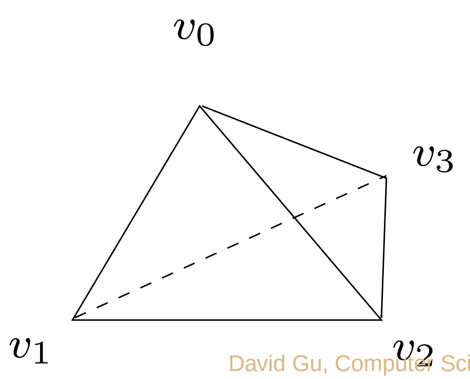
Simplical Complex

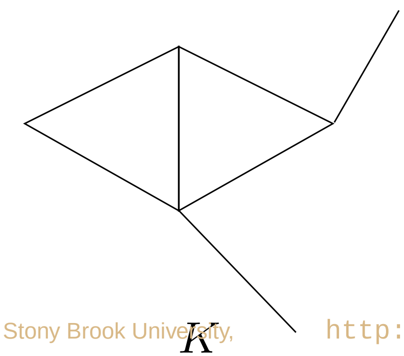
Suppose $k+1$ points in the general positions in $\mathcal{R}^n$, $v_0, v_1, \cdots, v_k$, the *standard simplex* $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{x \in \mathcal{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} \lambda_i = 1, \lambda_i \geq 0\},$$

we call $v_0, v_1, \cdots, v_k$ as the *vertices* of the simplex σ.
Suppose $\tau \subset \sigma$ is also a simplex, then we say τ is a *facet* of σ.
A *simplicial complex* K is a union of simplices, such that

1. If a simplex σ belongs to K, then all its facets also belongs to K.
2. If $\sigma_1, \sigma_2 \subset K$, $\sigma_1 \cap \sigma_2 \neq$, then the intersection of σ_1 and σ_2 is also a common facet.





Simplicial Homology

Associate a sequence of groups with a finite simplicial complex.
A k chain is a linear combination of all k simplicies in K,

$$\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathcal{Z}.$$

The n dimensional *chain space* is a linear space formed by all the n chains, we denote k dimensional chain space as $C_n(K)$
The *boundary* operator defined on a simplex is

$$\partial_n [v_0, v_1, \cdots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n],$$

The boundary operator acts on a chain is a linear operator

$$\partial_n : C_n \to C_{n-1}, \partial_n \sum_i \lambda_i \sigma_i = \sum_i \lambda_i \partial_n \sigma_i$$

Simplicial Homology Group

A chain σ is called a *closed* chain, if it has no boundary, namely $\partial\sigma = 0$.
A chain σ is called a *exact* chain, if it is the boundary of some other chain, namely $\sigma = \partial\tau$.
It can be easily shown that all exact chains are closed. Namely

$$\partial_{n-1} \circ \partial_n \equiv 0.$$

The topology of the surface is indicated by the differences between closed chains and the exact chains. For example, on a genus zero surface, all closed chains are boundaries (exact). But on a torus, there are some closed curves, which are not the boundaries of any surface patch.

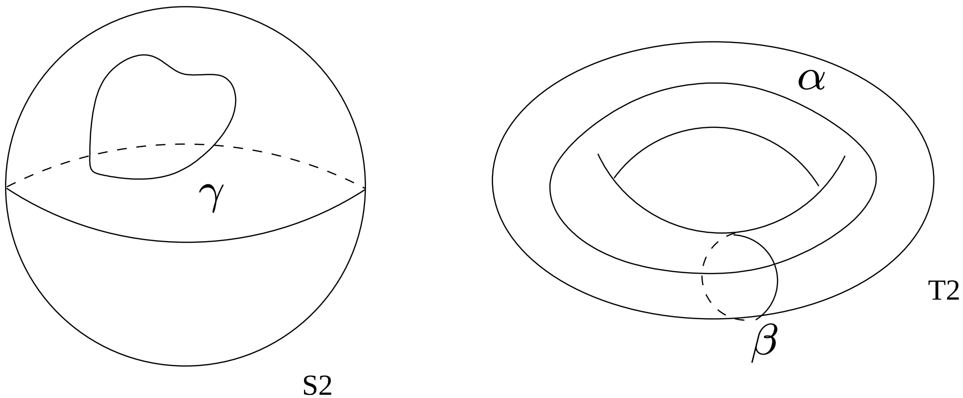


Figure 12: Idea of homology.

Simplicial Complex (Mesh)

Simplicial Complex (Mesh)

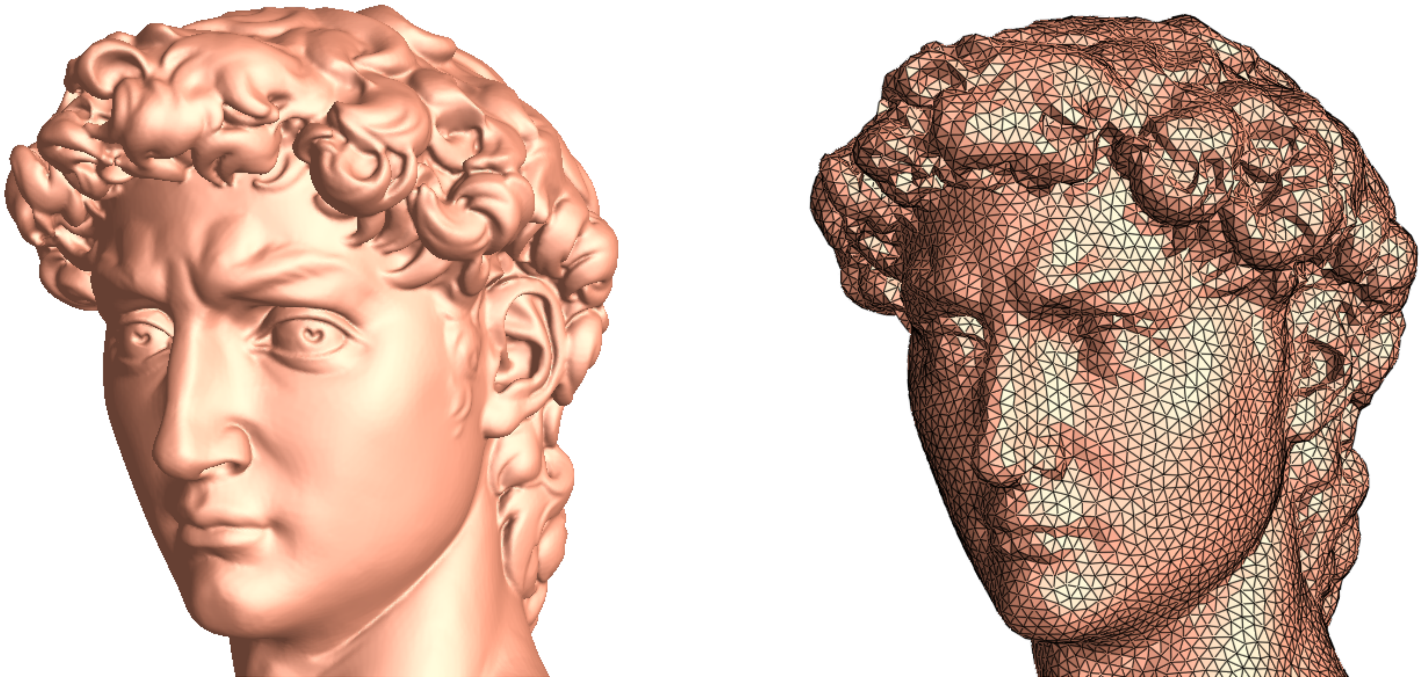


Figure 14: Triangle mesh.

Simplicial Homology

The n-th homology group $H_k(M, Z)$ of a simplical complex K is

$$H_n(K, \mathcal{Z}) = \frac{ker\partial_n}{img\partial_{n+1}}.$$

For example, two closed curves γ_1, γ_2 are homologous if and only if their difference is a boundary of some 2dimensional patch, $\gamma_1 - \gamma_2 = \partial_1 \Sigma, \Sigma \subset S$.

Simplicial Cohomology

A k cochain is a linear function

$$\omega : C_k \rightarrow \mathcal{Z}.$$

The k cochain space $C^k(M, \mathcal{Z})$ is linear space formed by all linear functionals defined on $C_k(M, \mathcal{Z})$. The k-cochain is also called k form.
The coboundary operator $\delta_k : C^k(M, \mathcal{Z}) \rightarrow C^{k+1}(M, \mathcal{Z})$ is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \omega \in C^k(M, \mathcal{Z}).$$

For example, ω is a 1-form, then $\delta_1 \omega$ is a 2-form, such that

$$\begin{aligned} \delta_1 \omega([v_0, v_1, v_2]) &= \omega(\partial_2 [v_0, v_1, v_2]) \\ &= \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0]) \end{aligned}$$

Simplicial Cohomology Group

A k-form ω is called a closed k-form, if $\delta\omega = 0$. If there is a $k-1$-form τ, such that $\delta_{k-1}\tau = \omega$, then ω is exact.
The set of all closed k-forms is the kernal of δ_k, denoted as $ker\delta_k$; the set of all exact k-forms is the image set of δ_{k-1}, denoted as $img\delta_{k-1}$.
The k-th cohomology group $H^k(M,\mathcal{Z})$ is defined as the quotient group

$$H^k(M,\mathcal{Z}) = \frac{ker\delta_k}{img\delta_{k-1}}.$$

Different one-forms

Suppose S is a surface with a differential structure $\{U_\alpha, \phi_\alpha\}$ with (u_α, v_α), then a real different one-form ω has the parametric representation on local chart

$$\omega = f_\alpha(u_\alpha, v_\alpha)du_\alpha + g(u_\alpha, v_\alpha)dv_\alpha,$$

where f_α, g_α are functions with C^∞ continuity.
On different chart $\{U_\beta, \phi_\beta\}$,

$$\omega = f_\beta(u_\beta, v_\beta)du_\beta + g(u_\beta, v_\beta)dv_\beta$$

then

$$(f_\alpha, g_\alpha)\begin{pmatrix} \frac{\partial u_\alpha}{\partial u_\beta} & \frac{\partial u_\alpha}{\partial v_\beta} \\ \frac{\partial v_\alpha}{\partial u_\beta} & \frac{\partial v_\alpha}{\partial v_\beta} \end{pmatrix} = (f_\beta, g_\beta)$$

Exterior Differentiation

A special operator $\wedge$ can be defined on differential forms, such that

$$\begin{aligned} f \wedge \omega &= f\omega \\ \omega \wedge \omega &= 0 \\ \omega_1 \wedge \omega_2 &= -\omega_2 \wedge \omega_1 \end{aligned}$$

The so called exterior differentiation operator d can be defined on differential forms, such that

$$\begin{aligned} df(u,v) &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv \\ d(\omega_1 \wedge \omega_2) &= d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2 \end{aligned}$$

The exterior differential operator d is the generalization of *curlex* and *divergence* on vector fields.
It can be verified that $d \circ d \equiv 0$,e.g,

$$\begin{aligned} d \circ df &= d(\frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv) \\ &= (\frac{\partial^2 f}{\partial v \partial u} - \frac{\partial^2 f}{\partial u \partial v})dv \wedge du \end{aligned}$$

de Rham Cohomology Group

- A *closed* 1-form ω satisfies
$$d\omega \equiv 0.$$
- An *exact* 1-form ω satisfies
$$\omega = df, f : S \rightarrow \mathcal{R}.$$
- All exact 1-forms are closed.
- The first de Rham cohomology group is defined as the quotient group
$$H^1(S,\mathcal{R}) = \frac{closed\ forms}{exact\ forms} = \frac{Ker\ d}{Img\ d}$$
- Two closed 1-forms ω_1 and ω_2 are *cohomologous*, if and only if the difference between them is a gradient of some function f:
$$\omega_1 - \omega_2 = df.$$
- de Rham cohomology groups are isomorphic to simplicial cohomology groups.

Pull back metric

Two surfaces M and N with Riemannian metrics, ds_M^2 and ds_N^2. Suppose (u,v) is a local parameter of M, $(\tilde{u},\tilde{v})$ of N. A map $\phi : M \to N$, represented as

$$(\tilde{u},\tilde{v}) = \phi(u,v),$$

then the metrics on M and N are

$$\begin{array}{rcl} ds_M^2 & = & E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dv^2, \\ ds_N^2 & = & \tilde{E}(\tilde{u},\tilde{v})d\tilde{u}^2 + 2\tilde{F}(\tilde{u},\tilde{v})d\tilde{u}d\tilde{v} + G(\tilde{u},\tilde{v})d\tilde{v}^2 \end{array}$$

The so called pull back metric on M induced by ϕ is denoted as $\phi^* ds_N^2$

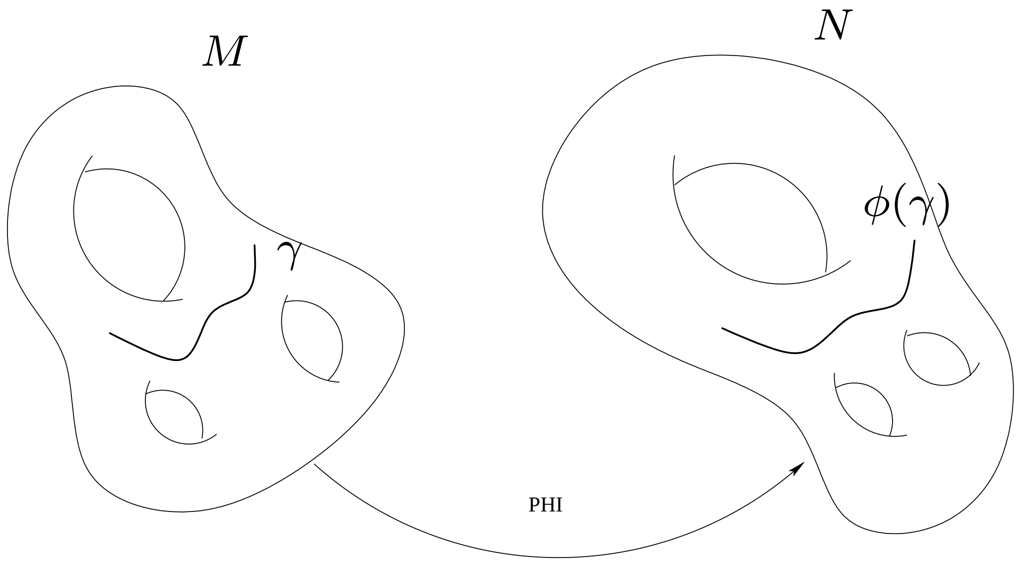
$$\begin{matrix} d\tilde{u} \\ d\tilde{v} \end{matrix} = \phi^* \begin{matrix} du \\ dv \end{matrix} = \begin{matrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{u}}{\partial v} \\ \frac{\partial \tilde{v}}{\partial u} & \frac{\partial \tilde{v}}{\partial v} \end{matrix} \quad \begin{matrix} du \\ dv \end{matrix}$$

pull back metric

Then the parametric representation of pull back metric is

$$\phi^* ds_N^2(u,v) = (du\ dv)(\phi^*)^T \begin{pmatrix} \tilde{E}(\phi(u,v)) & \tilde{F}(\phi(u,v)) \\ \tilde{F}(\phi(u,v)) & \tilde{G}(\phi(u,v)) \end{pmatrix} \phi^* \begin{pmatrix} du \\ dv \end{pmatrix}.$$

Intuitively, a curve segment $\gamma \subset M$ is mapped to a curve segment $\phi(\gamma) \subset N$, the length of γ on M is defined as the length of $\phi(\gamma)$ on N, this metric is the pull back metric.



Conformal Map

Two surfaces M and N with Riemannian metrics, ds_M^2 and ds_N^2. A map $\phi : M \to N$ is *conformal*, if the pull back metric $\phi^* ds_N^2$ satisfies

$$ds_M^2 = \lambda^2 \phi^* ds_N^2,$$

where λ is a positive function $\lambda : M \to \mathcal{R}^+$.

Harmonic map

Suppose a smooth map $\mathbf{f} : M \to N$ is a map, N is embedded in $\mathcal{R}^3$, then $\mathbf{f} = (f_1, f_2, f_3)$, the map is *harmonic*, if it minimizes the following harmonic energy

$$E(\mathbf{f}) = \quad {}_{k} \quad {}_{M} < \nabla f_k, \nabla f_k > dA_M$$

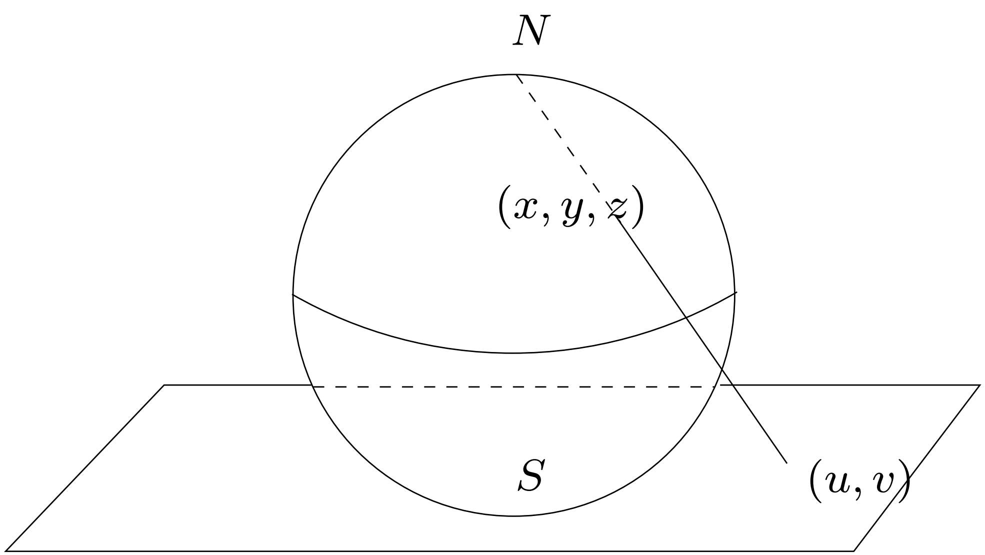
Equivalence between harmonic maps and conformal maps, $g = 0$

A map $\mathbf{f} : M \to N$, where M and N are genus zero closed surfaces, $\mathbf{f}$ is harmonic if and only if $\mathbf{f}$ is conformal.

Stereo graphic projection

The stereo graphic projection $\phi : S^2 \rightarrow \mathcal{R}^2$ is a conformal map

$$
\begin{aligned}
u &= \frac{2}{1-z}x \\
v &= \frac{2}{1-z}y
\end{aligned}
$$



Möbius Transformation Group

All the conformal map from sphere to sphere $\phi : S^2 \to S^2$ form a 6 dimensional *Möbius* group. Suppose S^2 is mapped to the complex plane using stereo-graphic projection. Then each map can be represented as

$$\phi(z) = \frac{az+b}{cz+d}, ad - bc = 1.0,$$

where a, b, c, d and z are complex numbers.
The conformal map from disk to disk form a 3 dimensional Möbius transformation group,

$$\phi(z) = \frac{az+b}{cz+d}, ad - bc = 1.0,$$

where $a, b, c,$ are real numbers.

Conformal map of topological disk

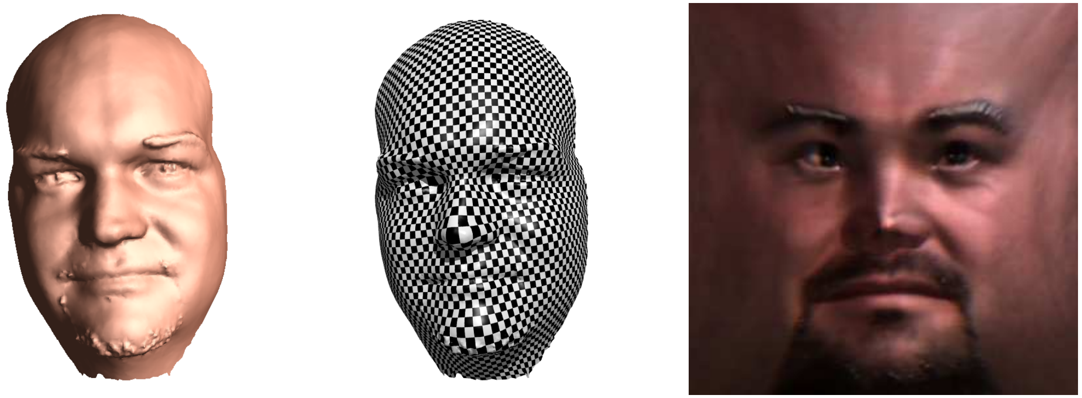


Figure 15: Conformal Map.

Mobius Transformation

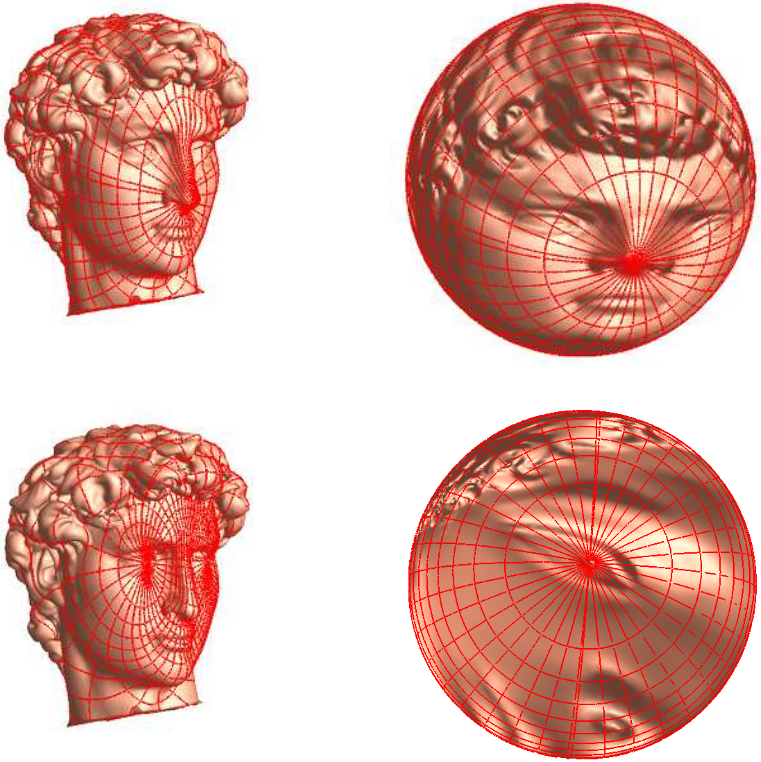


Figure 16: Conformal Map.

Mobius Transformation

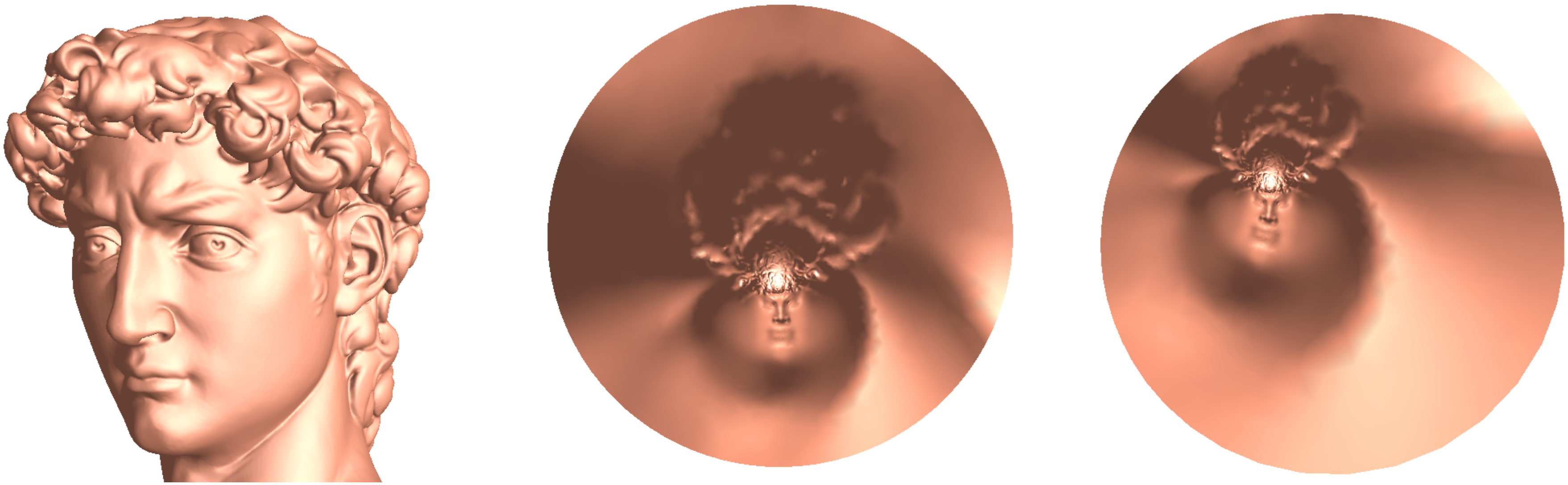


Figure 17: Mobius Transformation.

Analytic function

A function $\phi : \mathcal{C} \to \mathcal{C}$

$$f : (x, y) \to (u, v)$$

is analytic, if it satisfies the Riemann-Cauchy equation

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

holomorphic differentials on the plane

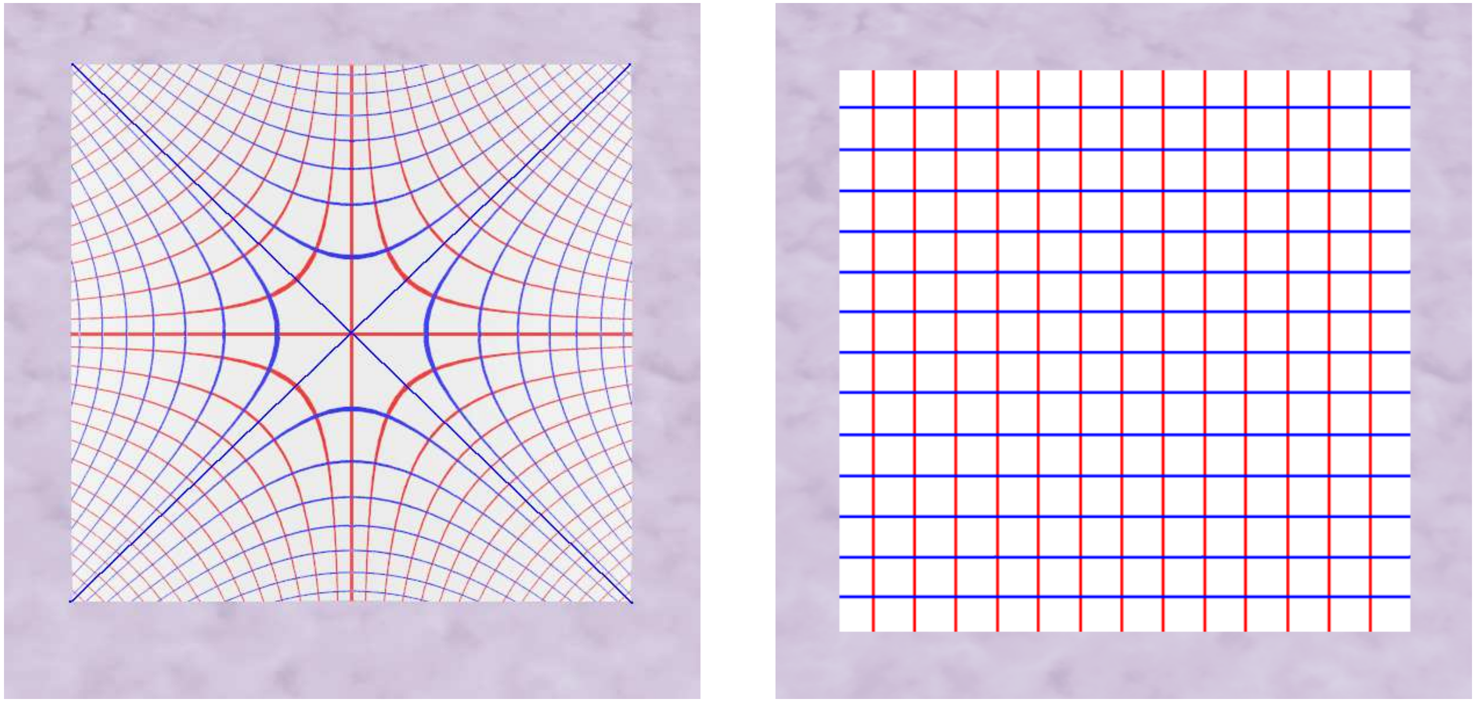


Figure 18: $w = z^2$.

Conformal Atlas

A manifold M with an atlas $\mathcal{A} = \{U_\alpha, \phi_\alpha\}$, if all chart transition functions

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$$

are holomorphic, then $\mathcal{A}$ is a conformal atlas for M.

Conformal Structure

A chart $\{U_\alpha, \phi_\alpha\}$ is *compatible* with an atlas $\mathcal{A}$, if the union $\mathcal{A} \cup \{U_\alpha, \phi_\alpha\}$ is still a conformal atlas.
Two conformal atlas are compatible if their union is still a conformal atlas.
Each conformal compatible equivalent class is a conformal structure.

Riemann surface

A surface S with a conformal structure $\mathcal{A} = \{U_\alpha, \phi_\alpha\}$ is called a Riemann surface. The definition domains of holomorphic functions and differential forms can be generalized from the complex plane to Riemann surfaces directly.

Harmonic Function

A function $f : S \rightarrow \mathcal{R}$ is harmonic, if it minimizes the harmonic energy

$$E(f) = \quad {}_{M} < \nabla f, \nabla f > dA.$$

Harmonic one-form

A differential one-form ω is *harmonic*, if and only if for each point $p \in M$, there is a neighborhood of p, U_p, there is a harmonic function

$$f : U_p \to \mathcal{R},$$

such that

$$\omega = \nabla f$$

on U_p.

Hodge Theorem

There exists a unique harmonic one-form in each cohomology class in $H^1(S, \mathcal{R})$.

Holomorphic one-forms

A holomorphic one-form is a differential form ω, on each chart $\{U_\alpha, \phi_\alpha\}$ with complex coordinates z_α,

$$\omega = f_\alpha(z_\alpha)dz_\alpha,$$

where f_α is a holomorphic function. On a different chart $\{U_\beta, \phi_\beta\}$ with complex coordinates z_β,

$$\begin{aligned} \omega &= f_\beta(z_\beta)dz_\beta \\ &= f_\beta(z_\beta(z_\alpha))\frac{dz_\beta}{dz_\alpha}dz_\alpha. \end{aligned}$$

then $f_\beta\frac{dz_\beta}{dz_\alpha}$ is still a holomorphic function.

Holomorphic differentials on surface

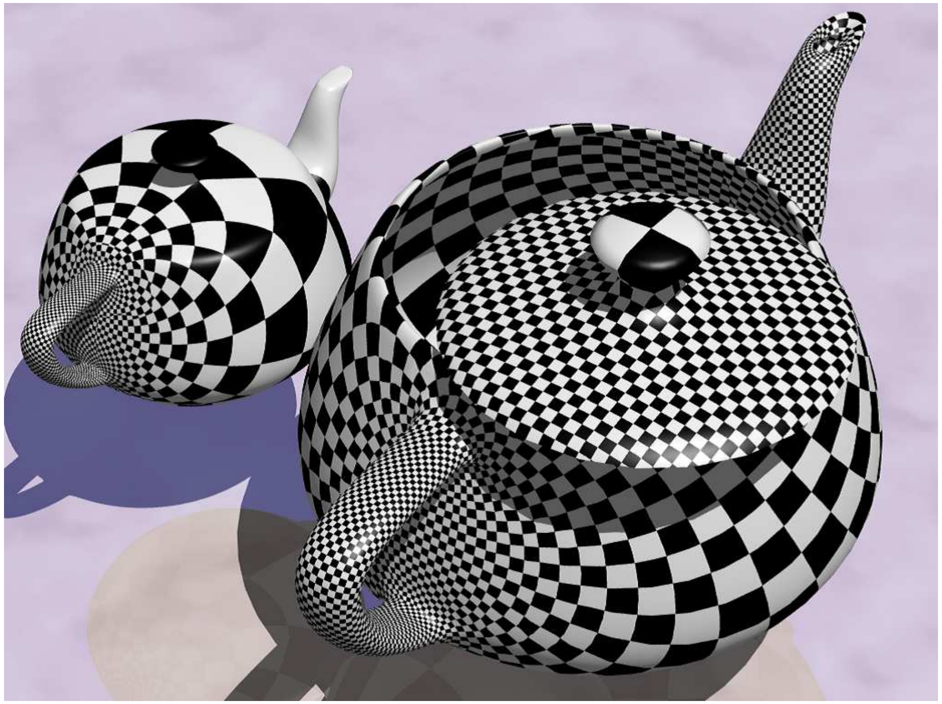


Figure 19: Holomorphic 1-forms on surfaces.

Holomorphic 1-form , Hodge Star Operator

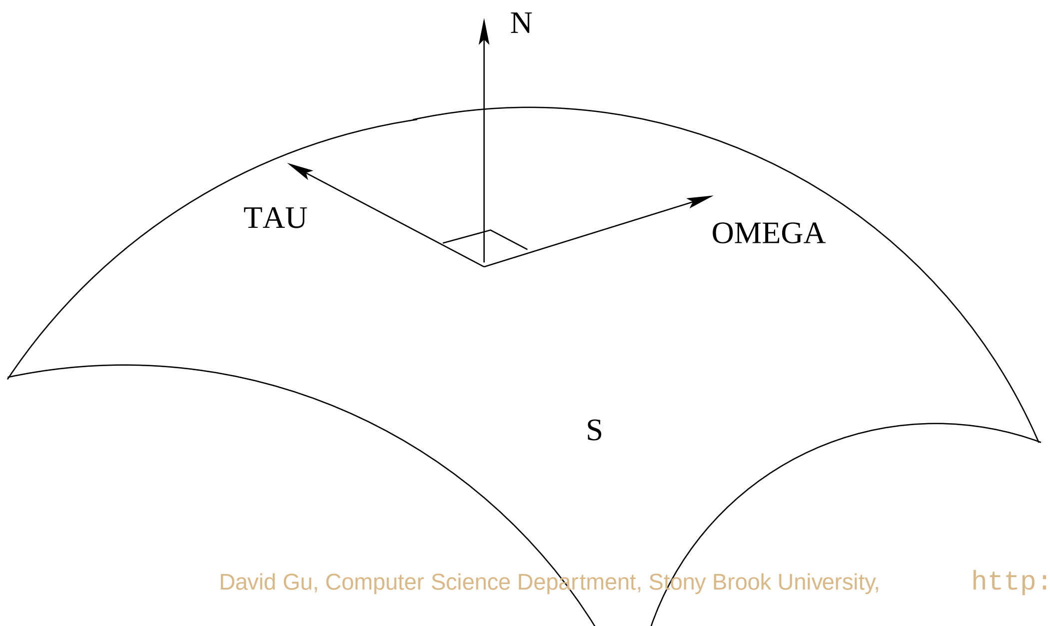
Suppose ω is a holomorphic 1-form, then

$$\omega = \tau + \sqrt{-1}\,{}^*\tau,$$

where τ is a real harmonic 1-form, $\tau = f(u,v)du + g(u,v)dv$, ${}^*\tau$ is a harmonic 1-form conjugate to τ,

$$^*\tau = -g(u,v)du + f(u,v)dv$$

the operate * is called the *Hodge Star Operator*. If we illustrate the operator intuitively as follows:



Zero Points

A holomorphic 1-form ω, on one local coordinates $\omega = f(z_\alpha)dz_\alpha$ on a surface, if at point $p \in S$, $f(p) = 0$, then point p is called a *zero point*.

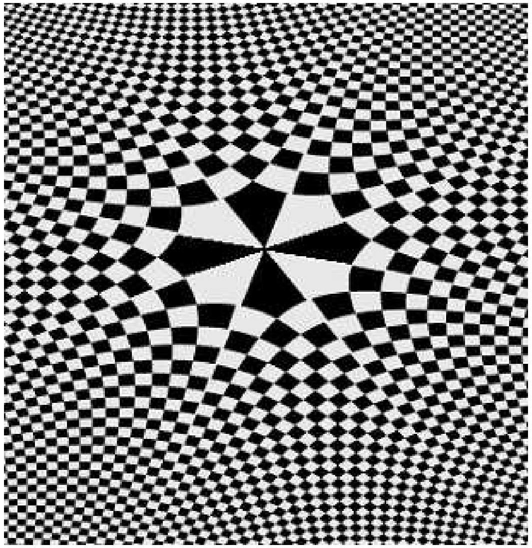
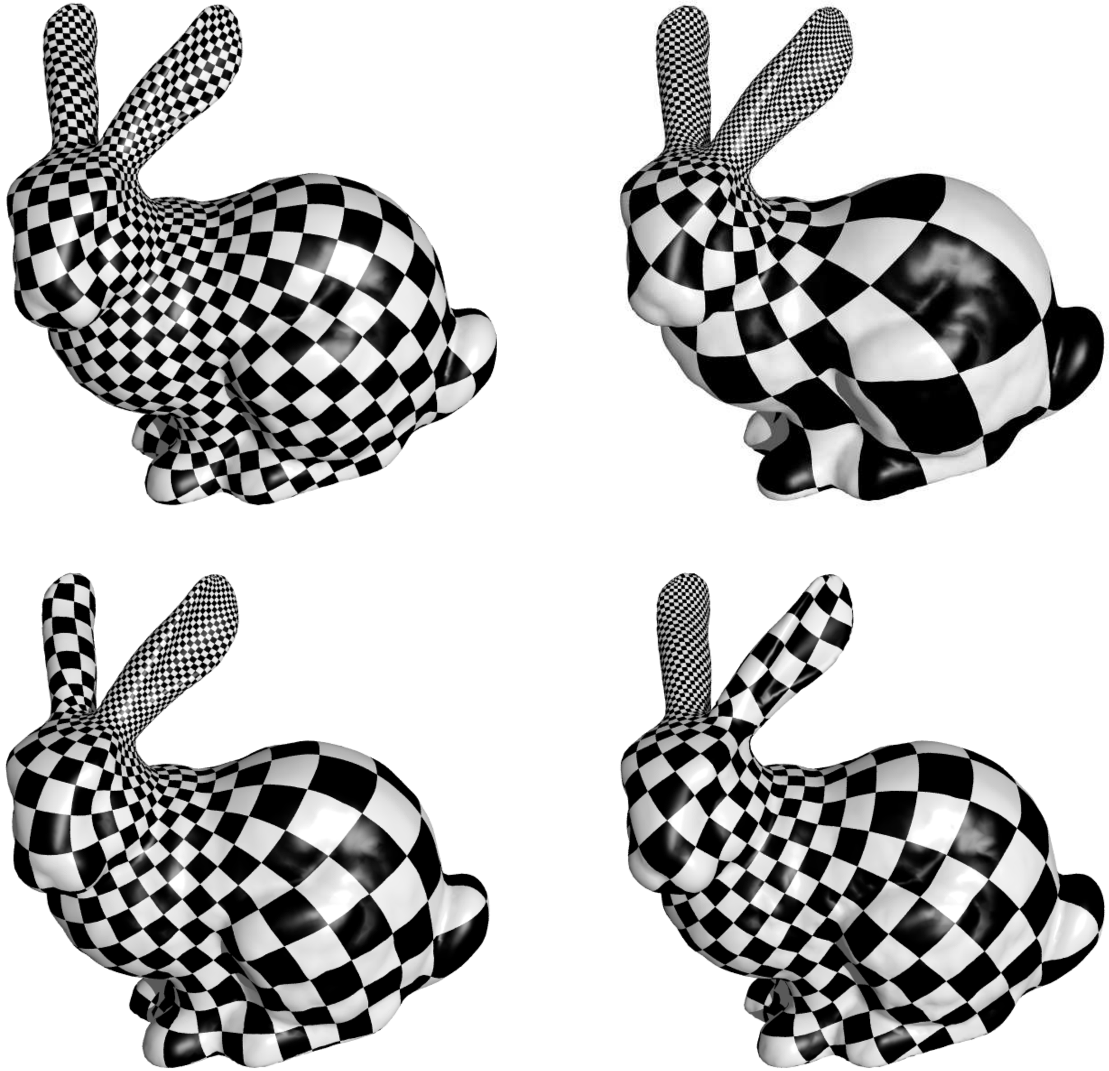


Figure 21: The zero point of a holomorphic 1-form.

The definition of zero point doesn't depend on the choice of the local coordinates.

Holomorphic differentials



Holomorphic quadratic differential forms

A holomorphic quadratic form is a differential form ω, on each chart $\{U_\alpha, \phi_\alpha\}$ with complex coordinates z_α,

$$\omega = f_\alpha(z_\alpha)dz_\alpha^2,$$

where f_α is a holomorphic function. On a different chart $\{U_\beta, \phi_\beta\}$ with complex coordinates z_β,

$$\begin{aligned}\omega &= f_\beta(z_\beta)dz_\beta \\ &= f_\beta(z_\beta(z_\alpha))(\frac{dz_\beta}{dz_\alpha})^2 dz_\alpha^2.\end{aligned}$$

then $f_\beta(\frac{dz_\beta}{dz_\alpha})^2$ is still a holomorphic function.

Holomorphic Trajectories

Suppose ω is a holomorphic 1-form on a Riemann surface S,

- A curve γ is called a *horizontal trajectory*, if along γ, $\omega^2 > 0$.
- A curve γ is called a *vertical trajectory*, if along γ, $\omega^2 < 0$.
- The trajectories through zero points are called *critical trajectories*.

Trajectories

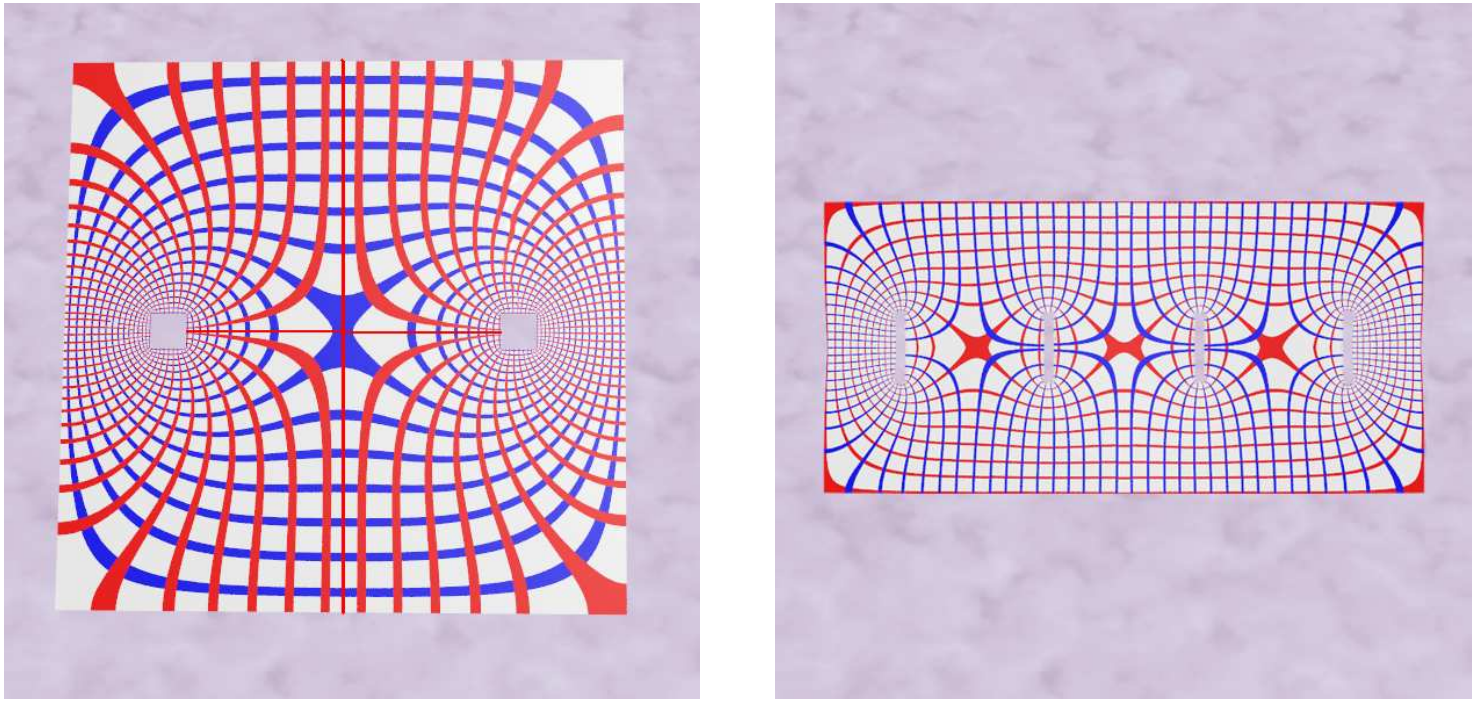


Figure 23: The red curves are the horizontal trajectories, the blue curves are vertical trajectories.

Finite Trajectories

A trajectory is *finite*, if its total length is finite. A finite trajectory is

- either a closed circle.
- finite curve segment connecting zero points.
- finite curve segment intersecting boundaries.
- finite curve segment connecting zero point and a boundary.

Finite Curve System

If all the horizontal of a holomorphic quadratic form ω^2 are finite, then they are called *finite curve system*.

The horizontal trajectories through zero points, and the zero points form the so called *critical graph*.

If the critical graph is finite, then the curve system is finite.

Holomorphic differentials on surface

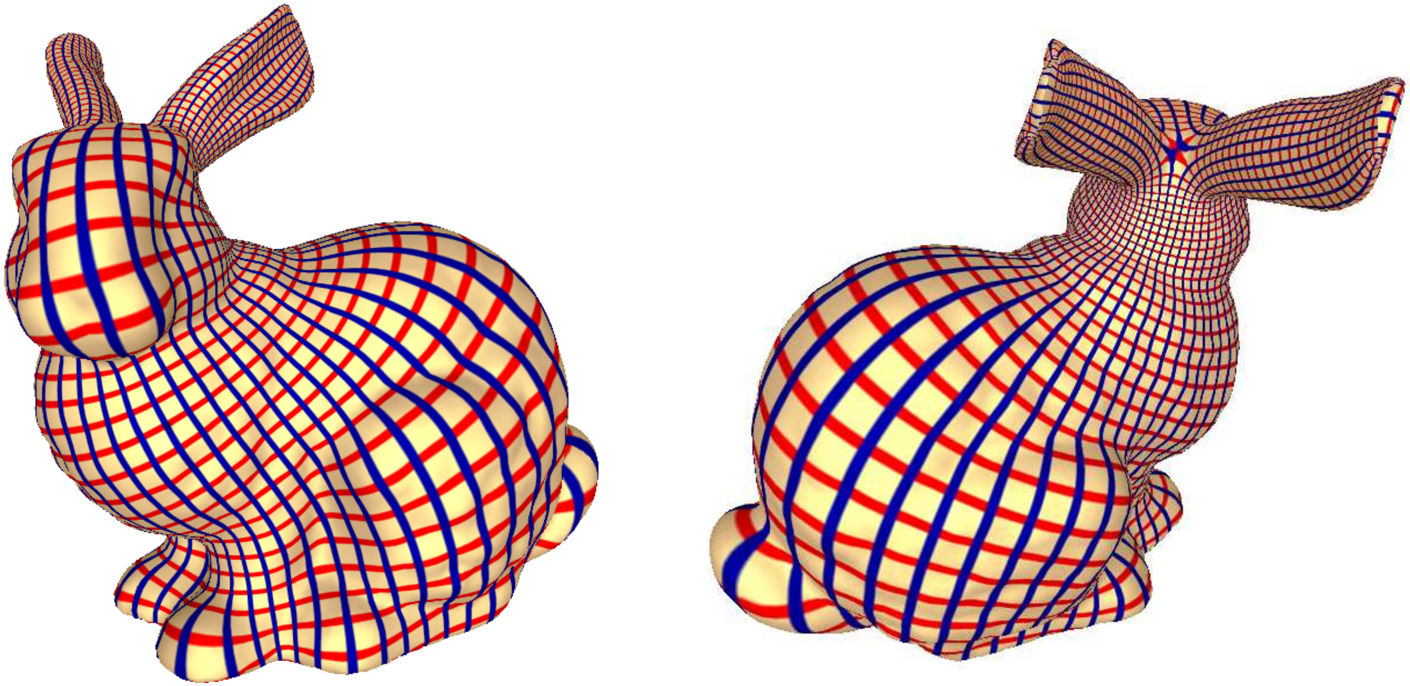


Figure 24: Holomorphic 1-forms on surfaces.

Decomposition Theorem

Suppose a Riemann surface S with a quadratic holomorphic form ϕdz^2, which induces a finite curve system, then the critical horizontal trajectories partition the surface to topological disks and cylinders, each segment can be conformally mapped to a parallelogram by integrating

$$w(p) = \int_{p_0}^{p} \sqrt{\phi} dz.$$

Decomposition

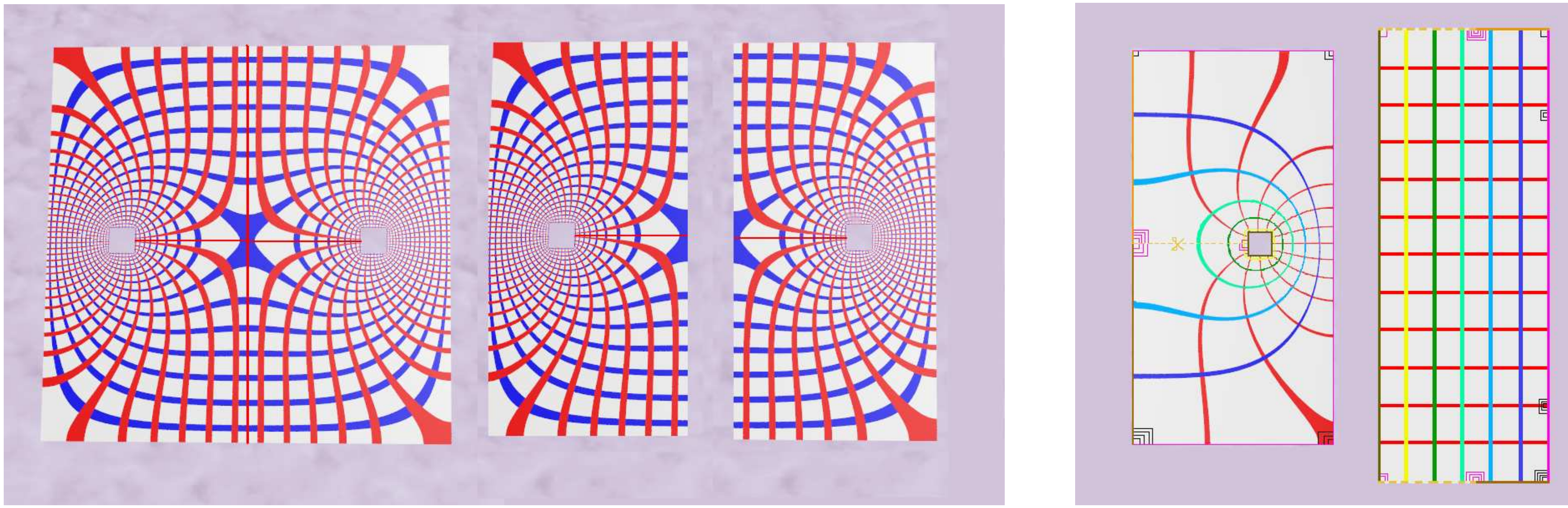


Figure 25: Critical graph of a finite curve system will partition the surface to topological disks, each segment is conformally mapped to a parallelogram.

Decomposition

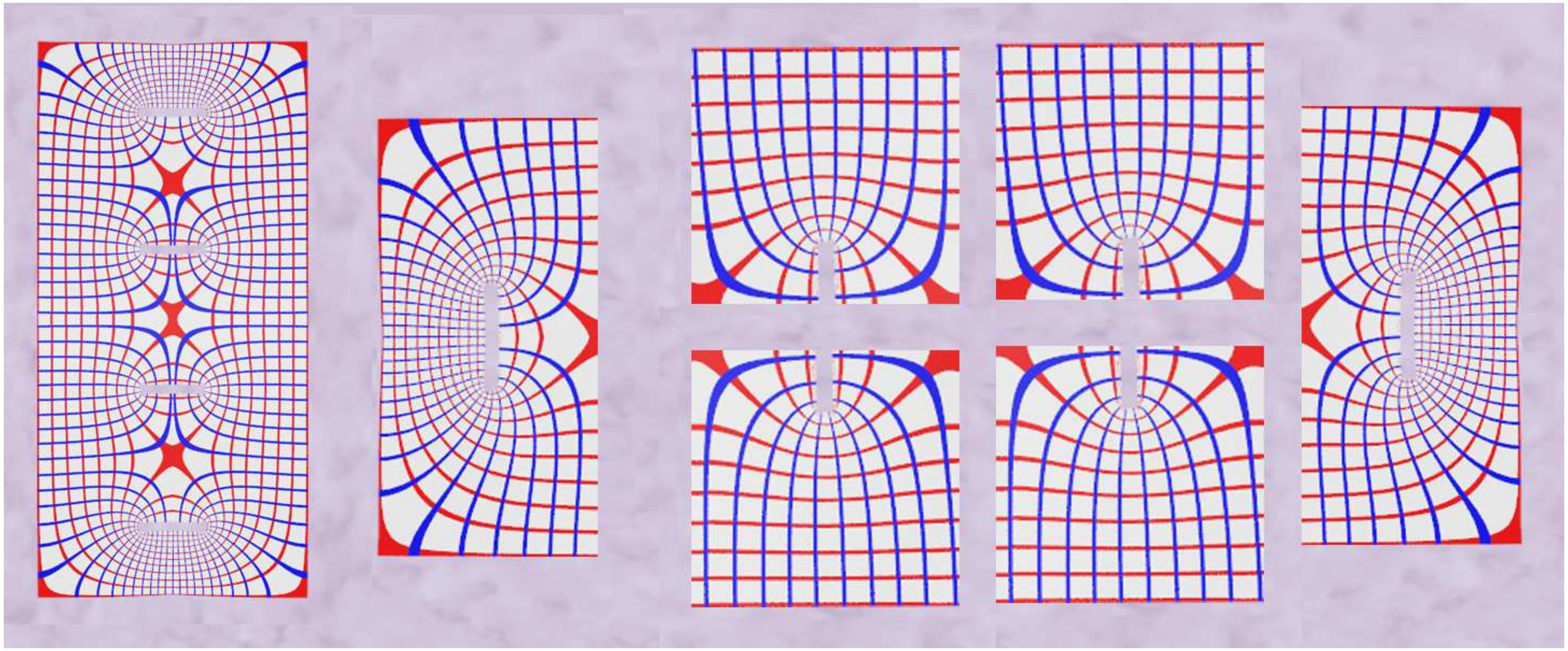


Figure 26: Critical graph of a finite curve system will partition the surface to topological disks, each segment is conformally mapped to a parallelogram.

Decomposition

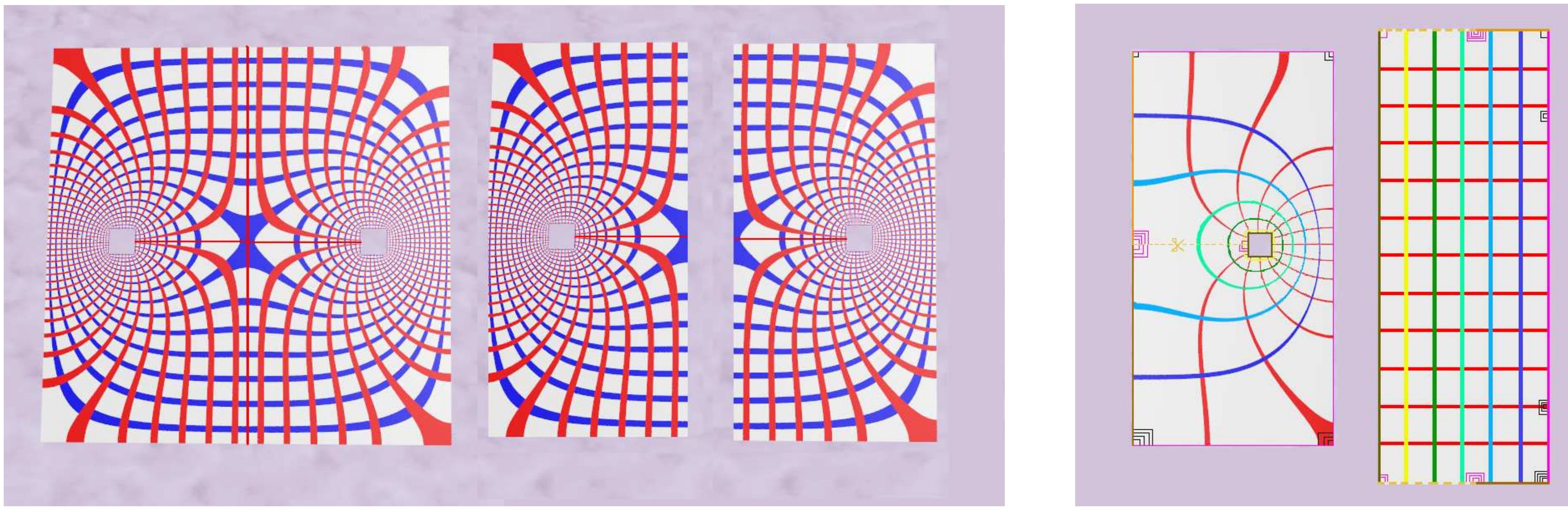
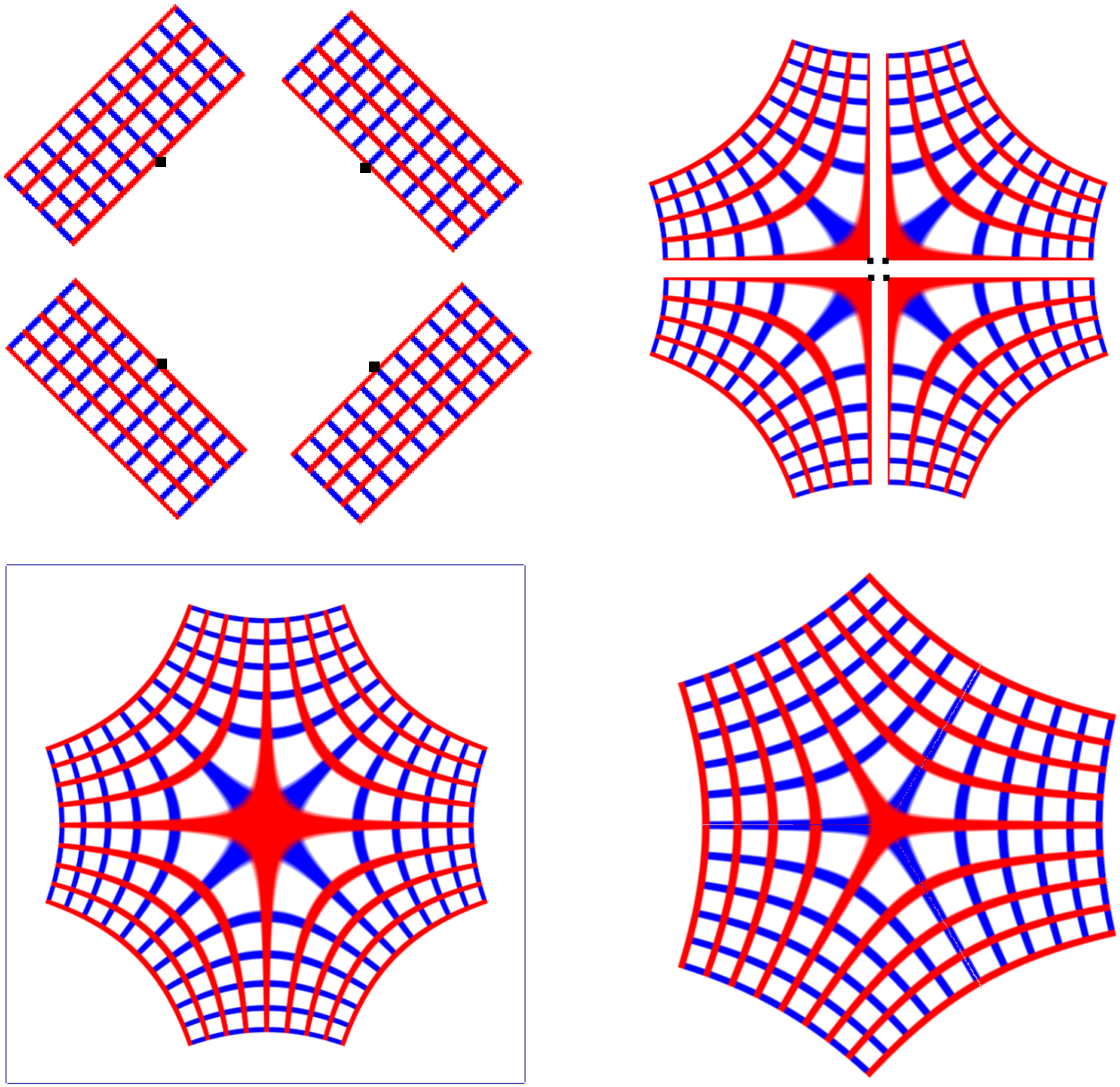


Figure 27: Critical graph of a finite curve system will partition the surface to topological disks, each segment is conformally mapped to a parallelogram.

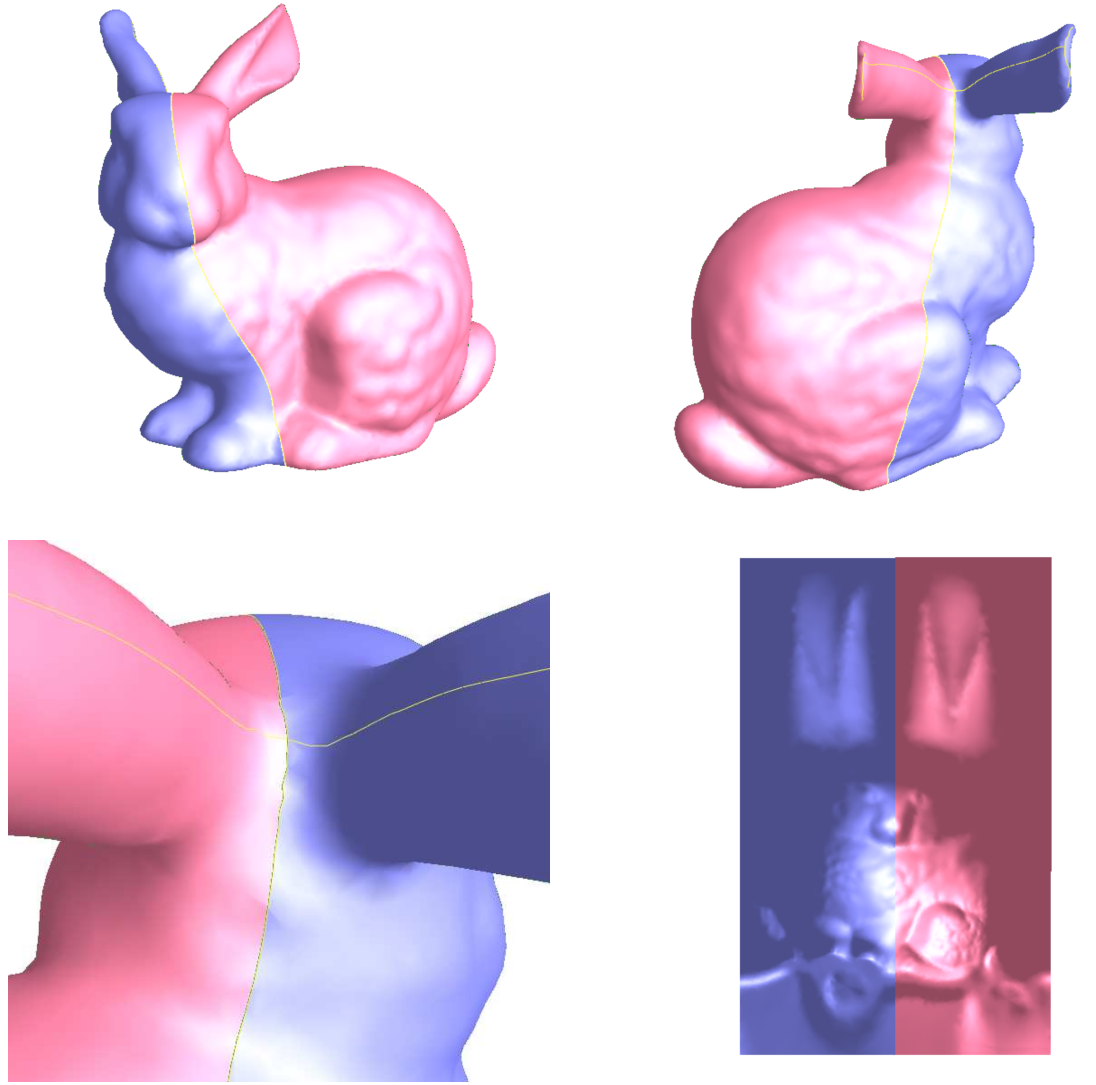
Global structure of finite circle system

Different parallelograms are glued together along their edges, and different patches are met at the zero points. The edges and zero points form the critical points.

Global structure of finite circle system



Global structure of finite circle system



Global structure of finite circle system

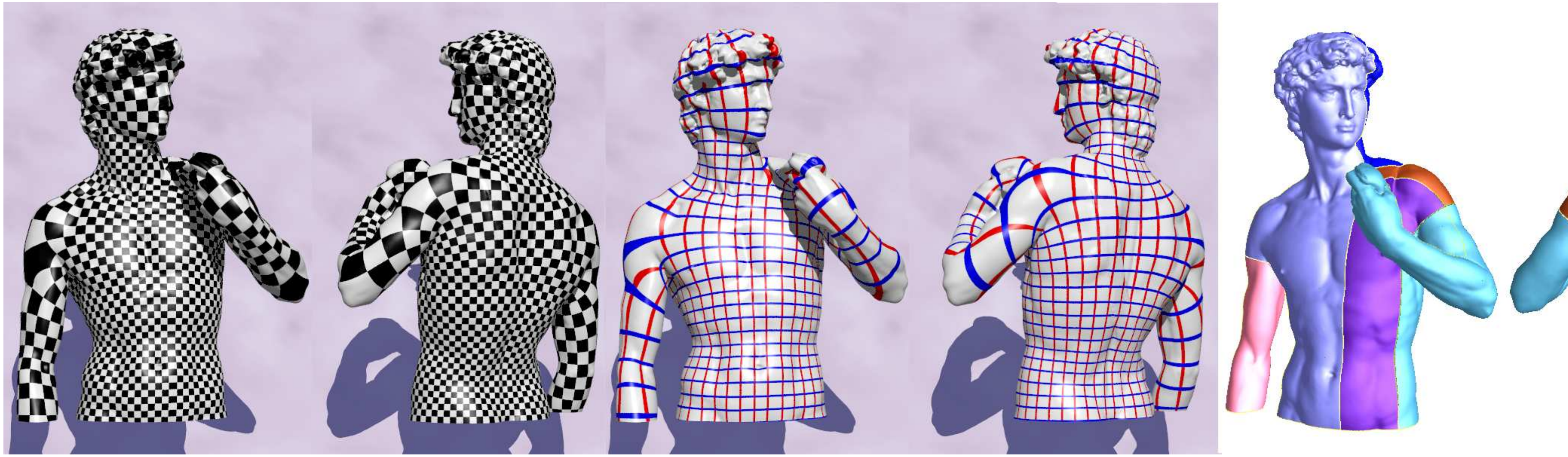


Figure 30: The ciritical graph partition the surface to 6 segments, each segment is conformally parameterized by a rectangle.

Global structure of finite circle system

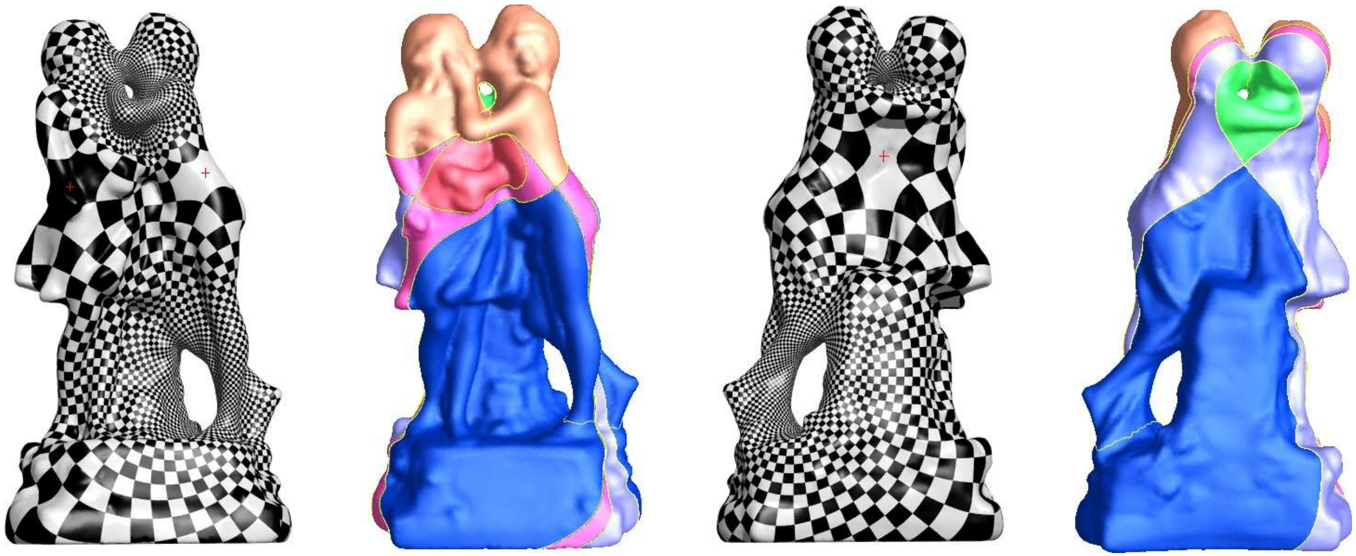


Figure 31: The ciritical graph partition the surface to 6 segments, each segment is a cylinder.

Global structure of finite circle system

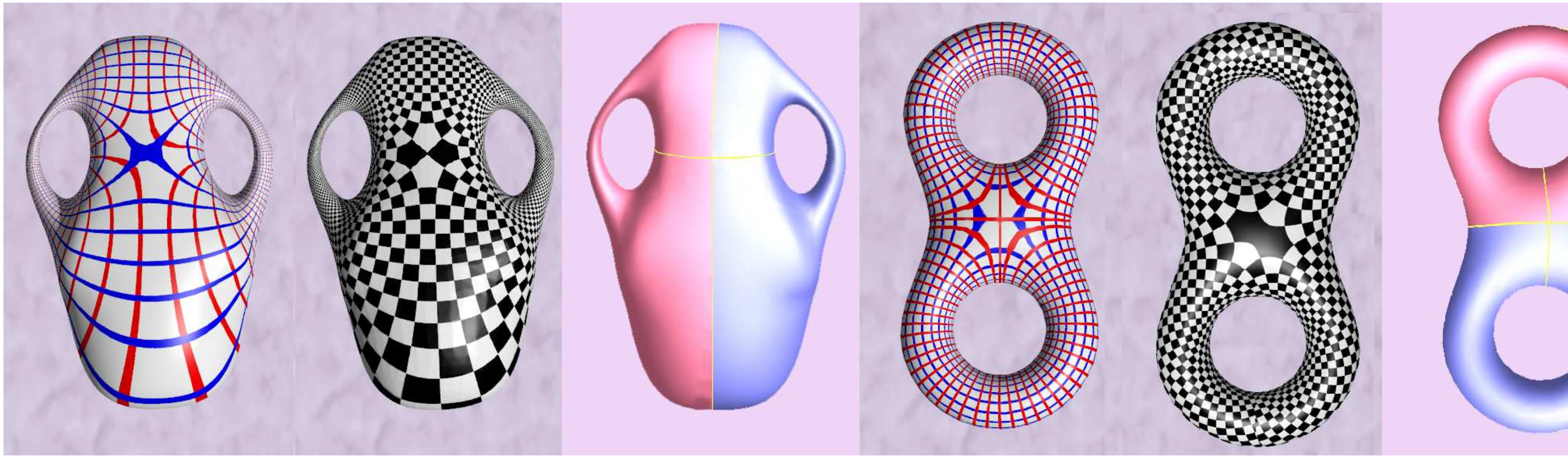


Figure 32: The ciritical graph partition the surface to 2 segments, each segment is a cylinder, and can be conformally mapped to a rectangle.

Applications

Medical Imaging-Conformal Brain Mapping

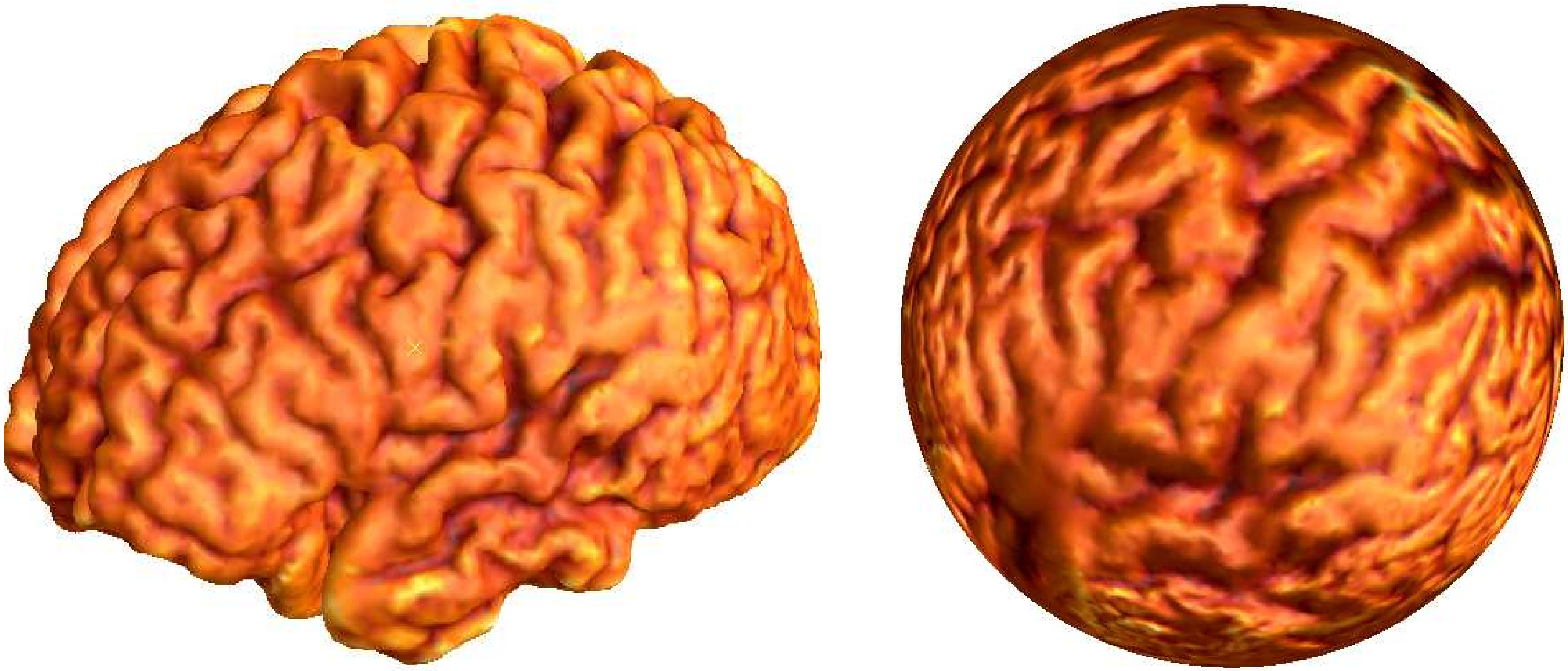


Figure 33: Thanks Yalin wang

Medical Imaging-Colon Flattening

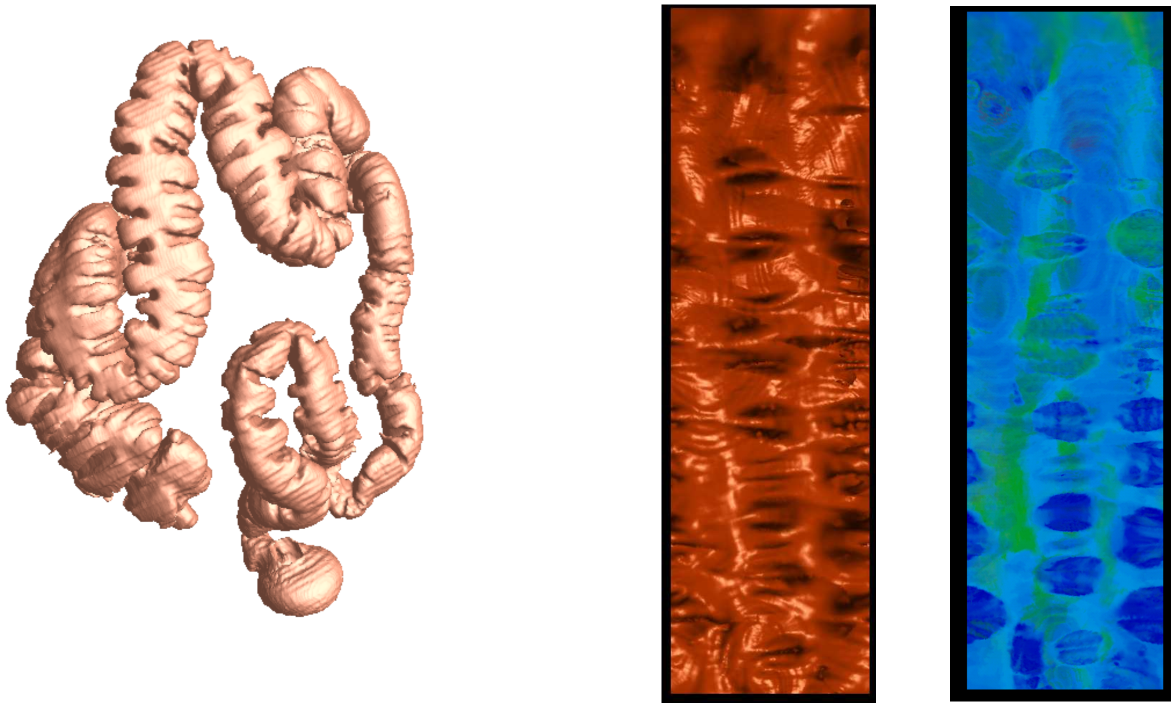


Figure 34: Thanks Wei Hong, Miao Jin

Manifold Spline

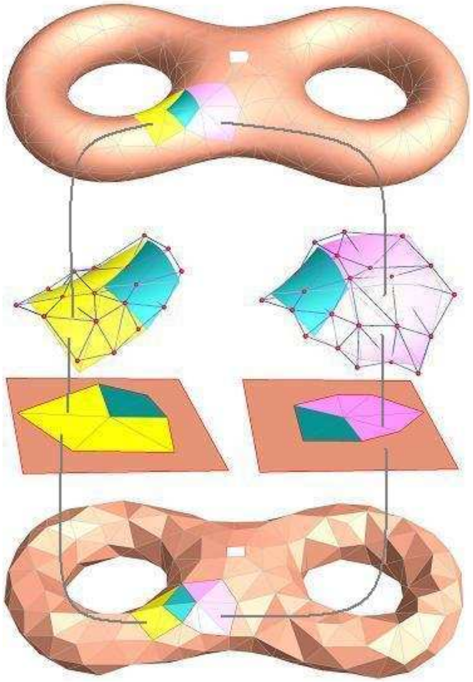


Figure 35: Thanks Ying

Manifold Spline

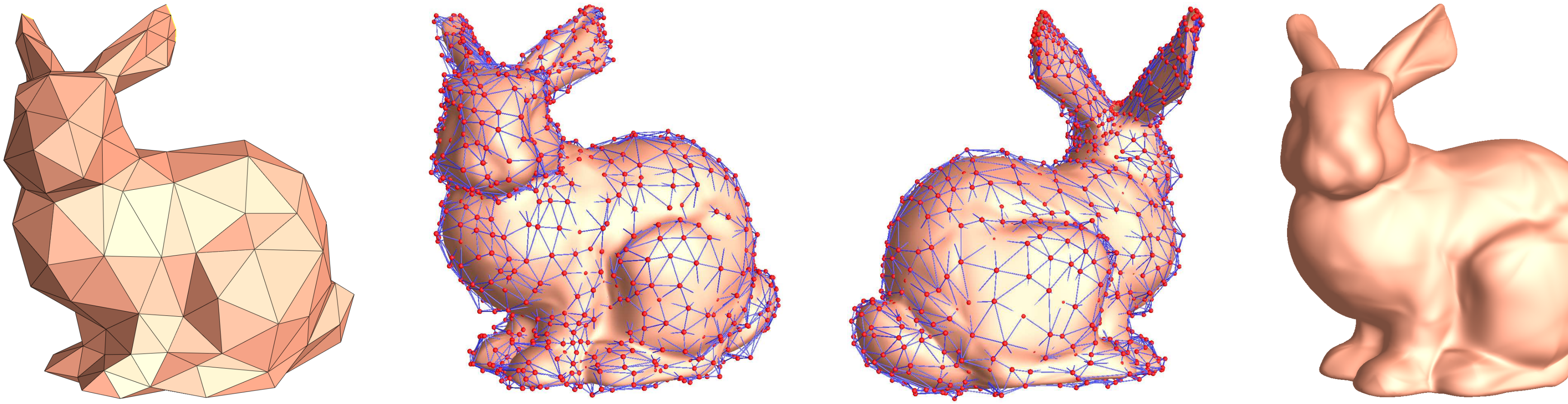
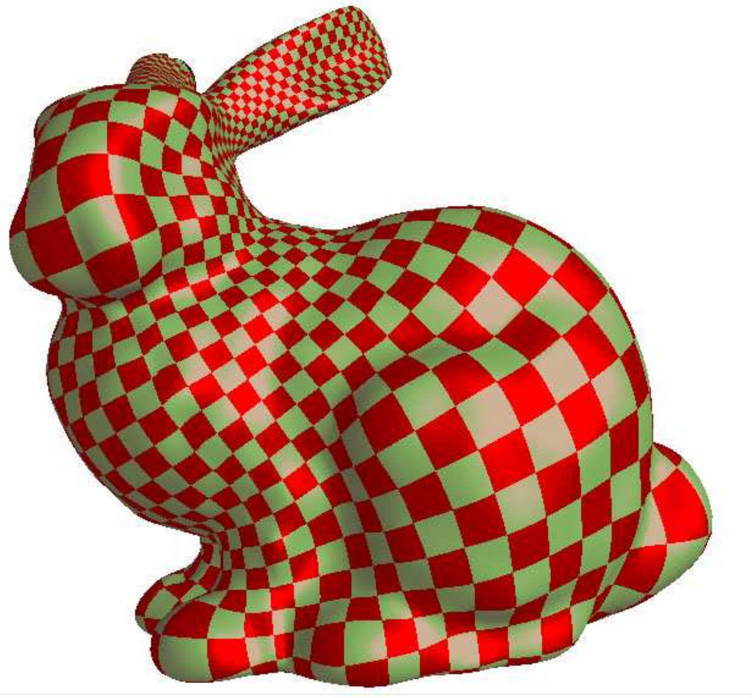
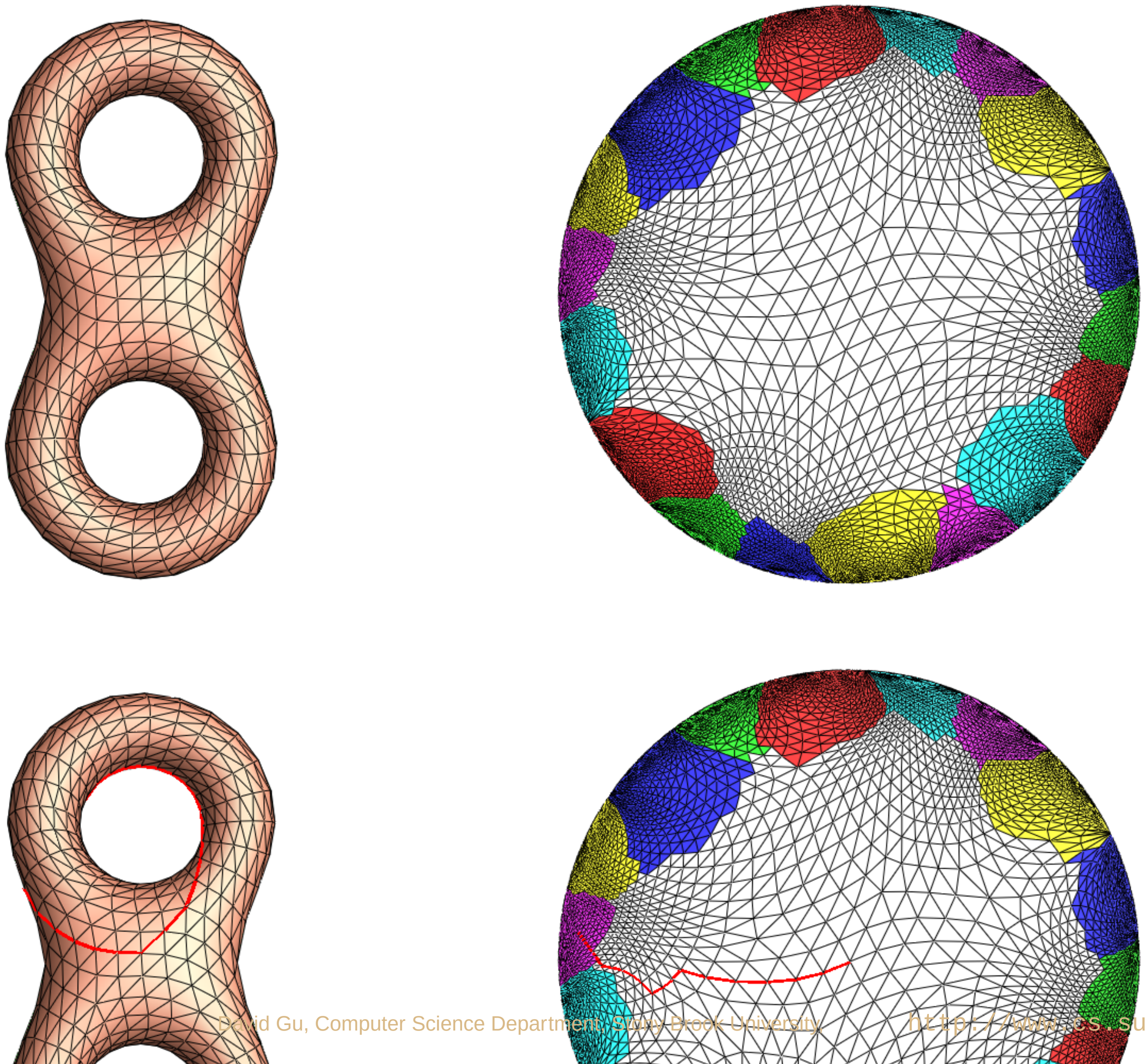


Figure 36: Thanks Ying

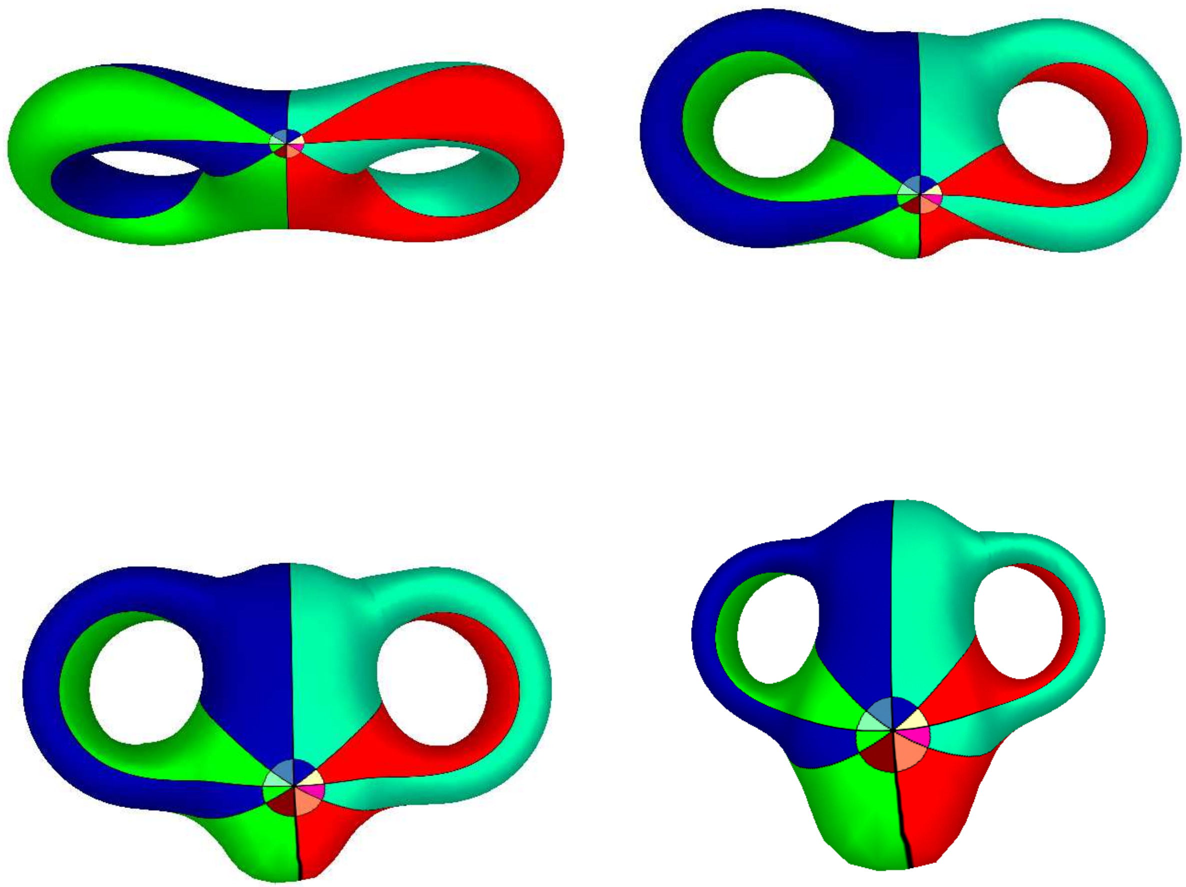
Manifold TSpline



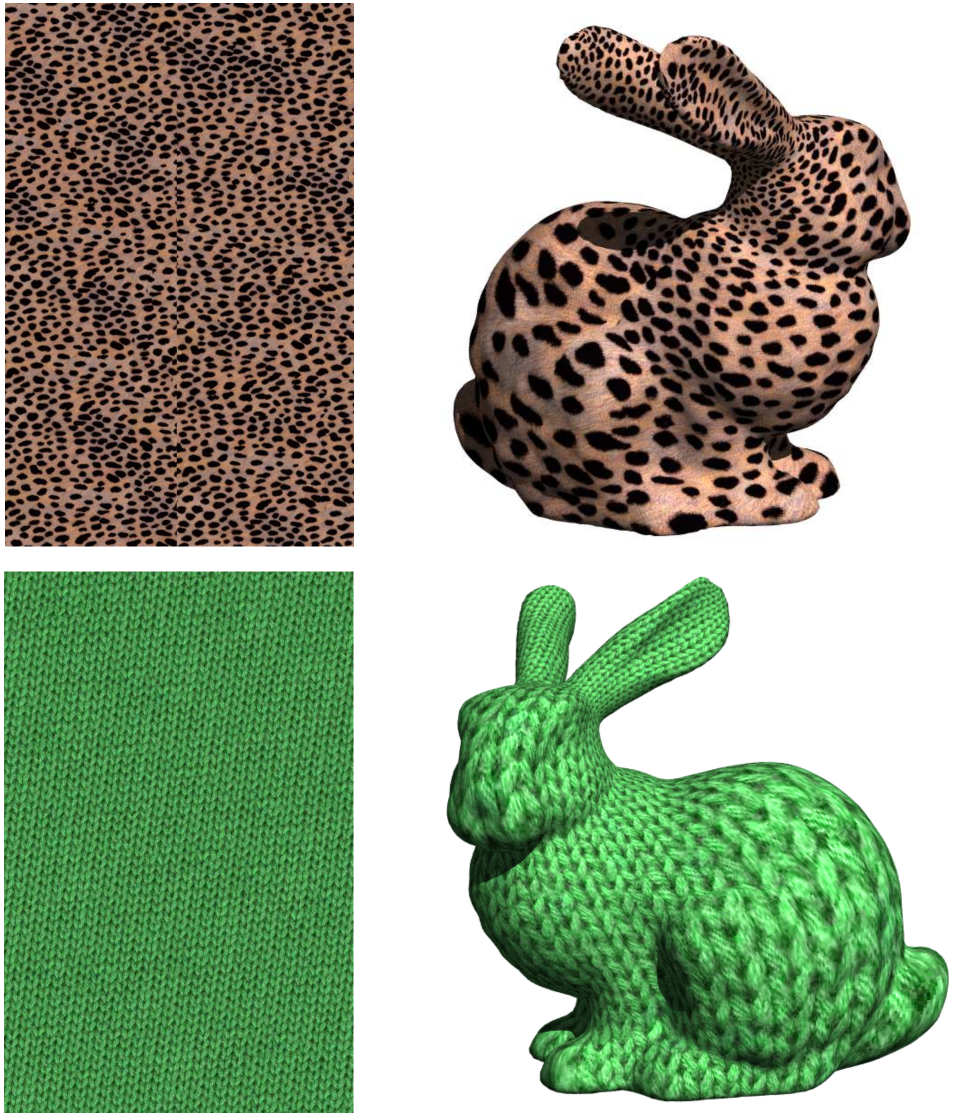
Surface Matching



Surface Matching



Texture Synthesis



Texture Synthesis