Surface and Volume Parameterization Methods

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Isogeometric Analysis 2011: Integrating Design and Analysis

Thanks

Thanks for the invitation.

Collaborators

The work is collaborated with Shing-Tung Yau, Feng Luo, Tony Chan, Paul Thompson, Yalin Wang, Ronald Lok Ming Lui, Hong Qin, Dimitris Samaras, Jie Gao, Arie Kaufman, and many other mathematicians, computer scientists and medical doctors.

Outline

- Surface Parameterization based on conformal geometry
- Volumetric Parameterization

Conformal Map

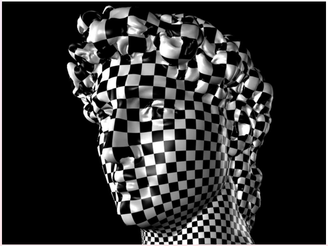
Conformal Geometry

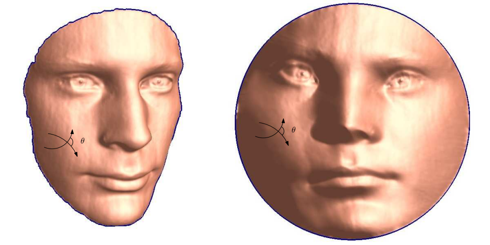
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \rightarrow (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

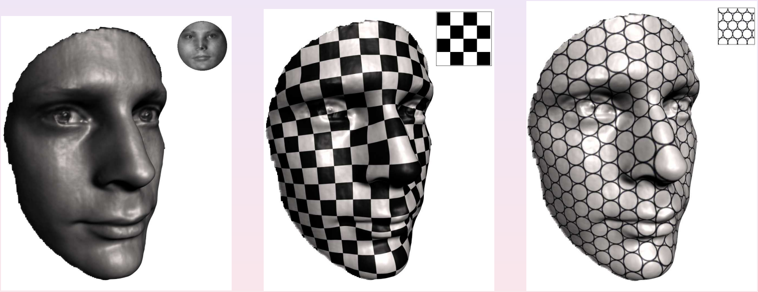
Conformal Mapping preserves angles.





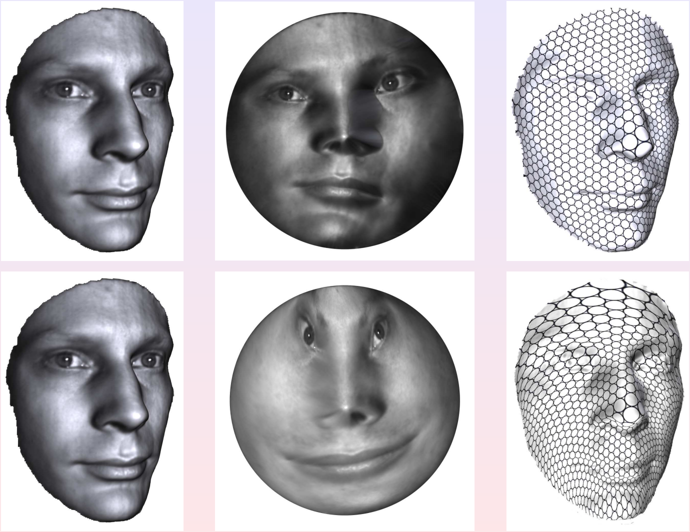
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.



Quasi-Conformal Map

Diffeomorphisms: map ellipse field to circle field.

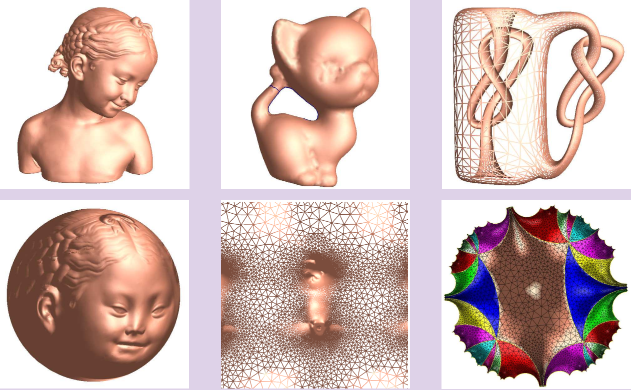


Theoretic Foundation

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

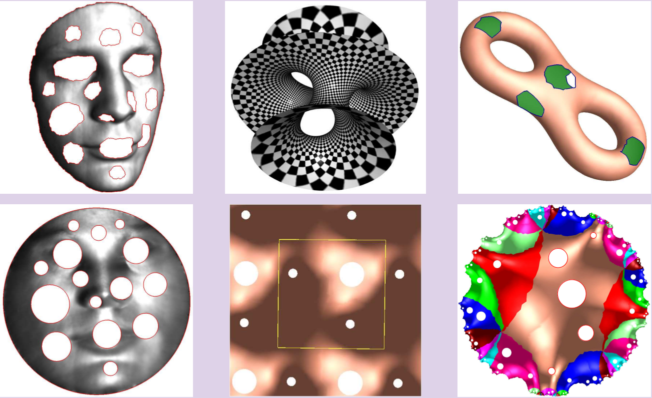
Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Uniformization of Open Surfaces

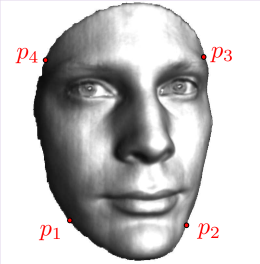


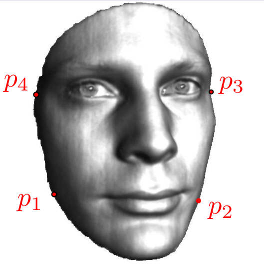
Spherical Euclidean Hyperbolic

Conformal Canonical Representation

Simply Connected Domains

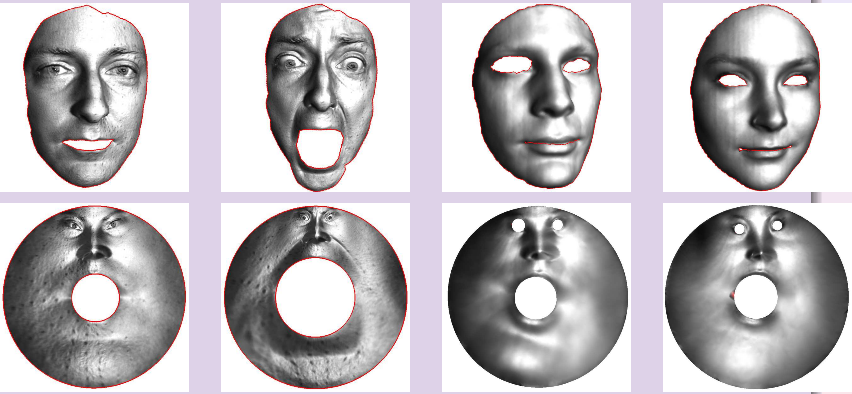
Topological Quadrilaterals





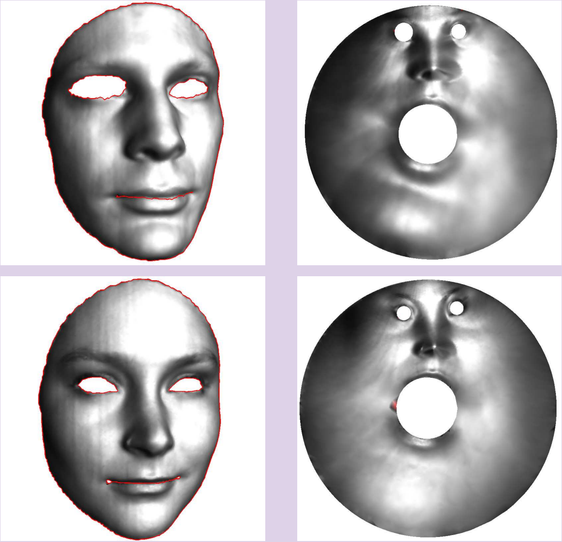
Conformal Canonical Forms

Multiply Connected Domains

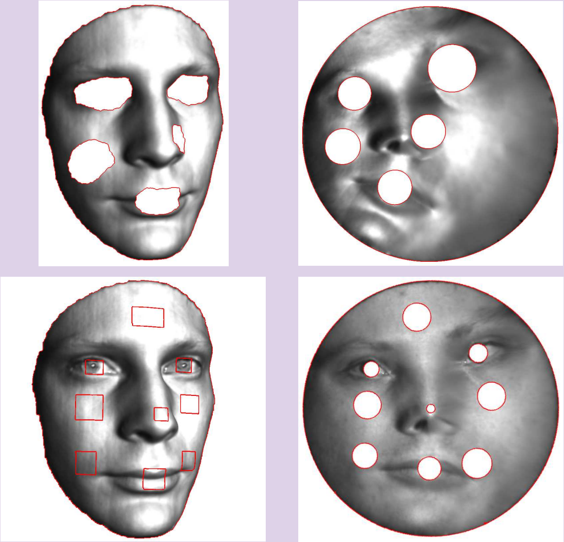


Conformal Canonical Forms

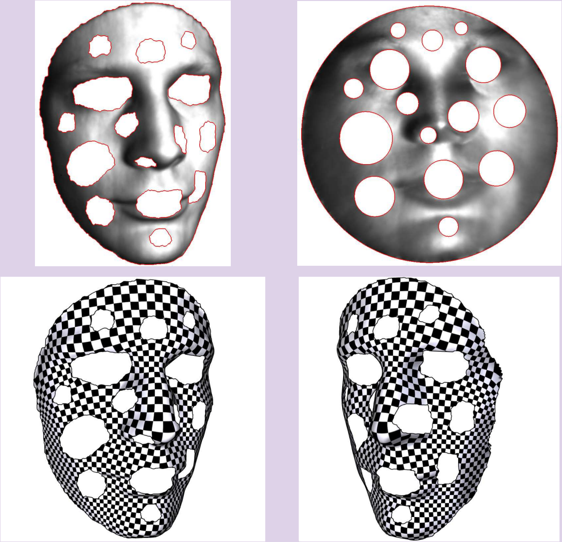
Multiply Connected Domains



Multiply Connected Domains



Multiply Connected Domains



Conformal Canonical Representations

Definition (Circle Domain in a Riemann Surface)

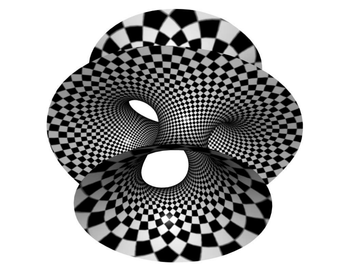
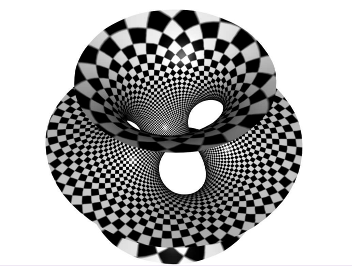
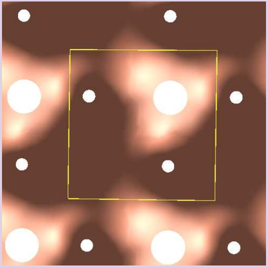
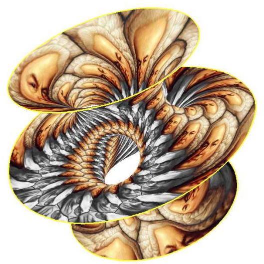
A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$) are round.

Theorem

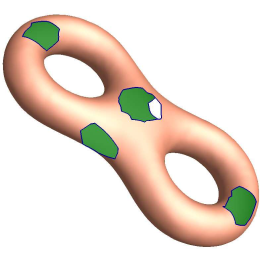
Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

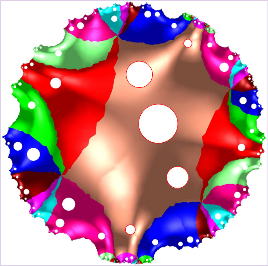
Conformal Canonical Form

Tori with Holes

High Genus Surface with Holes





Computational Methods

Computational Method

Methods for Surface Parameterization

1. Holomorphic differential method
2. Surface Ricci flow

Holomorphic Differential Method

All the holomoprhic 1-forms on a metric surface form a group, which is isomorphic to the first de Rham cohomology group. Our method computes the basis of the holomorphic 1-form group.

Harmonic 1-form

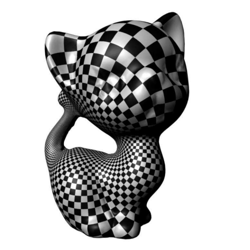
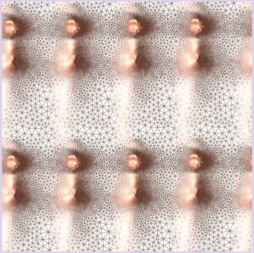
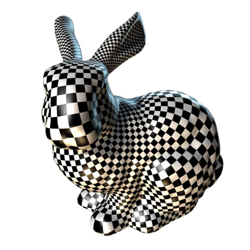
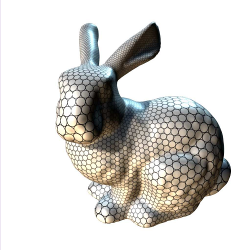
Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H_1(M)$.

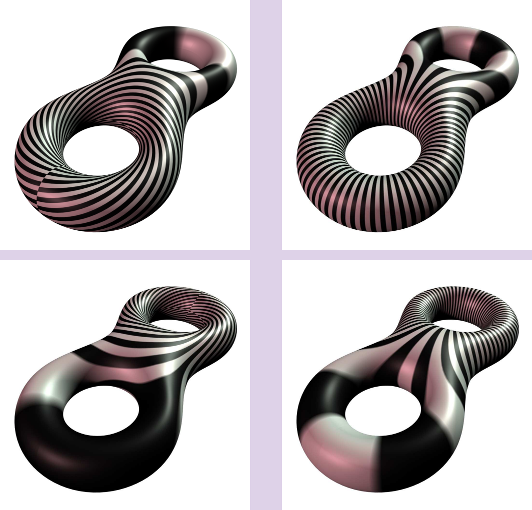
Holomorphic 1-form

Holomorphic 1-form - Global Conformal Parameterization

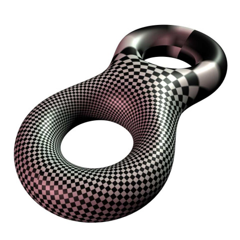
Harmonic 1-form

Harmonic 1-form Basis



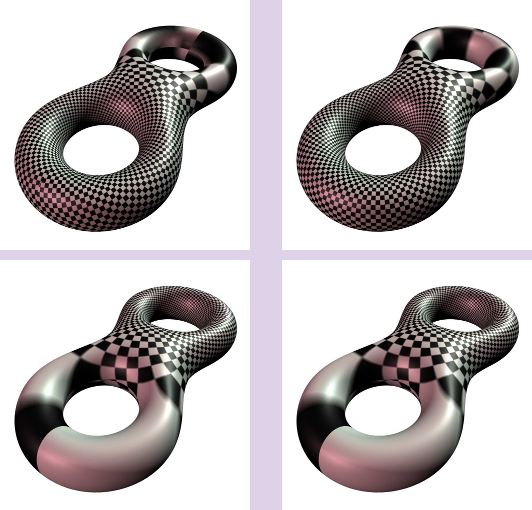
Harmonic 1-form

Harmonic 1-form Basis



Holomorphic 1-form

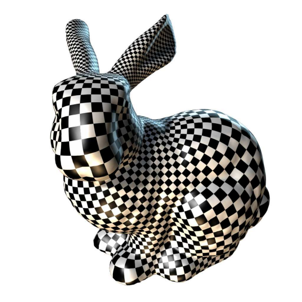
Holomorphic 1-form Basis



Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

The Gaussian curvature is given by

$$K(u,v) = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda(u,v)}}\Delta\lambda(u,v),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$.

Conformal Metric Deformation

Definition

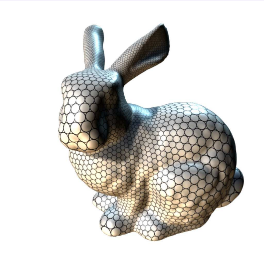
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabe Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

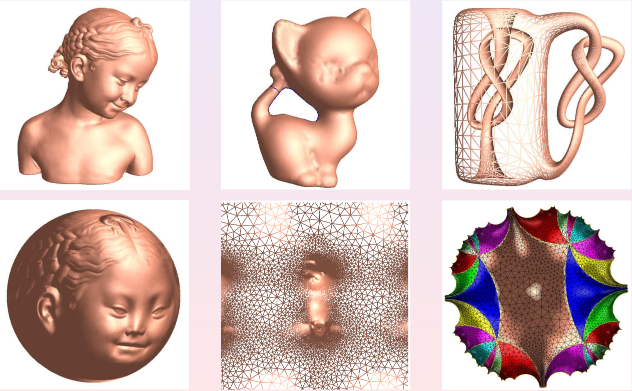
$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ *be a compact 2-dimensional Riemannian manifold. Then there is a metric* $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ *conformal to* $\mathbf{g}$ *which has constant Gauss curvature.*



Spherical Euclidean Hyperbolic

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric **g**, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -Kg_{ij}.$$

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant every where.

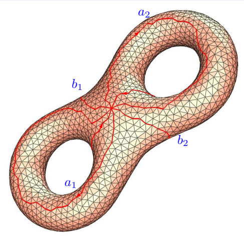
Key Idea

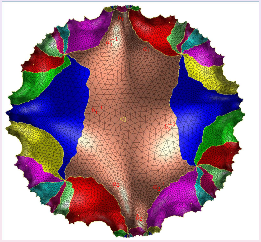
$K = -\Delta_{\mathbf{g}}\lambda$, Roughly speaking, $\frac{dK}{dt} = \Delta_{\mathbf{g}}\frac{d\lambda}{dt}$. Let $\frac{d\lambda}{dt} = -K$, then

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + K^2$$

Heat equation!

Computational results for genus 2 and genus 3 surfaces.





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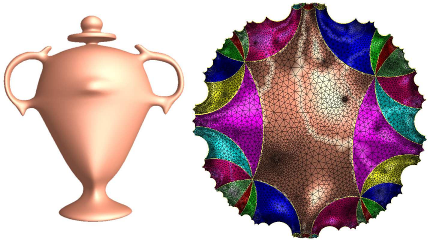
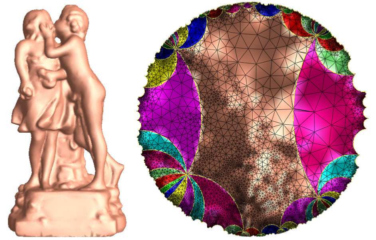
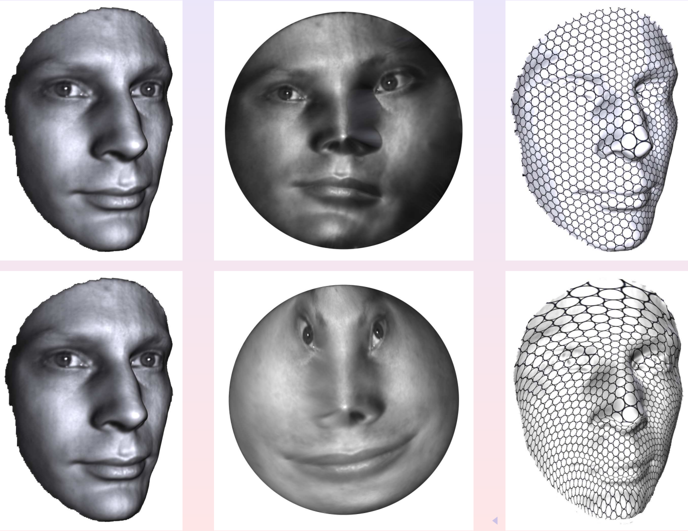




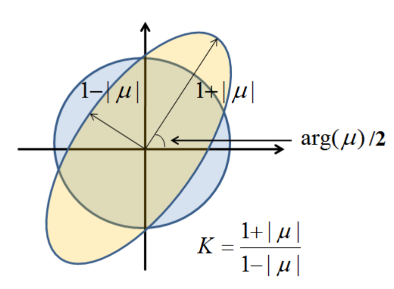
Quasi-Conformal Mappings

Quasi-Conformal Map

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.



Beltrami-Equation



Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

Solving Beltrami Equation

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

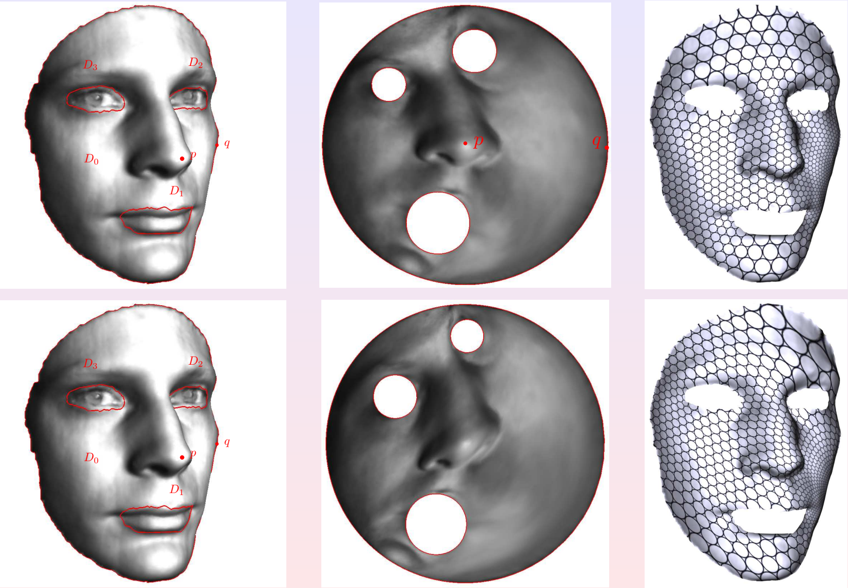
Given metric surfaces $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, let z, w be isothermal coordinates of S_1, S_2, $w = \phi(z)$.

$$\mathbf{g}_1 = e^{2u_1} dz d\bar{z} \quad (1)$$
$$\mathbf{g}_2 = e^{2u_2} dw d\bar{w}, \quad (2)$$

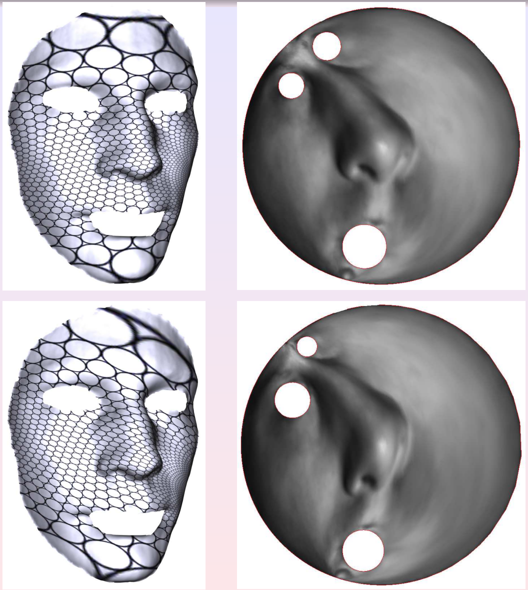
Then

- $\phi : (S_1, \mathbf{g}_1) \rightarrow (S_2, \mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi : (S_1, \phi^* \mathbf{g}_2) \rightarrow (S_2, \mathbf{g}_2)$ is isometric
- $\phi^* \mathbf{g}_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{z}|^2$.
- $\phi : (S_1, |dz + \mu d\bar{z}|^2) \rightarrow (S_2, \mathbf{g}_2)$ is conformal.

Quasi-Conformal Map Examples



Quasi-Conformal Map Examples



Volumetric Parameterization

Based on surface parameterization method

1. Volumetric harmonic map
2. Green's function on Star Shape
3. Direct product decomposition
4. Volumetric curvature flow

Volumetric Harmonic Map

Suppose we want to compute a volumetric mapping $\phi : V \to D$,

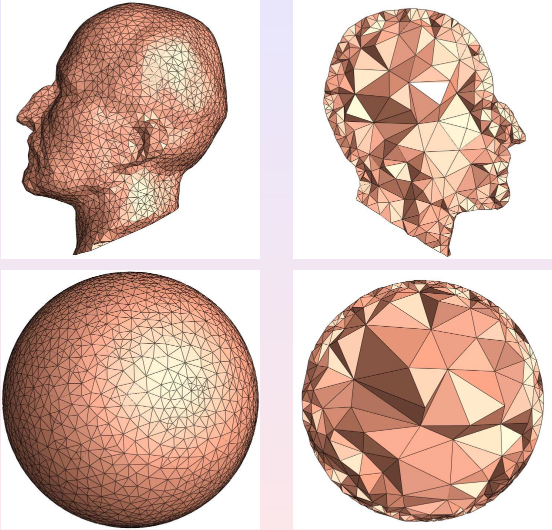
1. Compute a conformal mapping between the boundary surfaces

$$\psi : \partial V \to \partial D$$

2. Compute a volumetric harmonic mapping with Dirichlet boundary condition

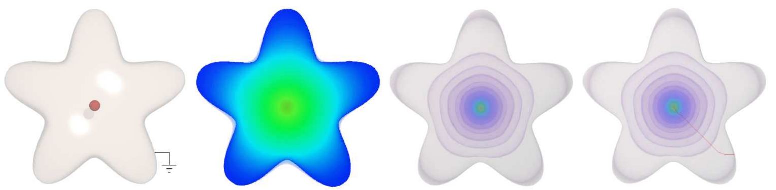
$$\Delta \phi = 0, \phi|_{\partial V} = \psi.$$

Volumetric Harmonic Mapping



Green's Function on Star Shape

Suppose V is a star shape with the center o, then each ray from the center intersects the boundary of V only once.



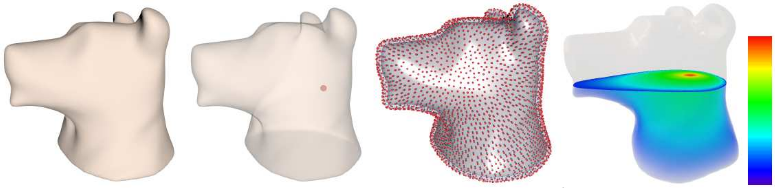
Green's Function on Star Shape

1. Compute the Green's function G on V, centered at o, such that $G|_{\partial V} = 0$.
2. Trace the gradient line from the center to the boundary.
3. Conformal map the boundary to the unit sphere.
4. The level sets of G are mapped to concentric spheres, the gradient lines are mapped to the radii of the unit ball.

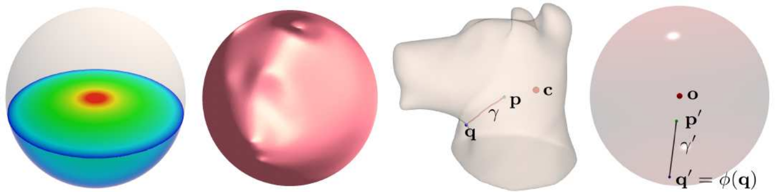
Theorem

The mapping must be a diffeomorphism.

Green's Function on Star Shape



Green's Function on Star Shape



Green's Function on Star Shape

Direct Product Method

The volume can be decomposed as the direct product of a surface and a curve (or circle), $V = S \times C$,

1. Decompose the boundary surface of V to top T, bottom B and wall W.
2. Conformal parameterize the top B, $\phi : B \to S$.
3. Compute a harmonic function f, such that

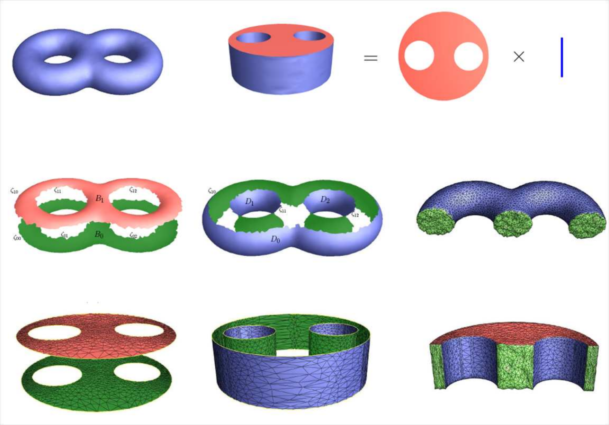
$$\begin{cases} f(p) = 1 & p \in T \\ f(p) = 0 & p \in B \\ \Delta f = 0 & \end{cases}$$

4. Trace the gradient line of f, then maps the volume to $S \times [0,1]$.

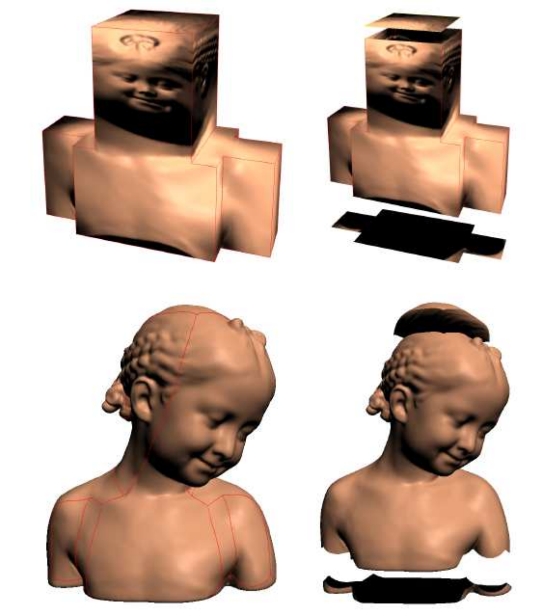
Theorem

The mapping must be a diffeomorphism.

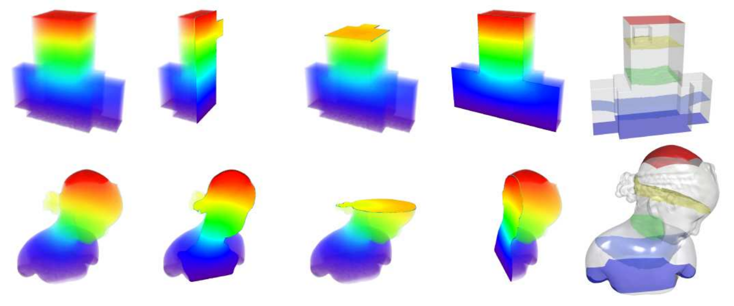
Direct Product



Direct Product



Direct Product



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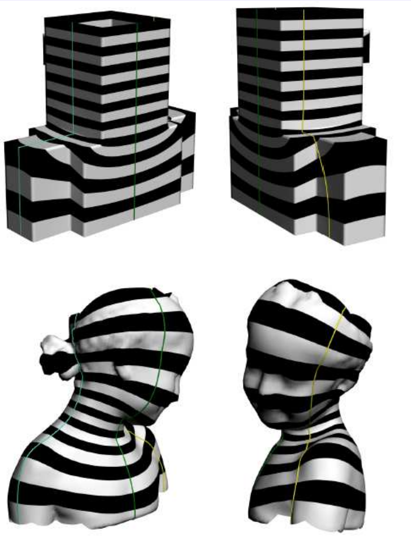
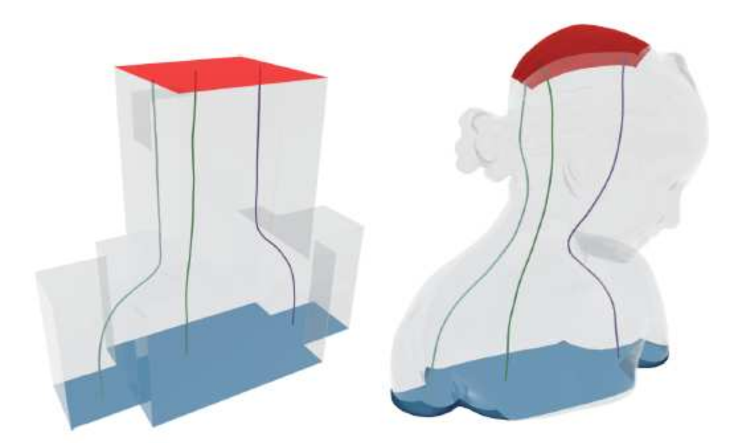


Direct Product



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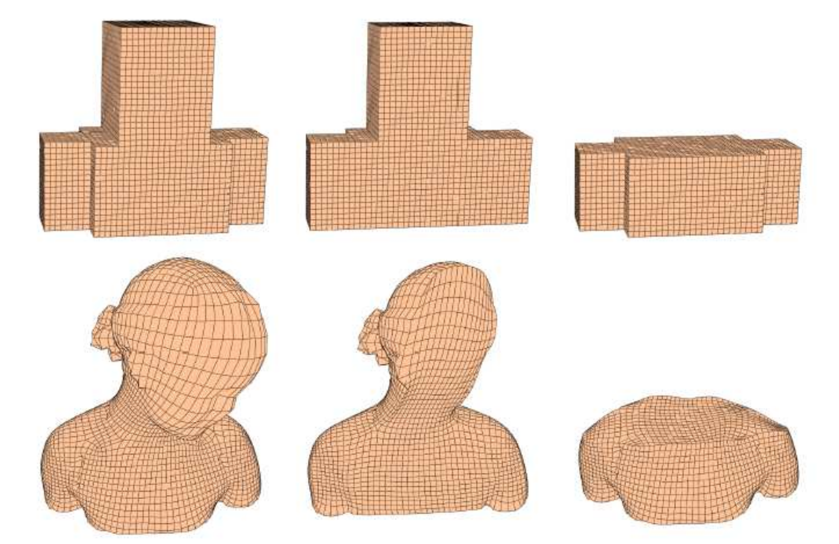

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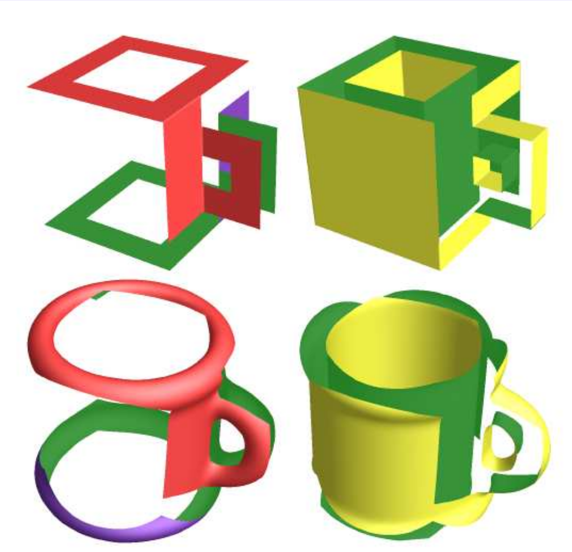

Direct Product

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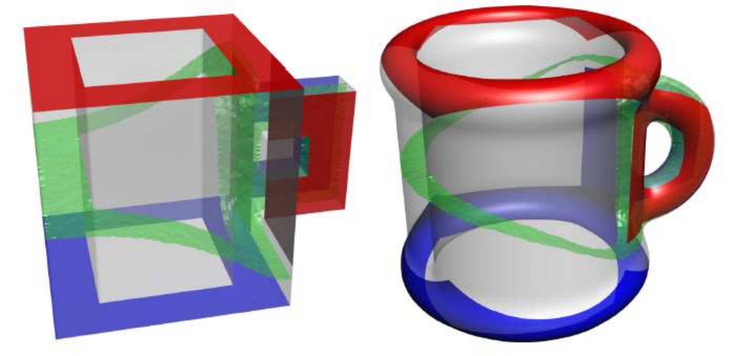


Direct Product

Suppose the given volume has complicated topology, such that the boundary surfaces are with high genus. Then we can compute the canonical hyperbolic Riemannian metric of the volume, and embed the universal covering space of the volume in three dimensional hyperbolic space $\mathbb{H}^3$.

Hyperbolic Volumetric Curvature Flow

1. Triangulate the volume to truncated tetrahedra.
2. Compute the curvature on each edge of the tetrahedra mesh

$$K(e_{ij}) = 2\pi - \sum_{kl} \theta_{ij}^{kl},$$

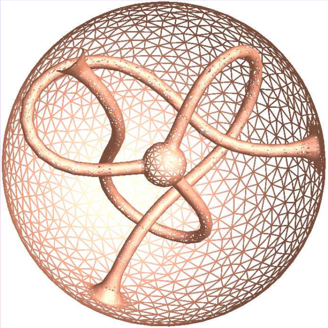
   where θ_{ij}^{kl} is the dihedral angle on the edge $(e_{ij}$ in the tetrahedron $[v_i, v_j, v_k, v_l]$.
3. Run curvature flow,

$$\frac{dl_{ij}}{dt} = K_{ij}$$

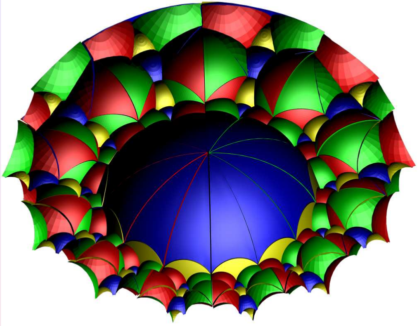
   where l_{ij} is the edge length of e_{ij}.

Theorem

If the input 3-manifold is a hyperbolic 3-manifold with complete geodesic boundary, then the curvature flow will converge to the canoical hyperbolic metric.



a. input 3-manifold



b. embedding of its UCS in $\mathbb{H}^3$

Manifold Spline

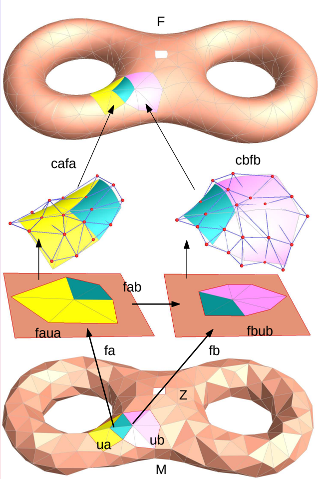
- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.

Manifold Spline

Extraordinary Points

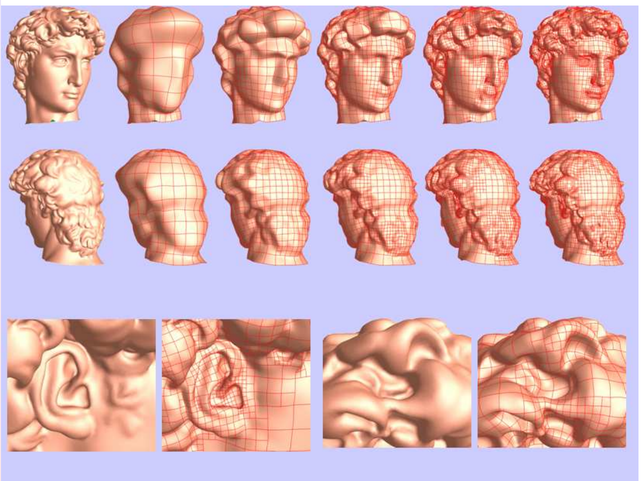
- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.

Manifold Spline



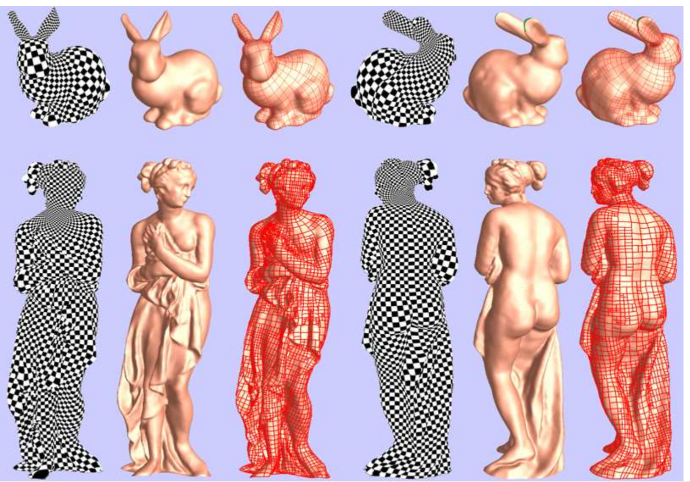
Manifold Spline

Converting a polygonal mesh to TSplines with multiple resolutions.



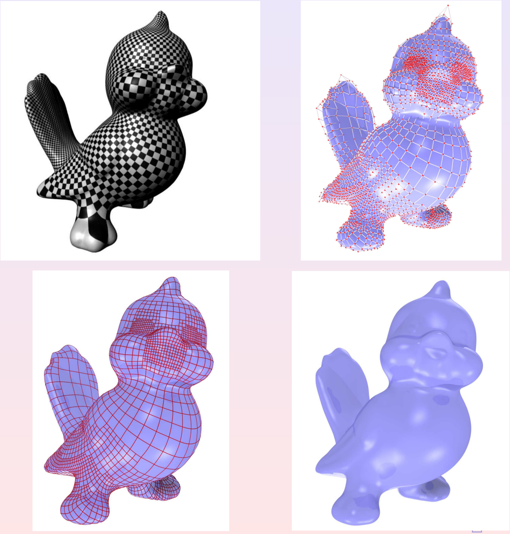
Manifold Spline

Converting scanned data to spline surfaces.



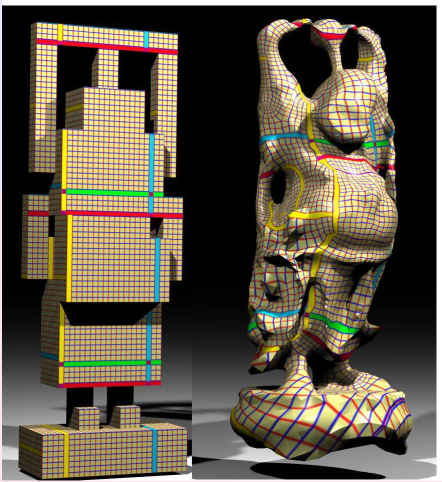
Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.



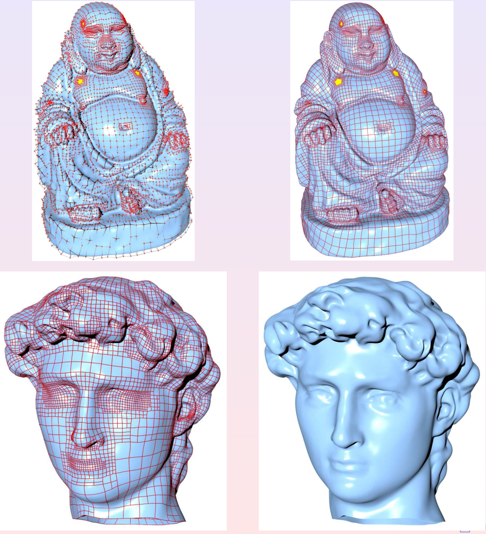
Polycube Map

Compute polycube maps for high genus surfaces.



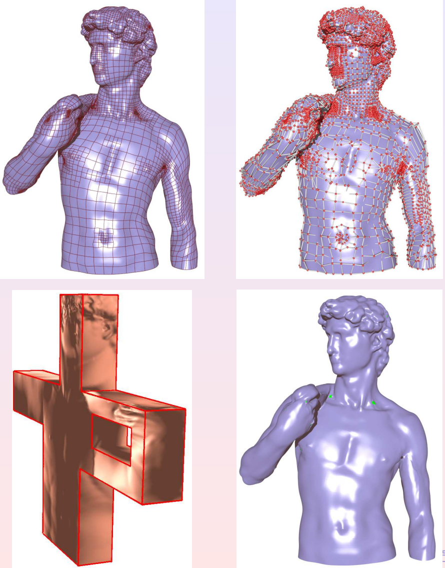
Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.

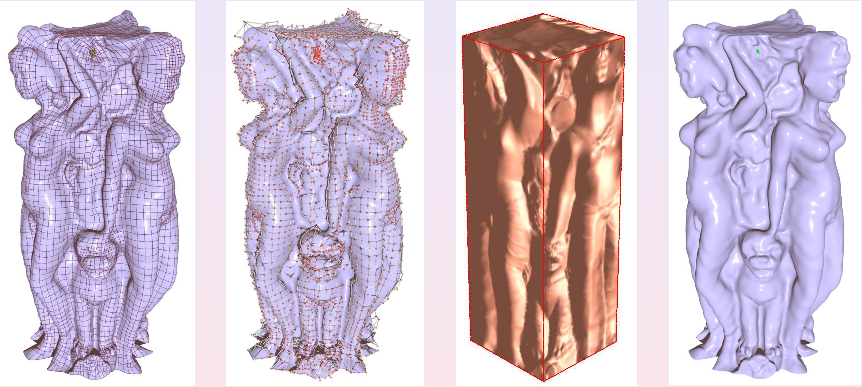


Manifold Spline

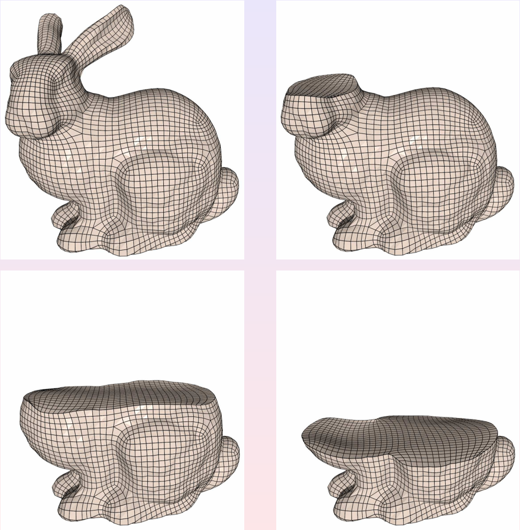
Polygonal mesh to spline, control net and the knot structure.



Manifold Spline

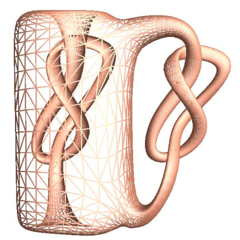


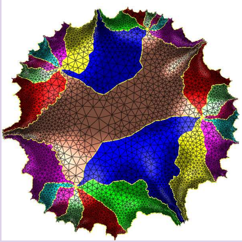
Volumetric spline.



Thanks

For more information, please email to zengwei@cs.sunysb.edu and gu@cs.sunysb.edu.





Thank you!