Computational Conformal Geometry: Theories, Algorithms and Applications

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Thanks for the invitation.
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Klein’s Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion
Conformal geometry lays down the theoretic foundation for

- Surface mapping
- Geometry classification
- Shape analysis

Applied in computer graphics, computer vision, geometric modeling, wireless sensor networking and medical imaging, and many other engineering, medical fields.
In pure mathematics, conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, differential geometry, partial differential equation.

In applied mathematics, computational complex function theory has been developed, which focuses on the conformal mapping between planar domains.

Recently, computational conformal geometry has been developed, which focuses on the conformal mapping between surfaces.
Conventional conformal geometric method can only handle the mappings among planar domains.

- Applied in thin plate deformation (biharmonic equation)
- Membrane vibration
- Electro-magnetic field design (Laplace equation)
- Fluid dynamics
- Aerospace design
Reasons for Booming

Data Acquisition

3D scanning technology becomes mature, it is easier to obtain surface data.
System Layout

- Object point
- Phase line
- Camera pixel
- Projector fringe
- Baseline
- Object

- CCD camera
- Object
- DLP projector
- RGB fringe
- PC
3D Scanning Results
Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Computational power has been increased tremendously. With the incentive in graphics, GPU becomes mature, which makes numerical methods for solving PDE’s much easier.
Fundamental Problems

1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.

2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmüller shape space.

3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics.

4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.

5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.

6. Compute the extremal quasi-conformal maps.

7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.
Complete Tools

Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation
The theory, algorithms and sample code can be found in the following books.

You can find them in the book store.
Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.
Conformal Structure
**Definition (Manifold)**

$M$ is a topological space, $\{U_\alpha\} \alpha \in I$ is an open covering of $M$, $M \subset \bigcup \alpha U_\alpha$. For each $U_\alpha$, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair $(U_\alpha, \phi_\alpha)$ is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then $M$ is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.
Manifold

\[ \phi \alpha \]

\[ U_\alpha \]

\[ U_\beta \]

\[ S \]

\[ \phi_\alpha (U_\alpha) \]

\[ \phi_\beta (U_\beta) \]

\[ \phi_{\alpha\beta} \]
Holomorphic Function

Definition (Holomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex function, $f : x + iy \to u(x, y) + iv(x, y)$, if $f$ satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then $f$ is a holomorphic function.

Denote

$$dz = dx + idy, \quad d\bar{z} = dx - idy, \quad \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

then if $\frac{\partial f}{\partial \bar{z}} = 0$, then $f$ is holomorphic.
Definition (Conformal Atlas)
Suppose $S$ is a topological surface, (2 dimensional manifold), $\mathcal{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta} : \mathbb{C} \to \mathbb{C}$ are bi-holomorphic, then $\mathcal{A}$ is called a conformal atlas.

Definition (Compatible Conformal Atlas)
Suppose $S$ is a topological surface, (2 dimensional manifold), $\mathcal{A}_1$ and $\mathcal{A}_2$ are two conformal atlases. If their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is still a conformal atlas, we say $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible.
The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose $S$ is a topological surface, consider all the conformal atlases on $M$, classified by the compatible relation

$$\{ \text{all conformal atlas} \} / \sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.
Definition (Conformal equivalent metrics)
Suppose $g_1, g_2$ are two Riemannian metrics on a manifold $M$, if

$$g_1 = e^{2u} g_2, u : M \rightarrow \mathbb{R}$$

then $g_1$ and $g_2$ are conformal equivalent.

Definition (Conformal Structure)
Consider all Riemannian metrics on a topological surface $S$, which are classified by the conformal equivalence relation,

$$\{ \text{Riemannian metrics on } S \} / \sim,$$

each equivalence class is called a conformal structure.
Definition (Isothermal coordinates)

Suppose \((S, g)\) is a metric surface, \((U_\alpha, \phi_\alpha)\) is a coordinate chart, \((x, y)\) are local parameters, if

\[
g = e^{2u}(dx^2 + dy^2),
\]

then we say \((x, y)\) are isothermal coordinates.

Theorem

Suppose \(S\) is a compact metric surface, for each point \(p\), there exists a local coordinate chart \((U, \phi)\), such that \(p \in U\), and the local coordinates are isothermal.
Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

All oriented compact metric surfaces are Riemann surfaces.
Conformal Mapping
Definition (biholomorphic Function)

Suppose \( f : \mathbb{C} \rightarrow \mathbb{C} \) is invertible, both \( f \) and \( f^{-1} \) are holomorphic, then \( f \) is a biholomorphic function.
The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.
Conformal Mapping
Conformal Geometry

Definition (Conformal Map)

Let $\phi : (S_1, g_1) \to (S_2, g_2)$ is a homeomorphism, $\phi$ is conformal if and only if

$$\phi^* g_2 = e^{2u} g_1.$$ 

Conformal Mapping preserves angles.
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.
Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.
Uniformization
Theorem (Poincaré Uniformization Theorem)

Let \((\Sigma, g)\) be a compact 2-dimensional Riemannian manifold. Then there is a metric \(\tilde{g} = e^{2\lambda} g\) conformal to \(g\) which has constant Gauss curvature.
Definition (Circle Domain)

A domain in the Riemann sphere \( \hat{\mathbb{C}} \) is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain \( \Omega \) in \( \hat{\mathbb{C}} \), whose boundary \( \partial \Omega \) has at most countably many components, is conformally homeomorphic to a circle domain \( \Omega^* \) in \( \hat{\mathbb{C}} \). Moreover \( \Omega^* \) is unique up to Möbius transformations, and every conformal automorphism of \( \Omega^* \) is the restriction of a Möbius transformation.
Uniformization of Open Surfaces

Spherical

Euclidean

Hyperbolic
Simply Connected Domains
Conformal Canonical Forms

Topological Quadrilateral

$\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \quad \mathbf{p}_4$
Conformal Canonical Forms

Multiply Connected Domains
Conformal Canonical Forms

Multiply Connected Domains

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Conformal Canonical Forms

Multiply Connected Domains

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Definition (Circle Domain in a Riemann Surface)

A circle domain in a Riemann surface is a domain, whose complement’s connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

Theorem

Let $\Omega$ be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface $R^*$ such that $\Omega$ is conformally homeomorphic to a circle domain $\Omega^*$ in $R^*$. More over, the pair $(R^*, \Omega^*)$ is unique up to conformal homeomorphism.
Conformal Canonical Form

Tori with holes

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High Genus Surface with holes
Teichmüller Space
Teichmüller Theory: Conformal Mapping

**Definition (Conformal Mapping)**

Suppose \((S_1, g_1)\) and \((S_2, g_2)\) are two metric surfaces, \(\phi : S_1 \to S_2\) is conformal, if on \(S_1\)

\[ g_1 = e^{2\lambda} \phi^* g_2, \]

where \(\phi^* g_2\) is the pull-back metric induced by \(\phi\).

**Definition (Conformal Equivalence)**

Suppose two surfaces \(S_1, S_2\) with marked homotopy group generators, \([a_i, b_i]\) and \([\alpha_i, \beta_i]\). If there exists a conformal map \(\phi : S_1 \to S_2\), such that

\[ \phi_* [a_i] = [\alpha_i], \phi_* [b_i] = [\beta_i], \]

then we say two marked surfaces are conformal equivalent.
Definition (Teichmüller Space)

Fix the topology of a marked surface $S$, all conformal equivalence classes sharing the same topology of $S$, form a manifold, which is called the Teichmüller space of $S$. Denoted as $T_S$.

- Each point represents a class of surfaces.
- A path represents a deformation process from one shape to the other.
- The Riemannian metric of Teichmüller space is well defined.
Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.
Multiply Connected Domains

Conformal Module: centers and radii, with Möbius ambiguity. The Teichmüller space is $3n - 3$ dimensional, $n$ is the number of holes.
Genus 0 surface with 3 boundaries is conformally mapped to the hyperbolic plane, such that all boundaries become geodesics.
The lengths of 3 boundaries are the conformal module.
Topological Pants Decomposition - $2g - 2$ pairs of Pants

[Diagram showing the decomposition of a surface into pants, with labels $p_1$, $p_2$, and $\theta$.]
Compute Teichmüller coordinates

Step 1. Compute the hyperbolic uniformization metric.

Step 2. Compute the Fuchsian group generators.
Step 3. Pants decomposition using geodesics and compute the twisting angle.
Compute Teichmüller coordinates

Compute the pants decomposition using geodesics and compute the twisting angle.
Quasi-Conformal Maps
Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.
Beltrami-Equation

Let $\phi : S_1 \rightarrow S_2$ be the map, $z, w$ are isothermal coordinates of $S_1, S_2$, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$
Diffeomorphism Space

Theorem

Given two genus zero metric surface with a single boundary,

\[
\{ \text{Diffeomorphisms} \} \cong \frac{\{ \text{Beltrami Coefficient} \}}{\{ \text{Mobius} \}}.
\]
Solving Beltrami Equation

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

Given metric surfaces \((S_1, g_1)\) and \((S_2, g_2)\), let \(z, w\) be isothermal coordinates of \(S_1, S_2, w = \phi(z)\).

\[
\begin{align*}
g_1 & = e^{2u_1} dz d\bar{z} \\
g_2 & = e^{2u_2} dw d\bar{w},
\end{align*}
\]

Then
- \(\phi : (S_1, g_1) \rightarrow (S_2, g_2)\), quasi-conformal with Beltrami coefficient \(\mu\).
- \(\phi : (S_1, \phi^* g_2) \rightarrow (S_2, g_2)\) is isometric
- \(\phi^* g_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{z}|^2\).
- \(\phi : (S_1, |dz + \mu d\bar{z}|^2) \rightarrow (S_2, g_2)\) is conformal.
Quasi-Conformal Map Examples
Quasi-Conformal Map Examples
Computational Method - Harmonic Mapping
Let \((M,g)\) and \((N,h)\) be Riemannian manifolds, \(u : M \to N\) is a \(C^1\) mapping.

\[
\begin{align*}
    ds^2_M &= \sum g_{\alpha\beta} dx^\alpha dx^\beta, \\
    ds^2_N &= \sum h_{ij}(u(x)) du^i du^j.
\end{align*}
\]

The pull back metric of \(h\) induced by \(u\) is \(u^*(ds^2_N)\) is a symmetric bilinear form

\[
    u^*(dS^2_N) = \sum_{\alpha,\beta} \left( \sum_{i,j} h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} \right) dx^\alpha dx^\beta.
\]

The **energy density** of mapping \(u\) is defined as

\[
    |du|^2 = \sum_{i,j,\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.
\]
Equivalently, choose an orthogonal frame field under $u^*(ds_N^2)$, each basis vector field is of unit length under $g$, the dual 1-forms are $\{\omega_1, \omega_2, \cdots, \omega_n\}$, such that

$$u^*(ds_N^2) = \sum_{\alpha=1}^{n} \lambda_\alpha (\omega_\alpha)^2.$$  

The energy density of the mapping $u$ is given by

$$|du|^2 = \sum_{\alpha=1}^{n} \lambda_\alpha.$$
**Definition (Harmonic Energy)**

The harmonic energy functional $E(u)$ is defined as

$$E(u) = \int_M |du|^2 dv_M,$$

where $dv_M = (\text{det}g)^{\frac{1}{2}} dx$ is the volume element of $M$.

**Definition (Harmonic Mapping)**

In the space of mappings, the critical points of $E(u)$ are called harmonic mappings.
Harmonic Energy Conformal Invariant

Suppose $u$ is a mapping from a surface $(S, g)$ to $(N, h)$. Suppose $	ilde{g} = e^{2\lambda}g$ is another metric of $S$, conformal to $g$, then

$$|\tilde{du}|^2 = e^{-2\lambda}|du|^2, \sqrt{\text{det}\tilde{g}} = e^{2\lambda}\sqrt{\text{det}g},$$

Then $\tilde{g} = g$. Harmonic energy is invariant under conformal metric transformation.

**Theorem**

Harmonic energy only depends on the conformal structure of the surface, independent of the choice of Riemannian metric.
Suppose $N$ is embedded in $\mathbb{R}^3$, $u : S \to N$ is a harmonic mapping, then

$$\Delta g_u^{T_uN} \equiv 0.$$ 

where $\Delta g_u = (\Delta g u_1, \Delta g u_2, \Delta g u_3)$. Namely, $\Delta g u$ is orthogonal to the tangent plane at the target space.
Heat Flow method

Definition (Heat flow)

Let \( u : S \to N \subset \mathbb{R}^3 \), the heat flow is given by

\[
\frac{du(x, t)}{dt} = - (\Delta_g u)^Tu(x)N
\]

The heat flow method will deform a mapping to the harmonic mapping under special normalization conditions.
**Theorem**

*Harmonic mapping from a genus zero closed surface to the unit sphere must be a conformal mapping.*

**Proof.**

Let $u : S \rightarrow \mathbb{S}^2$. Choose isothermal coordinates of both surfaces, define

$$\phi(z) = \langle \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \rangle$$

then

$$\phi(z) = \frac{1}{4} (|\frac{\partial u}{\partial x}|^2 - |\frac{\partial u}{\partial y}|^2 - \langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \rangle).$$

If $\phi(z) = 0$, then the mapping is conformal.

On the other hand, $\frac{\partial \phi(z)}{\partial z} = 0$, then $\phi(z)$ is holomorphic.

$\phi(z)dz^2$ is globally defined, the so-called Hopf differential.

Sphere has no non-zero holomorphic quadratic differentials. Therefore $\phi(z) = 0$. 

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Theorem

*The conformal automorphism from a sphere to itself must be a Möbius transformation*

\[ z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1, \ a, b, c, d \in \mathbb{C}. \]
Theorem (Rado)

Let $\Omega \subset \mathbb{R}^2$ is a convex domain with smooth boundary. For any homeomorphism $\phi : S^1 \to \partial \Omega$, there exists a unique harmonic mapping $u : D \to \Omega$, such that $u|_{\partial D} = \phi$, furthermore, $u$ is a diffeomorphism.
We use piecewise linear triangle mesh to approximate the original surface. Suppose $u : M \rightarrow \mathbb{R}$ the harmonic energy is given by

$$E(u) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij} (f(v_i) - f(v_j))^2.$$

The discrete Laplace-Beltrami operator is given by

$$\Delta f(v_i) = \sum_j w_{ij} (f(v_j) - f(v_i)).$$

where $w_{ij}$ is the cotangent formula.
Computational Method - Holomorphic Form
Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Theorem (Hodge)

All the harmonic 1-forms form a group, which is isomorphic to $H^1(M)$.

Theorem (Hodge Decomposition)

$$
\Omega^k(M) = \text{Img} d^{k-1} \oplus \text{Img} \delta^{k+1} \oplus H^k_\Delta(M).
$$
Let $\omega$ be a closed 1-form. Compute a function $f \in C^0(M, \mathbb{R})$, such that

$$\delta^1(\omega + df) = 0,$$

then $\omega + df$ is the unique harmonic 1-form, cohomologous to $\omega$. 

**Harmonic 1-form Basis**
Let $\omega$ be a closed 1-form. Compute a function $f : V \rightarrow \mathbb{R}$, such that

$$\sum_j w_{ij} (\omega + df)([v_i, v_j]) = 0, \forall v_i \in V.$$ 

Then $\omega + df$ is the unique harmonic 1-form, cohomologous to $\omega$. 

Harmonic 1-form Basis

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Let \((S, g)\) be a metric surface, \(\{e_1, e_2\}\) be an orthonormal frame field, \(\left\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\}\) be the base vector fields, \(\{du, dv\}\) be the dual differential 1-form fields.

\[ *du = dv, *dv = -du. \]
If $\omega$ is a harmonic 1-form, so is $\ast \omega$. Suppose $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ is the set basis of harmonic 1-forms, then $\ast \omega = \sum_k \lambda_k \omega_k$. Locally, on each triangle $\ast (adx + bdy) = ady - bdx$. Solve linear system

$$\int_M \omega_i \wedge \ast \omega = \sum_k \lambda_k \int_M \omega_i \wedge \omega_k, \ i = 1, 2, \cdots, 2g.$$

to solve $\lambda_k$'s, where $\ast \omega$ on the left hand side is locally evaluated.
Holomorphic 1-form

Holomorphic 1-form Basis
Topological Quadrilateral
Figure: Topological quadrilateral.
Definition (Topological Quadrilateral)
Suppose $S$ is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then $S$ is called a topological quadrilateral, and denoted as $Q(p_1, p_2, p_3, p_4)$.

Theorem
Suppose $Q(p_1, p_2, p_3, p_4)$ is a topological quadrilateral with a Riemannian metric $g$, then there exists a unique conformal map $\phi : S \rightarrow \mathbb{C}$, such that $\phi$ maps $Q$ to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.
Assume the boundary of $Q$ consists of four segments 
$\partial Q = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, such that

$$
\begin{align*}
\partial \gamma_1 & = p_2 - p_1, \\
\partial \gamma_2 & = p_3 - p_2, \\
\partial \gamma_3 & = p_4 - p_3, \\
\gamma_4 & = p_1 - p_4.
\end{align*}
$$

We compute two harmonic functions $f_1, f_2 \to \mathbb{R}$, such that

$$
\begin{align*}
\Delta f_1 & = 0, \\
\left.f_1\right|_{\gamma_1} & = 0, \\
\left.f_1\right|_{\gamma_3} & = 1, \\
\left.\frac{\partial f_1}{\partial n}\right|_{\gamma_2 \cup \gamma_4} & = 0
\end{align*}
$$

and

$$
\begin{align*}
\Delta f_2 & = 0, \\
\left.f_2\right|_{\gamma_2} & = 0, \\
\left.f_2\right|_{\gamma_4} & = 1, \\
\left.\frac{\partial f_2}{\partial n}\right|_{\gamma_1 \cup \gamma_3} & = 0
\end{align*}
$$
The $df_1$ and $df_2$ are two exact harmonic 1-forms. We need to find a scalar $\lambda$, such that $\ast df_1 = \lambda df_2$, this can be achieved by solving the following equation,

$$\int_S df_1 \wedge \ast df_2 = \lambda \int_S df_1 \wedge df_2.$$ 

Then the desired holomorphic 1-form $\omega = df_1 + i\lambda df_2$. The conformal mapping is given by

$$\phi(p) = \int_q^p \omega,$$

where $q$ is the base point, the path from $q$ to $p$ is arbitrarily chosen.
Topological Annulus
Figure: Topological annulus.
Definition (Topological Annulus)

Suppose $S$ is a surface of genus zero with two boundaries, the $S$ is called a topological annulus.

Theorem

Suppose $S$ is a topological annulus with a Riemannian metric $g$, the boundary of $S$ are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius $\gamma$. Then $-\log \gamma$ is the conformal module of $S$. The mapping $\phi$ is unique up to a planar rotation.
Topological Annulus

First, we compute a harmonic function $f : S \to \mathbb{R}$, such that

$$\begin{cases} 
  f|_{\gamma_1} & = 0 \\
  f|_{\gamma_2} & = 1 \\
  \Delta f & = 0
\end{cases}$$

Then $df$ is an exact harmonic 1-form. Then we compute a harmonic 1-form $\tau$, such that $\int_{\gamma_1} \tau = 1$. 

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Then we compute a constant $\lambda$, such that $\ast df = \lambda \tau$, by solving the following equation,

$$\int_S df \wedge \ast df = \lambda \int_S df \wedge \tau.$$  

Then $\omega = df + i\lambda \tau$ is a holomorphic 1-form. Let $\text{Im}g(\int_{\gamma_1} \omega) = k$. The conformal mapping is given by

$$\phi(p) = \exp^{\frac{2\pi}{k} \int_q^p \omega}.$$
Figure: Topological annulus.
Riemann Mapping
Conformal Module

### Simply Connected Domains

<table>
<thead>
<tr>
<th>![Image 1]</th>
<th>![Image 2]</th>
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<tbody>
<tr>
<td>![Image 3]</td>
<td>![Image 4]</td>
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Definition (Topological Disk)

Suppose $S$ is a surface of genus zero with one boundary, the $S$ is called a topological disk.

Theorem

Suppose $S$ is a topological disk with a Riemannian metric $g$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the canonical disk. The mapping $\phi$ is unique up to a Möbius transformation,

$$z \rightarrow e^{i\theta} \frac{z - Z_0}{1 - \bar{Z_0}z}.$$
Punch a small hole in the disk, then use the algorithm for topological annulus to compute the conformal mapping. The punched hole will be mapped to the center.
Multiply connected domains
Definition (Multiply-Connected Annulus)
Suppose $S$ is a surface of genus zero with multiple boundaries, then $S$ is called a multiply connected annulus.

Theorem
Suppose $S$ is a multiply connected annulus with a Riemannian metric $g$, then there exists a conformal mapping $\phi : S \rightarrow \mathbb{C}$, which maps $S$ to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of $S$. Such kind of conformal mapping are unique up to Möbius transformations.
Figure: Harmonic forms and holomorphic forms.
Suppose there are $n + 1$ boundary components $\{\gamma_0, \gamma_1, \gamma_2, \cdots, \gamma_n\}$. \{\omega_1, \omega_2, \cdots, \omega_n\} are the holomorphic 1-form basis. Choose two boundary components, $\gamma_0, \gamma_1$, solve linear equation $\omega = \sum_{k=1}^{n} \lambda_k \omega_k$,

$$\text{img}(\int_{\gamma_0} \omega) = 2\pi, \text{img}(\int_{\gamma_1} \omega) = -2\pi, \text{img}(\int_{\gamma_k} \omega) = 0, 2 \leq k \leq n.$$ 

Then the mapping is given by

$$p \to \exp \int_{q}^{p} \omega,$$

where $q$ is the base point on the surface.
Figure: Conformal circular slit mapping.
Hole Filling

Adding sample points in the center hole, use Delaunay triangulation to fill in with boundary constraints.

Figure: Fill interior holes.
Figure: Koebe’s method for computing conformal maps for multiply connected domains.
Figure: Koebe’s method for computing conformal maps for multiply connected domains.
Figure: Koebe’s method for computing conformal maps for multiply connected domains.
Theorem (Gu and Luo 2009)

Suppose genus zero surface has \( n \) boundaries, then there exists constants \( C_1 > 0 \) and \( 0 < C_2 < 1 \), for step \( k \), for all \( z \in \mathbb{C} \),

\[
|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2 \left[ \frac{k}{n} \right]},
\]

where \( f \) is the desired conformal mapping.
Topological Torus
Figure: Genus one closed surface.
1. We compute a basis for the fundamental group $\pi_1(S)$, \(\{\gamma_1, \gamma_2\}\).

2. Compute the holomorphic 1-form basis $\omega_1, \omega_2$, such that $\int_{\gamma_i} \omega_j = \delta_{ij}$.

3. Slice the surface along $\gamma_1, \gamma_2$ to get a fundamental domain $\tilde{S}$.

4. The conformal mapping $\phi: \tilde{S} \to \mathbb{C}$ is given by

$$
\phi(p) = \int_q^p \omega_1,
$$

where $q$ is the base point, the path from $q$ to $p$ in $\tilde{S}$ can be arbitrarily chosen.
Suppose $a + ib = \int_{\gamma_2} \omega_1$, then $a + ib$ is the conformal module of the torus. The deck transformation group generators are

$$T_1(z) = z + 1, \ T_2(z) = z + a + ib.$$ 

By using all deck transformations to translate $\phi(\tilde{S})$, we can conformally map the universal covering space of $S$ onto the whole complex plane $\mathbb{C}$, each fundamental domain is a parallelogram.
Computational Method - Surface Ricci Flow
Discrete Curvature Flow

Isothermal Coordinates

A surface $\Sigma$ with a Riemannian metric $g$, a local coordinate system $(u, \nu)$ is an isothermal coordinate system, if

$$g = e^{2\lambda(u, \nu)} (du^2 + dv^2).$$

Gaussian Curvature

The Gaussian curvature is given by

$$K(u, \nu) = -\Delta_g \lambda = -\frac{1}{e^{2\lambda(u, \nu)}} \Delta \lambda(u, \nu),$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \nu^2}$. 
**Definition**

Suppose $\Sigma$ is a surface with a Riemannian metric, 

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}g$ is also a Riemannian metric on $\Sigma$ and called a **conformal metric**. $\lambda$ is called the conformal factor.

$$g \rightarrow e^{2\lambda}g$$

Conformal metric deformation.

Angles are invariant measured by conformal metrics.
Yamabe Equation

Suppose $\bar{g} = e^{2\lambda} g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda} (-\Delta_g \lambda + K),$$

geodesic curvature on the boundary

$$\bar{k}_g = e^{-\lambda} (-\partial_n \lambda + k_g).$$
Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, g)$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{g} = e^{2\lambda} g$ conformal to $g$ which has constant Gauss curvature.

Spherical  Euclidean  Hyperbolic
Surface Ricci Flow

Definition (Hamilton’s Surface Ricci Flow)

A closed surface with a Riemannian metric $g$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$  

If the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant everywhere.

Key Idea

$$K = -\Delta_g \lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = \Delta_g \frac{d\lambda}{dt}.$$  

Let $\frac{d\lambda}{dt} = -K$, then

$$\frac{dK}{dt} = \Delta_g K + 2K^2$$

Heat equation!
Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 
A Discrete Metric on a triangular mesh is a function defined on the vertices, \( l : E = \{\text{all edges}\} \rightarrow \mathbb{R}^+ \), satisfies triangular inequality.

A mesh has infinite metrics.
### Definition (Discrete Curvature)

Discrete curvature: \( K : V = \{ \text{vertices} \} \to \mathbb{R}^1 \).

\[
K(v) = 2\pi - \sum_i \alpha_i, \quad v \notin \partial M; \\
K(v) = \pi - \sum_i \alpha_i, \quad v \in \partial M
\]

### Theorem (Discrete Gauss-Bonnet theorem)

\[
\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).
\]
Discrete Metrics Determines the Curvatures

\[ v_i \quad v_j \quad v_k \]

\[ l_j \quad \theta_k \quad l_i \]

\[ v_i \quad v_j \quad v_k \]

\[ l_j \quad \theta_k \quad l_i \]

\[ v_i \quad v_j \quad v_k \]

\[ l_j \quad \theta_k \quad l_i \]

\[ R^2 \quad H^2 \quad S^2 \]

Angle and edge length relations: cosine laws \( R^2, H^2, S^2 \)

\[
\cos l_i = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \tag{3}
\]

\[
\cosh l_i = \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} \tag{4}
\]

\[
1 = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} \tag{5}
\]
Discrete Conformal Metric Deformation

Conformal maps Properties
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation
Replace infinitesimal circles by circles with finite radii.
Circle Patterns

There are many local settings for circle patterns. The radius is variable, the intersection angles do not change.
Circle Patterns

\[
\begin{align*}
&\text{(a)} & &\text{(b)} & &\text{(c)} & &\text{(d)} \\
&\gamma_i & &\gamma_i & &\gamma_i & &\gamma_i \\
&\gamma_j & &\gamma_j & &\gamma_j & &\gamma_j \\
&\gamma_k & &\gamma_k & &\gamma_k & &\gamma_k \\
&\theta_i & &\theta_i & &\theta_i & &\theta_i \\
&\theta_j & &\theta_j & &\theta_j & &\theta_j \\
&\theta_k & &\theta_k & &\theta_k & &\theta_k \\
&l_{ij} & &l_{ij} & &l_{ij} & &l_{ij} \\
&l_{ji} & &l_{ji} & &l_{ji} & &l_{ji} \\
&l_{kj} & &l_{kj} & &l_{kj} & &l_{kj} \\
&l_{jk} & &l_{jk} & &l_{jk} & &l_{jk} \\
&l_{ki} & &l_{ki} & &l_{ki} & &l_{ki} \\
&l_{ik} & &l_{ik} & &l_{ik} & &l_{ik} \\
&\Theta_{ij} & &\Theta_{ij} & &\Theta_{ij} & &\Theta_{ij} \\
&\Theta_{ji} & &\Theta_{ji} & &\Theta_{ji} & &\Theta_{ji} \\
&\Theta_{jk} & &\Theta_{jk} & &\Theta_{jk} & &\Theta_{jk} \\
&\Theta_{kj} & &\Theta_{kj} & &\Theta_{kj} & &\Theta_{kj} \\
&\gamma_i & &\gamma_i & &\gamma_i & &\gamma_i \\
&\gamma_j & &\gamma_j & &\gamma_j & &\gamma_j \\
&\gamma_k & &\gamma_k & &\gamma_k & &\gamma_k \\
&\theta_i & &\theta_i & &\theta_i & &\theta_i \\
&\theta_j & &\theta_j & &\theta_j & &\theta_j \\
&\theta_k & &\theta_k & &\theta_k & &\theta_k \\
&l_{ij} & &l_{ij} & &l_{ij} & &l_{ij} \\
&l_{ji} & &l_{ji} & &l_{ji} & &l_{ji} \\
&l_{kj} & &l_{kj} & &l_{kj} & &l_{kj} \\
&l_{jk} & &l_{jk} & &l_{jk} & &l_{jk} \\
&l_{ki} & &l_{ki} & &l_{ki} & &l_{ki} \\
&l_{ik} & &l_{ik} & &l_{ik} & &l_{ik} \\
&\Theta_{ij} & &\Theta_{ij} & &\Theta_{ij} & &\Theta_{ij} \\
&\Theta_{ji} & &\Theta_{ji} & &\Theta_{ji} & &\Theta_{ji} \\
&\Theta_{jk} & &\Theta_{jk} & &\Theta_{jk} & &\Theta_{jk} \\
&\Theta_{kj} & &\Theta_{kj} & &\Theta_{kj} & &\Theta_{kj} \\
\end{align*}
\]

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Conformal Geometry
Discrete Approach

Circle Packing

- Thurston introduced circle packing metric for studying 3-manifolds in 1978.
- Sullivan and Rodin proved Thurston’s circle packing conjecture in 1987.
- The first variational principle for CP metrics was founded by Colin de Verdiere (1991).
- Chow and Luo built the connection between Ricci flow and circle packing in 2003.
- Springborn, Bobenko and Schroder’s circle pattern in 2005.
Circle Packing Metric

**CP Metric**

We associate each vertex $v_i$ with a circle with radius $\gamma_i$. On edge $e_{ij}$, the two circles intersect at the angle of $\Phi_{ij}$. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos \Phi_{ij}$$

CP Metric $(\Sigma, \Gamma, \Phi)$, $\Sigma$ triangulation,

$$\Gamma = \{\gamma_i|\forall v_i\}, \Phi = \{\phi_{ij}|\forall e_{ij}\}$$
Discrete Conformal Factor

Conformal Factor

Defined on each vertex $u : V \to \mathbb{R}$,

$$u_i = \begin{cases} 
\log \gamma_i & \mathbb{R}^2 \\
\log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\
\log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 
\end{cases}$$

Properties

- Symmetry
  $$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation
  $$dK = \Delta du,$$

$\Delta$ is a discrete Laplace-Beltrami operator.

Discrete conformal metric deformation:

\[
\begin{align*}
\frac{y_k}{2} &= e^{u_i} \frac{l_k}{2} e^{u_j} & \mathbb{R}^2 \\
\sinh \frac{y_k}{2} &= e^{u_i} \sinh \frac{l_k}{2} e^{u_j} & \mathbb{H}^2 \\
\sin \frac{y_k}{2} &= e^{u_i} \sin \frac{l_k}{2} e^{u_j} & \mathbb{S}^2
\end{align*}
\]

Properties: \( \frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} \) and \( dK = \Delta du \).
Unified framework for both Discrete Ricci flow and Yamabe flow

- Curvature flow
  \[ \frac{du}{dt} = \bar{K} - K, \]

- Energy
  \[ E(u) = \int \sum_i (\bar{K}_i - K_i) du_i, \]

- Hessian of \( E \) denoted as \( \Delta \),
  \[ dK = \Delta du. \]
One Example: Discrete Yamabe Flow
Definition (Delaunay Triangulation)

Each PL metric $d$ on $(S, V)$ has a Delaunay triangulation $T$, such that for each edge $e$ of $T$,

$$a + a' \leq \pi,$$

It is the dual of Voronoi decomposition of $(S, V, d)$

$$R(v_i) = \{x|d(x, v_j) \leq d(x, v_i) \text{ for all } v_j\}$$
**Discrete Conformality**

**Definition (Conformal change)**

Conformal factor \( u : V \to \mathbb{R} \). Discrete conformal change is vertex scaling.

![Diagram of vertex scaling](image)

proposed by physicists Rocek and Williams in 1984 in the Lorenzian setting. Luo discovered a variational principle associated to it in 2004.
Discrete Yamabe Flow

**Definition (Discrete Yamabe Flow)**

\[
\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i)
\]

**Theorem (Luo)**

The discrete Yamabe flow converge exponentially fast, \( \exists c_1, c_2 > 0, \) such that

\[
|u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t}, |K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t},
\]
Definition (Discrete Conformal Equivalence)

PL metrics $d, d'$ on $(S, V)$ are discrete conformal,

$$d \sim d'$$

if there is a sequence $d = d_1, d_2, \ldots, d_k = d'$ and $T_1, T_2, \ldots, T_k$ on $(S, V)$, such that

1. $T_i$ is Delaunay in $d_i$
2. if $T_i \neq T_{i+1}$, then $(S, d_i) \simeq (S, d_{i+1})$ by an isometry homotopic to $id$
3. if $T_i = T_{i+1}$, $\exists u : V \to \mathbb{R}$, such that $\forall$ edge $e = [v_i, v_j]$,

$$l_{d_{i+1}}(e) = e^{u(v_i)} l_d e^{u(v_j)}$$
Discrete conformal metrics
Main Theorem

**Theorem (Gu-Luo-Sun-Wu (2013))**

∀ PL metrics $d$ on closed $(S, V)$ and $\forall \bar{K} : V \to (-\infty, 2\pi)$, such that $\sum \bar{K}(v) = 2\pi \chi(S)$, $\exists$ a PL metric $\bar{d}$, unique up to scaling on $(S, V)$, such that

1. $\bar{d}$ is discrete conformal to $d$
2. The discrete curvature of $\bar{d}$ is $\bar{K}$.

Furthermore, $\bar{d}$ can be found from $d$ from a discrete curvature flow.

**Remark**

$\bar{K} = \frac{2\pi \chi(S)}{|V|}$, discrete uniformization.
Main Theorem

Theorem (Gu-Luo-Wu (2016))

*Discrete conformal mapping produced by dynamic Yambe flow converges to the smooth conformal mapping.*
Main Theorem

1. The uniqueness of the solution is obtained by the convexity of discrete surface Ricci energy and the convexity of the admissible conformal factor space ($u$-space).

2. The existence is given by the equivalence between PL metrics on $(S, V)$ and the decorated hyperbolic metrics on $(S, V)$ and the Ptolemy identity.

Algorithm

Input: a closed triangle mesh $M$, target curvature $\bar{K}$, step length $\delta$, threshold $\varepsilon$
Output: a PL metric conformal to the original metric, realizing $\bar{K}$.

1. Initialize $u_i = 0$, $\forall v_i \in V$.
2. compute edge length, corner angle, discrete curvature $K_i$
3. update to Delaunay triangulation by edge swap
4. compute edge weight $w_{ij}$.
5. $u^+ = \delta \Delta^{-1}(\bar{K} - K)$
6. normalize $u$ such that the mean of $u_i$’s is 0.
7. repeat step 2 through 6, until the max $|\bar{K}_i - K_i| < \varepsilon$. 
Computational results for genus 2 and genus 3 surfaces.
Hyperbolic Koebe’s Method
Applications
Computer Graphics
Computer Graphics Application

Graphics

- Surface Parameterization, texture mapping
- Texture synthesis, transfer
- Vector field design
- Shape space and retrieval.
Surface Parameterization

Map the surfaces onto canonical parameter domains
Surface Parameterization

Applied for texture mapping.
Design vector fields on surfaces with prescribed singularity positions and indices.
Convert the surface to knot structure using smooth vector fields.
Texture Transfer

Transfer the texture between high genus surfaces.
Compute polycube maps for high genus surfaces.
Normal Map

[Images of two 3D models, possibly representing a gargoyle, with normal maps applied, showing the difference in lighting and shading.]
- Quasi-conformal geometry controls angle-distortion;
- Optimal mass transportation map controls area-distortion;
Figure: Volumetric morphing using our method.
Figure: Volumetric morphing using our method.
Computer Vision
## Vision

- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.
Isometric deformation is conformal. The mask is bent without stretching.
Facial expression change is not-conformal.
3D surface matching is converted to image matching by using conformal mappings.
Face Surfaces with Different Expressions are Matched
Face Expression Tracking
Face Expression Tracking
Single Mesh for facial expression transfer.
Face Expression Tracking

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Conformal Geometry
Face Expression Tracking
Surface Registration

(a) Armadillo #1
(b) Armadillo #2
(c) APP map #1
(d) APP map #2
(e) Conformal map #1
(f) Conformal map #2
\{2D \text{ Contours}\} \cong \frac{\{\text{Diffeomorphism on } S^1\} \cup \{\text{Conformal Module}\}}{\{\text{Mobius Transformation}\}}
Given a metric surface \((S, g)\), a Riemann mapping \(\varphi : (S, g) \rightarrow \mathbb{D}^2\), the conformal factor \(e^{2\lambda}\) gives a probability measure on the disk. The shape distance is given by the Wasserstein distance.

Fig. 9: The computation of Wasserstein distance
Fig. 10: Face surfaces for expression clustering. The first row is “sad”, the second row is “happy” and the third row is “surprise”.
Expression Classification

Compute the Wasserstein distances among all the facial surfaces, isometrically embed on the plane using MDS method, perform clustering.
Curvature Sensitive Remeshing
# Curvature Sensitive Remeshing

## Algorithm Pipeline

1. Compute the conformal parameterization of the input surface \( \varphi : (S, g) \rightarrow (\mathbb{D}, dzd\bar{z}) \),

2. Compute the optimal mass transportation map \( \psi : (\mathbb{D}, dxdy) \rightarrow (\mathbb{D}, \mu) \), where \( \mu \) is the combination of the surface area element \( e^{2\lambda} dxdy \) and the absolute value of the Gaussian curvature measure \( |K(x, y)| dxdy \),

3. Uniformly sample on the preimage of the OMT map \( \psi^{-1}(\mathbb{D}) \),

4. Pull back the samples to the conformal parameter domain \( \varphi(S) \), compute the Delaunay triangulation \( T \),

5. Pull back the triangulation \( T \) to the original surface \( S \), which induces the remeshing of \( S \).
Curvature Sensitive Parameterization (CSP)

Original face mesh

Conformal parameterization
Curvature Sensitive Parameterization (CSP)

Original face mesh

Area preserving parameterization
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.1$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature \times 0.2
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.4$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 0.8$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 1.0$
Curvature Sensitive Parameterization (CSP)

Original face mesh

CSP: area + curvature $\times 2.0$
Compare Multi-scale remeshing between APP and CSP

(a) APP
(b) original mesh (140K)
(c) CSP
Compare Multi-scale remeshing between APP and CSP

Remeshing: 1K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 2K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 4K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe

David Gu  Conformal Geometry
Compare Multi-scale remeshing between APP and CSP

Remeshing: 8K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 16K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 32K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Compare Multi-scale remeshing between APP and CSP

Remeshing: 64K vertices

(a) APP wireframe  (b) APP smooth  (c) CSP smooth  (d) CSP wireframe
Meshing

Theorem

Suppose $S$ is a surface with a Riemannian metric. Then there exist meshing method which ensures the convergence of curvatures.

Key idea: Delaunay triangulations on uniformization domains. Angles are bounded, areas are bounded.
Theorem

Let $M$ be a compact Riemannian surface embedded in $\mathbb{R}^3$ with the induced Euclidean metric, $T$ the triangulation generated by Delaunay refinement on conformal uniformization domain, with circumradius bound $\varepsilon$. If $B$ is the relative interior of a union of triangles of $T$, then

$$|\phi^G_T(B) - \phi^G_M(\pi(B))| \leq K\varepsilon$$

$$|\phi^H_T(B) - \phi^H_M(\pi(B))| \leq K\varepsilon$$

where $\pi : T \rightarrow M$ is the closest point projection, $\phi^H, \phi^G$ are the mean and Gaussian curvature measures, where

$$K = O(\text{area}(B)) + O(\text{length}(\partial B)).$$
Hexahedral Meshing
Genus Zero Case

(a) Stanford bunny  
(b) Spherical mapping  
(c) Cube mapping
(d) Solid bunny (e) Solid ball mapping (f) Solid cube mapping
Figure: A genus one closed surface can be conformally and periodically mapped onto the plane, each fundamental domain is a parallelogram. The subdivision of the parallelogram induces a quad-mesh of the surface.
Figure: The interior of the kitten surface is mapped onto a canonical solid cylinder.
Figure: Input Surface $\partial \Omega$. 
Figure: Tetrahedral meshing $\Omega$. 
Figure: Admissible curve system \( \{ \gamma_1, \gamma_2, \cdots, \gamma_{3g-3} \} \), pants decomposition \( \{ P_1, P_2, \cdots, P_{2g-2} \} \).
Figure: Pairs of pants \( \{P_1, P_2, \ldots, P_{2g-2}\} \).
Figure: Pants decomposition graph.
Figure: Foliation, Holomorphic quadratic differential.
Figure: Admissible curve system, pants decomposition.
Figure: Pants decomposition graph.
Figure: Foliation, Holomorphic quadratic differential.
**Figure:** Critical horizontal trajectories and vertical trajectories.
Figure: Quad-Mesh $\mathcal{Q}$ and the cylindrical decomposition.
Figure: Cylindrical decomposition.
Algorithmic Pipeline

Figure: Colorable quadrilateral mesh, all vertex valences are even.
Figure: Left solid cylinder, maps to the canonical solid cylinder.
Figure: Hexahedral meshing of solid cylinders.
Figure: Hexahedral mesh $\mathcal{H}$ of the interior volume $\Omega$. 
Figure: Hexahedral mesh $\mathcal{H}$ of the interior volume $\Omega$.
Manifold Spline

- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.
Extraordinary Points

- Fully control the number, the index and the position of extraordinary points.
- For surfaces with boundaries, splines without extraordinary point can be constructed.
- For closed surfaces, splines with only one singularity can be constructed.
Manifold Spline

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Conformal Geometry
Converting a polygonal mesh to TSplines with multiple resolutions.
Manifold Spline

Converting scanned data to spline surfaces.
Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.
Converting scanned data to spline surfaces, the control points, knot structure are shown.
Polygonal mesh to spline, control net and the knot structure.
Manifold Spline

volumetric spline.
Visualization

(a) Front view  
(b) Angle-preserving  
(c) Area-preserving  
(d) Back view

Angle-preserving parameterization vs. area-preserving parameterization

(a) 2x  
(b) 3x  
(c) 4x  
(d) 6x

Importance driven parameterization. The Buddha's head region is magnified by different factors
Wireless Sensor Network
Wireless Sensor Network

- Detecting global topology.
- Routing protocol.
- Load balancing.
- Isometric embedding.
Greedy Routing

Given sensors on the ground, because of the concavity of the boundaries, greedy routing doesn’t work.
Greedy Routing

Map the network to a circle domain, all boundaries are circles, greedy routing works.
Schoktty Group - Circular Reflection
Optimal Planar Graph Embedding.
Thurston-Andreev Theorem

A planar graph can be embedded on the unit sphere, such that the face circles are orthogonal to vertex circles; the circles at the vertices of an edge are tangent to each other. Such kind of embedding differ by a Möbius transformation.
Canonical Homotopy Class Representative
Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.
Shortest word Problem (NP Hard):

\[
\gamma = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = (a_3 b_3 a_3^{-1} b_3^{-1})^{-1}
\]
Loop Lifting
Lifting a loop from base surface to the universal covering space.
Birkoff Curve Shorting

Birkoff curve shortening deforms a loop to a geodesic.
Birkoff curve shortening deforms a loop to a geodesic.
1. Compute the uniformization metric using Ricci flow.
2. Compute the geodesic loop by Birkhoff curve shortening.
3. Lift the geodesic loop to the universal covering space.
4. Trace the lifted loop to compute the word.
Medical Imaging Application

Medical Imaging

Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.
Registration problem:

Moving

Target
Landmark only:

Target

Landmark
Intensity only:

Target

Intensity
Landmark + intensity only:

Target

Landmark + intensity
Intensity only:

Target

Intensity
Landmark + Intensity:

Target

Intensity + Landmark
Medical registration: Vertebral bone

Subject 1

(A)

Subject 2

(B)
Registration result:
Conformal Brain Mapping

Brain Cortex Surface

Conformal Brain Mapping for registration, matching, comparison.
Using conformal module to analyze shape abnormalities.
Automatic sulcal landmark Tracking

- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically.

![Extraction of high mean curvature region by Chan-Vese segmentation](image)

\[
F(c_1, c_2, \psi) = \int_S (I_f - c_1)^2 H(\psi) dS + \int_S (I_f - c_2)^2 (1 - H(\psi)) dS \\
+ \nu \int_S |\nabla_S H(\psi)| dS
\]

\[
c_1 = \frac{\int_D I_f \circ \phi(x, y) H(\psi \circ \phi(x, y)) \lambda(x, y) dxdy}{\int_D H(\psi \circ \phi(x, y)) \lambda(x, y) dxdy}
\]

\[
c_2 = \frac{\int_D I_f \circ \phi(x, y) (1 - H(\psi \circ \phi(x, y))) \lambda(x, y) dxdy}{\int_D (1 - H(\psi \circ \phi(x, y))) \lambda(x, y) dxdy}
\]

Euler Lagrange equation is:

\[
\frac{\partial \psi}{\partial t} = \lambda \delta(\psi) \left[ \text{div}_S \left( \frac{\nabla_S \psi}{||\nabla_S \psi||_S} \right) - (I_f - c_1)^2 - (I_f - c_2)^2 \right] \text{ or }
\]

\[
\frac{\partial \psi \circ \phi}{\partial t} = \lambda \delta(\psi \circ \phi) \left[ \text{div} \left( \frac{1}{\lambda} \sqrt{\frac{\nabla \psi \circ \phi}{||\nabla \psi \circ \phi||}} \right) - (I_f \circ \phi - c_1)^2 - (I_f \circ \phi - c_2)^2 \right]
\]
Abnormality detection on brain surfaces

The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.
Abnormality detection on brain surfaces

The brain is undergoing gyri thickening (commonly observed in Williams Syndrome). The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.

\[
\text{Beltrami Index (BI)} = \int_D \mu(z) \|dz\ /	ext{Area}(D)
\]

\(\mu\) = Beltrami coefficient

David Gu

Conformal Geometry
Alzheimer Study
Colon cancer is the 4th killer for American males. Virtual colonoscopy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.
Colon Flattening
Virtual Colonoscopy

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.
Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.
Colon Registration
Brain Morphometry

IQ from shape
IQ from shape

Cross validation accuracy (%) vs. k
3D Fabrication

Figure: Strips for the genus two surface model.

David Gu
Conformal Geometry
3D Fabrication

Figure: Fabrication by plain weaving.

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Conformal Geometry
Figure: Fabrication by plain weaving.
Summary

- Conformal structure is more flexible than Riemannian metric
- Conformal structure is more rigid than topology
- Conformal geometry can be used for a broad range of engineering applications.
For more information, please email to gu@cs.stonybrook.edu.

Thank you!