Surface and Volume Based Techniques for Shape Modeling and Analysis

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SIGGRAPH Asia 2013 Course

Surface Ricci Flow

Basic Concepts

Manifold

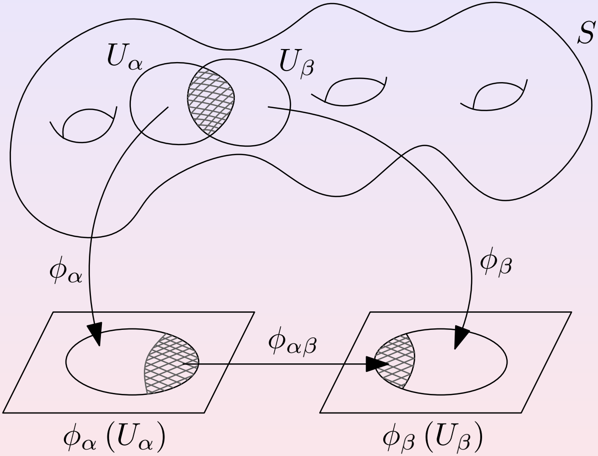
Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then M is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Manifold



S
U_α
U_β
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Holomorphic Function

Definition (Holomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex function, $f : x + iy \to u(x,y) + iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

$$dz = dx + idy, d\bar{z} = dx - idy,$$

then the dual operators

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

then if $\frac{\partial f}{\partial \bar{z}} = 0$, then f is holomorphic.

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta} : \mathbb{C} \to \mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

Conformal Structure

The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose S is a topological surface, consider all the conformal atlases on M, classified by the compatible relation

$$\{all\ conformal\ atlas\}/\sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

Smooth map

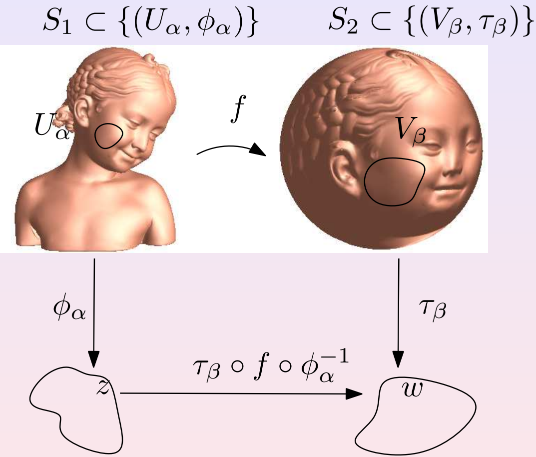
Definition (Smooth map)

Suppose $f : S_1 \to S_2$ is a map between two smooth manifolds. For each point p, choose a chart of S_1, (U_α, ϕ_α), $p \in U_\alpha$). The image $f(U_\alpha) \subset V_\beta$, (V_β, τ_β) is a chart of S_2. The local representation of f

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \tau_\beta(V_\beta)$$

is smooth, then f is a smooth map.

Map between Manifolds



Definition (Riemannian Metric)

A Riemannian metric on a smooth manifold M is an assignment of an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$, $\forall p \in M$, such that

1. $g_p(a_1X_1 + a_2X_2, b_1Y_1 + b_2Y_2) = \sum_{i,j=1}^{2} a_i b_j g_p(X_i, Y_j)$.
2. $g_p(X,Y) = g_p(Y,X)$
3. g_p is non-degenerate.
4. $\forall p \in M$, there exists local coordinates $\{x^i\}$, such that $g_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are C^∞ functions.

Pull back Riemannian Metric

Definition (Pull back Riemannian metric)

Suppose $f:(M,\mathbf{g})\to(N,\mathbf{h})$ is a smooth mapping between two Riemannian manifolds, $\forall p\in M$, $f_*: T_pM \to T_{f(p)}N$ is the tangent map. The pull back metric $f^*\mathbf{h}$ induced by the mapping f is given by

$$f^*h(X_1,X_2) := h(f_*X_1, f_*X_2), \forall X_1, X_2 \in T_pM.$$

Local representation of the pull back metric is given by

$$f^*\mathbf{h} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Conformal Structure

Definition (Conformal equivalent metrics)

Suppose $\mathbf{g}_1, \mathbf{g}_2$ are two Riemannian metrics on a manifold M, if

$$\mathbf{g}_1 = e^{2u}\mathbf{g}_2, u : M \to \mathbb{R}$$

then $\mathbf{g}_1$ and $\mathbf{g}_2$ are conformal equivalent.

Definition (Conformal Structure)

Consider all Riemannian metrics on a topological surface S, which are classified by the conformal equivalence relation,

$$\{\textit{Riemannian metrics on } S\}/\sim,$$

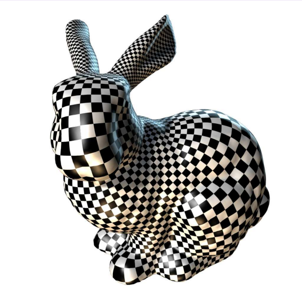
each equivalence class is called a conformal structure.

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface M with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Riemannian metric vs Conformal Structure

Definition (Isothermal coordinates)

Suppose $(S,\mathbf{g})$ is a metric surface, (U_α,ϕ_α) is a coordinate chart, (x,y) are local parameters, if

$$\mathbf{g} = e^{2u}(dx^2+dy^2),$$

then we say (x,y) are isothermal coordinates.

Theorem

Suppose S is a compact metric surface, for each point p, there exits a local coordinate chart (U,ϕ), such that $p\in U$, and the local coordinates are isothermal.

Riemannian metric and Conformal Structure

Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

All compact metric surfaces are Riemann surfaces.

Smooth Surface Ricci Flow

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface M with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

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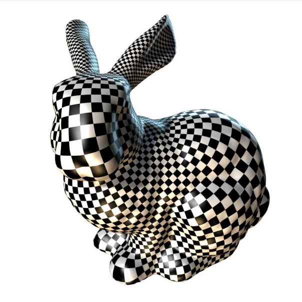


Gaussian Curvature

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

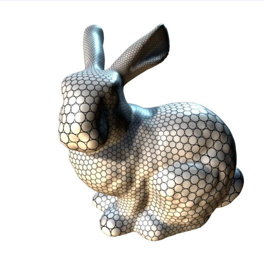
Suppose M is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(K - \Delta_{\mathbf{g}}\lambda),$$

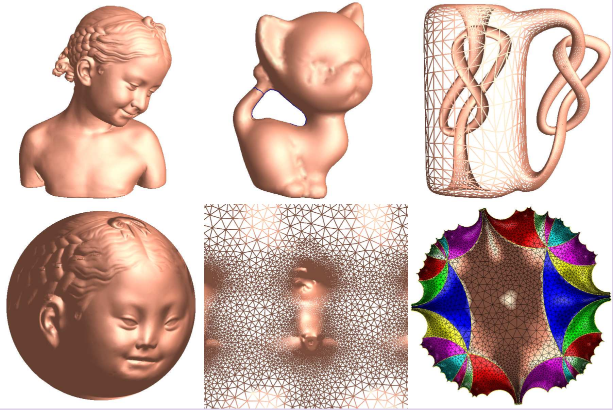
geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(k_g - \partial_{\mathbf{g},n}\lambda).$$

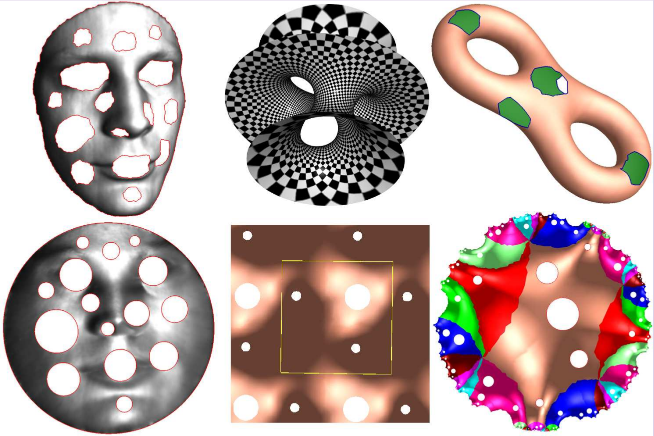
Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ *be a compact 2-dimensional Riemannian manifold. Then there is a metric* $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ *conformal to* $\mathbf{g}$ *which has constant Gauss curvature.*



Uniformization of Open Surfaces



Surface Ricci Flow

Key Idea

$$K = -\Delta_{\mathbf{g}}\lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}\frac{d\lambda}{dt}$$

Let $\frac{d\lambda}{dt} = -K$,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + K^2$$

Heat equation!

Surface Ricci Flow

Definition (Hamilton's Surface Ricci Flow)

A closed surface S with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = \left(\frac{4\pi\chi(S)}{A(0)} - 2K\right) g_{ij}.$$

where $\chi(S)$ is the Euler characteristic number of S, $A(0)$ is the initial total area.

The ricci flow preserves the total area during the flow, converge to a metric with constant Gaussian curvature $\frac{4\pi\chi(S)}{A(0)}$.

Ricci Flow

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + K^2$$

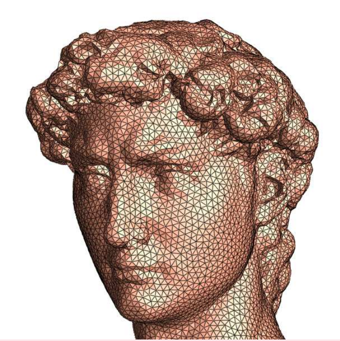
- Ricci flow

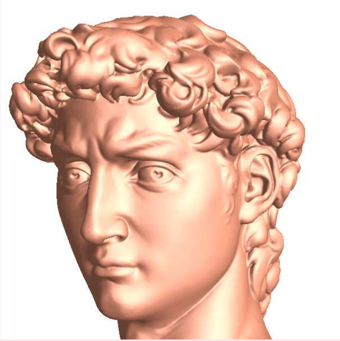
$$\frac{du}{dt} = \bar{K} - K.$$

Discrete Surface

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





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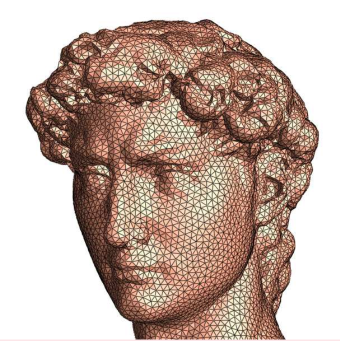

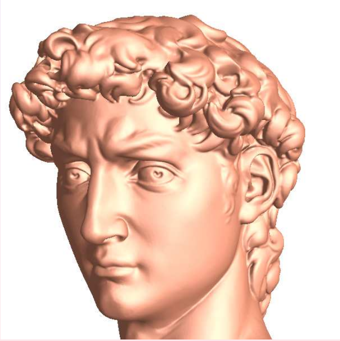

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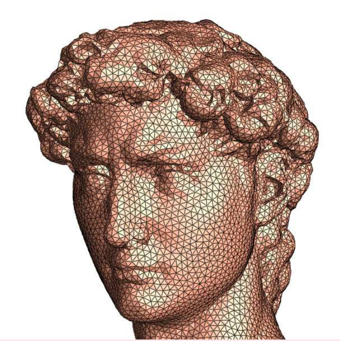
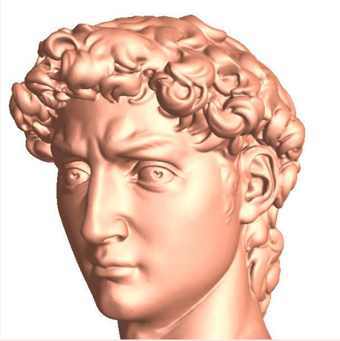




Discrete Generalization

Concepts

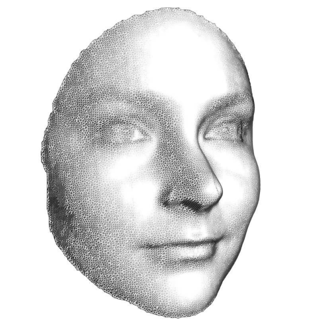
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

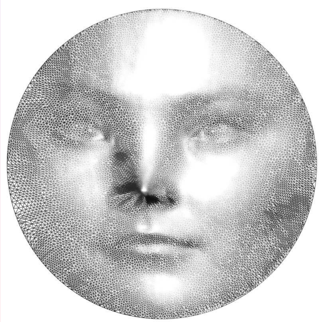
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

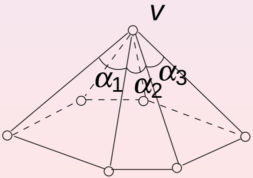
Definition (Discrete Curvature)

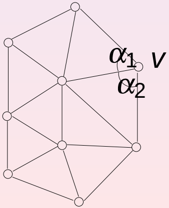
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

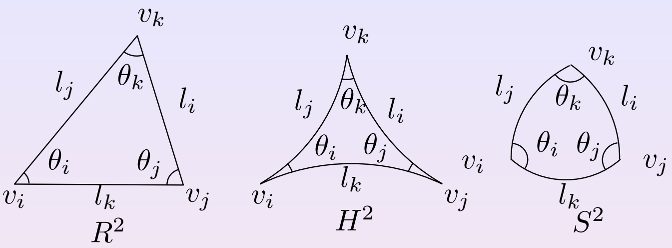
$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$







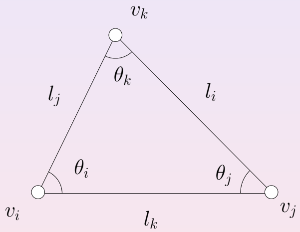
cosine laws

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j\cosh\theta_k}{\sinh\theta_j\sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (3)$$

Derivative cosine law



Lemma (Derivative Cosine Law)

Suppose corner angles are the functions of edge lengths, then

$$\frac{\partial \theta_i}{\partial l_i} = \frac{l_i}{A}$$

$$\frac{\partial \theta_i}{\partial l_j} = -\frac{\partial \theta_i}{\partial l_i} \cos \theta_k$$

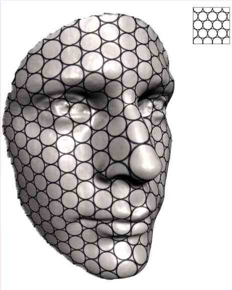
where $A = l_j l_k \sin \theta_i$.

Discrete Conformal Structure

Discrete Conformal Metric Deformation

Conformal maps Properties

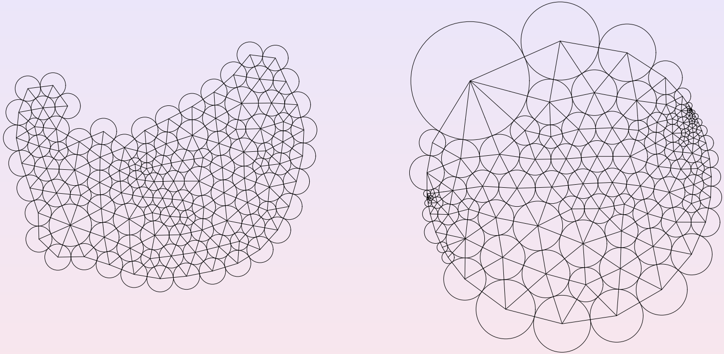
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.



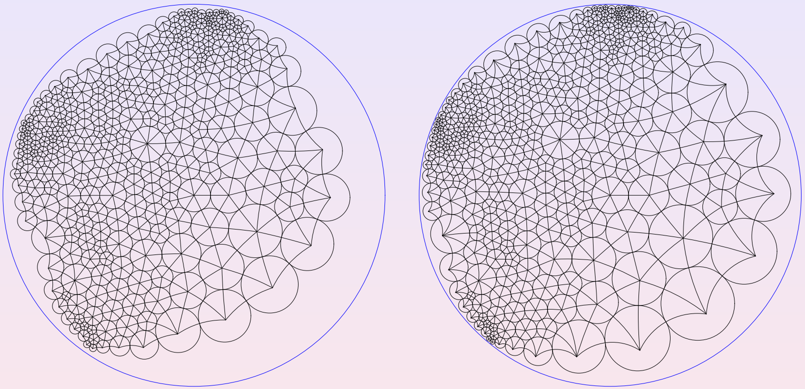
Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

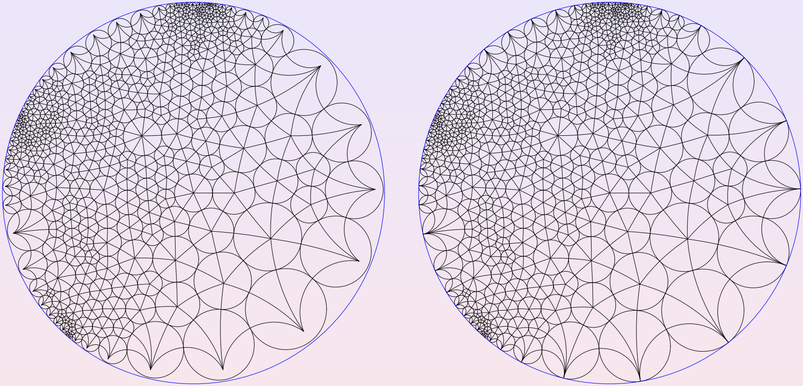
Discrete Conformal Metric Deformation vs CP



Discrete Conformal Metric Deformation vs CP



Discrete Conformal Metric Deformation vs CP



Thurston's Circle Packing Metric

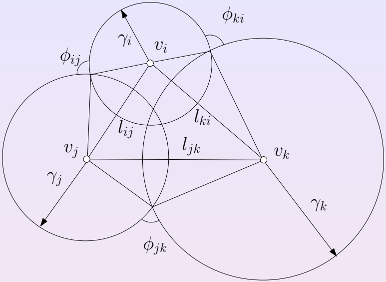
Thurston's CP Metric

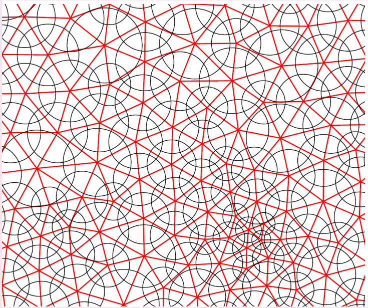
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (T, Γ, Φ), T triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$



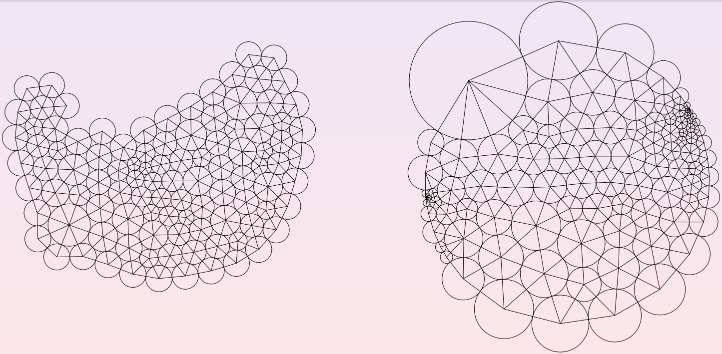


Discrete Conformal Equivalence Metrics

Definition

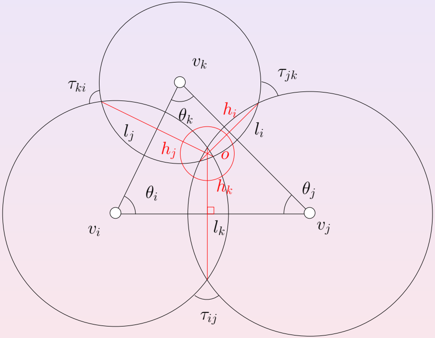
Conformal Equivalence Two CP metrics (T_1, Γ_1, Φ_1) and (T_2, Γ_2, Φ_2) are conformal equivalent, if they satisfy the following conditions

$$T_1 = T_2 \text{ and } \Phi_1 = \Phi_2.$$



Definition (Power Circle)

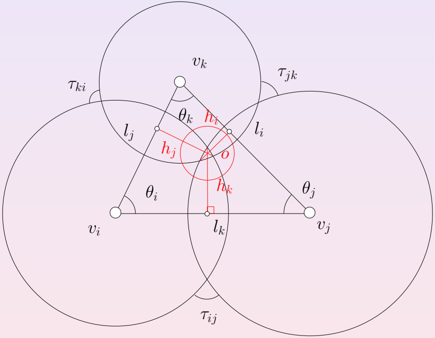
The unit circle orthogonal to three circles at the vertices (v_i,γ_i), (v_j,γ_j) and (v_k,γ_k) is called the power circle. The center is called the power center. The distance from the power center to three edges are denoted as h_i, h_j, h_k respectively.



Theorem (Symmetry)

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$
$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$
$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Therefore the differential 1-form $\omega = \theta_i du_i + \theta_j du_j + \theta_k du_k$ is closed.

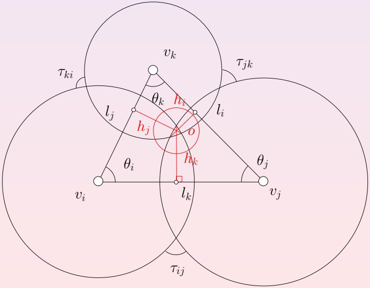


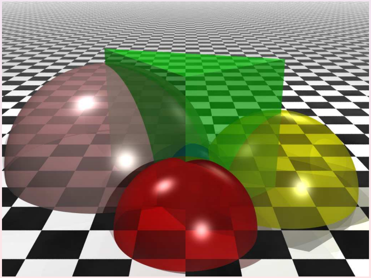
Definition (Discrete Ricci Energy)

The functional associated with a CP metric on a triangle is

$$E(\mathbf{u}) = \int_{(0,0,0)}^{(u_i,u_j,u_k)} \theta_i(\mathbf{u})du_i + \theta_j(\mathbf{u})du_j + \theta_k(\mathbf{u})du_k.$$

Geometrical interpretation: the volume of a truncated hyperbolic hyper-ideal tetrahedron.

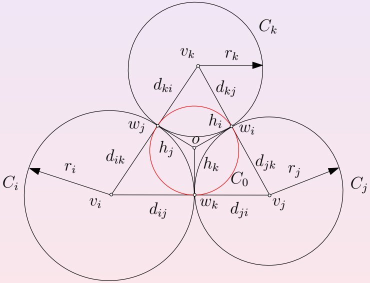


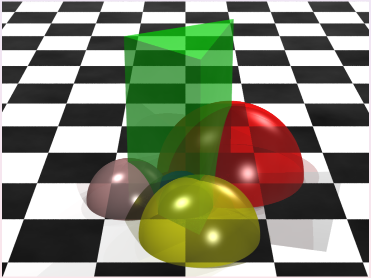


Generalized Circle Packing/Pattern

Definition (Tangential Circle Packing)

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j.$$



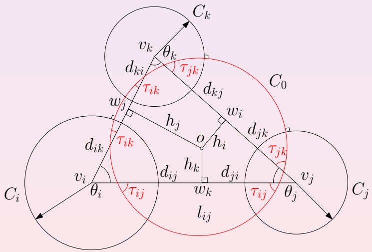


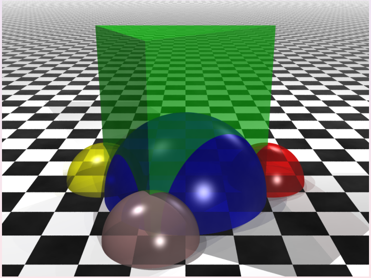
Generalized Circle Packing/Pattern

Definition (Inversive Distance Circle Packing)

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\eta_{ij}.$$

where $\eta_{ij} > 1$.



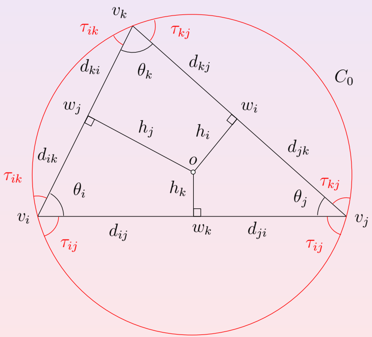


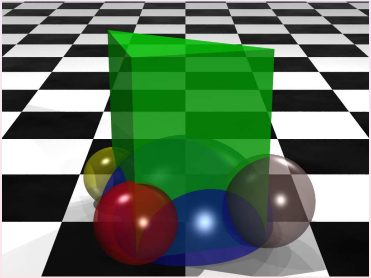
Generalized Circle Packing/Pattern

Definition (Discrete Yamabe Flow)

$$l_{ij}^2 = 2\gamma_i \gamma_j \eta_{ij}.$$

where $\eta_{ij} > 0$.





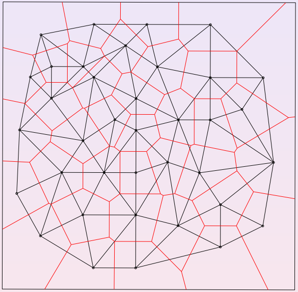
Voronoi Diagram

Definition (Voronoi Diagram)

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$, the Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | |\mathbf{x} - p_i|^2 \leq |\mathbf{x} - p_j|^2, \forall j\}.$$

The dual triangulation to the Voronoi diagram is called the Delaunay triangulation.

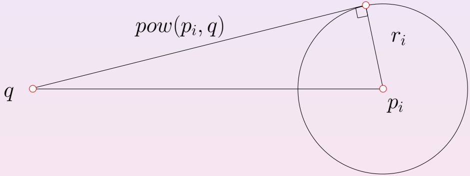


Power Distance

Power Distance

Given $\mathbf{p}_i$ associated with a sphere $(\mathbf{p}_i, r_i)$ the power distance from $\mathbf{q} \in \mathbb{R}^n$ to $\mathbf{p}_i$ is

$$pow(\mathbf{p}_i, \mathbf{q}) = |\mathbf{p}_i - \mathbf{q}|^2 - r_i^2.$$



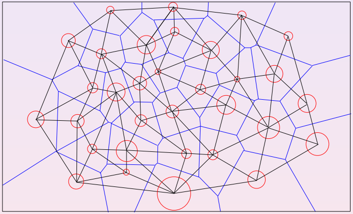
Power Diagram

Definition (Power Diagram)

Given $p_1, \cdots, p_k$ in $\mathbb{R}^n$ and sphere radii $\gamma_1, \cdots, \gamma_k$, the power Voronoi cell W_i at p_i is

$$W_i = \{\mathbf{x} | Pow(\mathbf{x}, p_i) \leq Pow(\mathbf{x}, p_j), \forall j\}.$$

The dual triangulation to Power diagram is called the Power Delaunay triangulation.



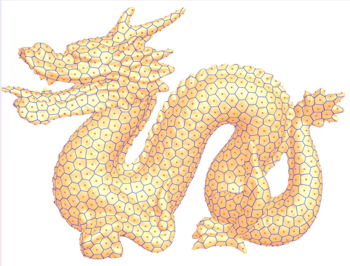
Voronoi Diagram Delaunay Triangulation

Definition (Voronoi Diagram)

Let (S,V) be a punctured surface, V is the vertex set. d is a flat cone metric, where the cone singularities are at the vertices. The Voronoi diagram is a cell decomposition of the surface, Voronoi cell W_i at v_i is

$$W_i = \{\mathbf{p} \in S | d(\mathbf{p}, v_i) \leq d(\mathbf{p}, v_j), \forall j\}.$$

The dual triangulation to the voronoi diagram is called the Delaunay triangulation.



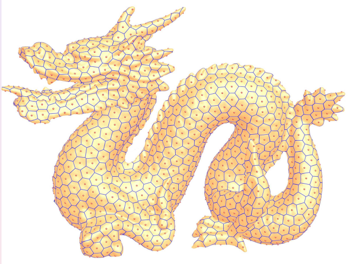
Power Voronoi Diagram Delaunay Triangulation

Definition (Power Diagram)

Let (S,V) be a punctured surface, with a generalized circle packing metric. The Power diagram is a cell decomposition of the surface, a Power cell W_i at v_i is

$$W_i = \{\mathbf{p} \in S | Pow(\mathbf{p}, v_i) \leq Pow(\mathbf{p}, v_j), \forall j\}.$$

The dual triangulation to the power diagram is called the power Delaunay triangulation.

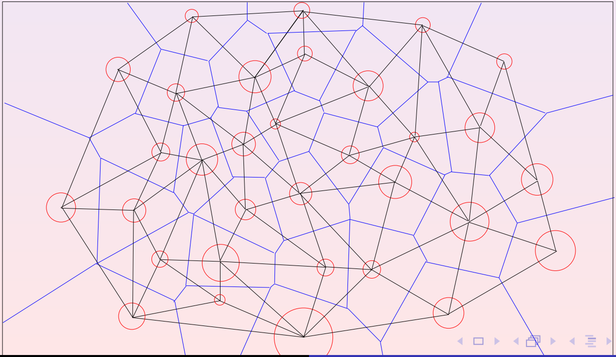


Edge Weight

Definition (Edge Weight)

(S, V, d), d a generalized CP metric. D the Power diagram, T the Power Delaunay triangulation. $\forall e \in D$, the dual edge $\bar{e} \in T$, the weight

$$w(e) = \frac{|e|}{|\bar{e}|}.$$



Discrete Surface Ricci Flow

Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u}: V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Discrete Surface Ricci Flow

Definition (Discrete Surface Ricci Flow with Surgery)

Suppose (S, V, d) is a triangle mesh with a generalized CP metric, the discrete surface Ricci flow is given by

$$\frac{du_i}{dt} = \bar{K}_i - K_i,$$

where $\bar{K}_i$ is the target curvature. Furthermore, during the flow, the Triangulation preserves to be Power Delaunay.

Theorem (Exponential Convergence)

The flow converges to the target curvature $K_i(\infty) = \bar{K}_i$. Furthermore, there exists $c_1, c_2 > 0$, such that

$$|K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t}, |u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t},$$

Discrete Conformal Metric Deformation

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i} = -w_{ij}$$

- Discrete Laplace Equation

$$dK_i = \sum_{[v_i,v_j]\in E} w_{ij}(du_i - du_j)$$

namely

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Discrete Laplace-Beltrami operator

Definition (Laplace-Beltrami operator)

Δ is the discrete Lapalce-Beltrami operator, $\Delta = (d_{ij})$, where

$$d_{ij} = \begin{cases} \sum_k w_{ik} & i = j \\ -w_{ij} & i \neq j, [v_i, v_j] \in E \\ 0 & otherwise \end{cases}$$

Lemma

Given (S, V, d) with generalized CP metric, if T is the Power Delaunay triangulation, then Δ is positive definite on the linear space $\sum_i u_i = 0$.

Because Δ is diagonal dominant.

Definition (Discrete Surface Ricci Energy)

Suppose (S, V, d) is a triangle mesh with a generalized CP metric, the discrete surface energy is defined as

$$E(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{u}} \sum_{i=1}^{k} (\bar{K}_i - K_i) du_i.$$

1. gradient $\nabla E = \bar{\mathbf{K}} - \mathbf{K}$,
2. Hessian
$$\left(\frac{\partial^2 E}{\partial u_i \partial u_j} \right) = \Delta,$$
3. Ricci flow is the gradient flow of the Ricci energy,
4. Ricci energy is concave, the solution is the unique global maximal point, which can be obtained by Newton's method.

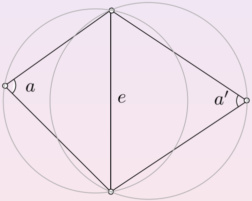
One Example: Discrete Yamabe Flow

Delaunay Triangulation

Definition (Delaunay Triangulation)

Each PL metric d on (S,V) has a Delaunay triangulation T, such that for each edge e of T,

$$a + a' \leq \pi,$$

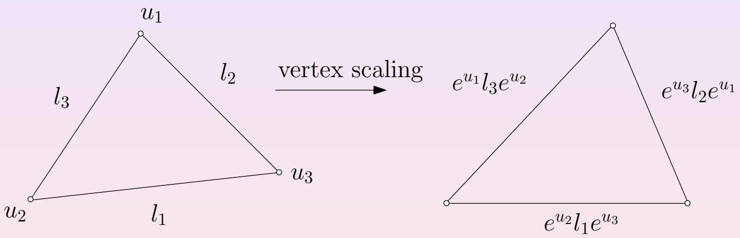


It is the dual of Voronoi decomposition of (S,V,d)

$$R(v_i) = \{x | d(x,v_i) \leq d(x,v_j) \text{ for all } v_j\}$$

Definition (Conformal change)

Conformal factor $u : V \to \mathbb{R}$. Discrete conformal change is vertex scaling.



proposed by physicists Rocek and Williams in 1984 in the Lorenzian setting. Luo discovered a variational principle associated to it in 2004.

Discrete Yamabe Flow

Definition (Discrete Yamabe Flow)

$$\frac{du(v_i)}{dt} = \bar{K}(v_i) - K(v_i)$$

Theorem (Luo)

The discrete Yamabe flow converge exponentially fast, $\exists c_1, c_2 > 0$*, such that*

$$|u_i(t) - u_i(\infty)| < c_1 e^{-c_2 t}, |K_i(t) - K_i(\infty)| < c_1 e^{-c_2 t},$$

Discrete Conformality

Definition (Discrete Conformal Equivalence)

PL metrics d, d' on (S, V) are discrete conformal,

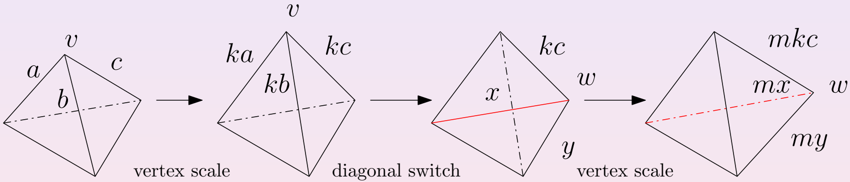
$$d \sim d'$$

if there is a sequence $d = d_1, d_2, \cdots, d_k = d'$ and $T_1, T_2, \cdots, T_k$ on (S, V), such that

1. T_i is Delaunay in d_i
2. if $T_i \neq T_{i+1}$, then $(S, d_i) \cong (S, d_{i+1})$ by an isometry homotopic to *id*
3. if $T_i = T_{i+1}$, $\exists u : V \to \mathbb{R}$, such that $\forall$ edge $e = [v_i, v_j]$,

$$l_{d_{i+1}}(e) = e^{u(v_i)} l_{d_i} e^{u(v_j)}$$

Discrete conformal metrics



Main Theorem

Theorem (Gu-Luo-Sun-Wu (2013))

$\forall$ PL metrics d on closed (S,V) and $\forall \bar{K}: V \to (-\infty, 2\pi)$, such that $\sum \bar{K}(v) = 2\pi\chi(S)$, $\exists$ a PL metric $\bar{d}$, unique up to scaling on (S,V), such that

1. *$\bar{d}$ is discrete conformal to d*
2. *The discrete curvature of $\bar{d}$ is $\bar{K}$.*

Furthermore, $\bar{d}$ can be found from d from a discrete curvature flow.

Remark

$\bar{K} = \frac{2\pi\chi(S)}{|V|}$, discrete uniformization.

Main Theorem

1. The uniqueness of the solution is obtained by the convexity of discrete surface Ricci energy and the convexity of the admissible conformal factor space (u-space).
2. The existence is given by the equivalence between PL metrics on (S, V) and the decorated hyperbolic metrics on (S, V) and the Ptolemy identity.

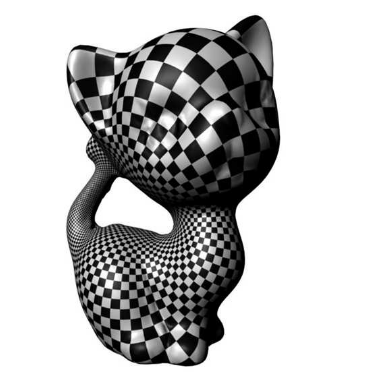
X. Gu, F. Luo, J. Sun, T. Wu, ”A discrete uniformization theorem for polyhedral surfaces”, arXiv:1309.4175.

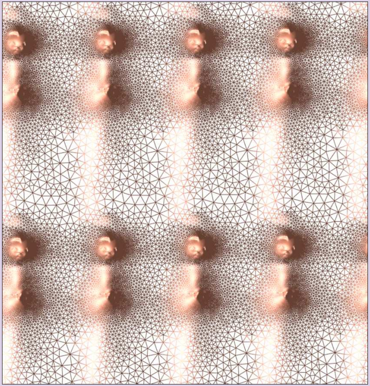
Algorithm

Input: a closed triangle mesh M, target curvature $\bar{K}$, step length δ, threshold ε
Output:a PL metric conformal to the original metric, realizing $\bar{K}$.

1. Initialize $u_i = 0, \forall v_i \in V$.
2. compute edge length, corner angle, discrete curvature K_i
3. update to Delaunay triangulation by edge swap
4. compute edge weight w_{ij}.
5. $\mathbf{u} += \delta \Delta^{-1}(\bar{\mathbf{K}} - \mathbf{K})$
6. normalize $\mathbf{u}$ such that the mean of u_i's is 0.
7. repeat step 2 through 6, until the max $|\bar{K}_i - K_i| < \varepsilon$.

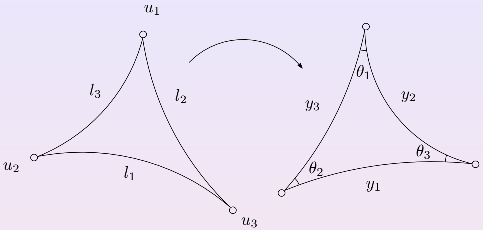
Genus One Example





Hyperbolic Discrete Surface Yamabe Flow

Discrete conformal metric deformation:



conformal factor

$$\begin{aligned} \frac{y_k}{2} &= e^{u_i} \frac{l_k}{2} e^{u_j} & \mathbb{R}^2 \\ \sinh \frac{y_k}{2} &= e^{u_i} \sinh \frac{l_k}{2} e^{u_j} & \mathbb{H}^2 \\ \sin \frac{y_k}{2} &= e^{u_i} \sin \frac{l_k}{2} e^{u_j} & \mathbb{S}^2 \end{aligned}$$

Properties: $\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$ and $d\mathbf{K} = \Delta d\mathbf{u}$.

Unified framework for both Discrete Ricci flow and Yamabe flow

- Curvature flow

$$\frac{du}{dt} = \bar{K} - K,$$

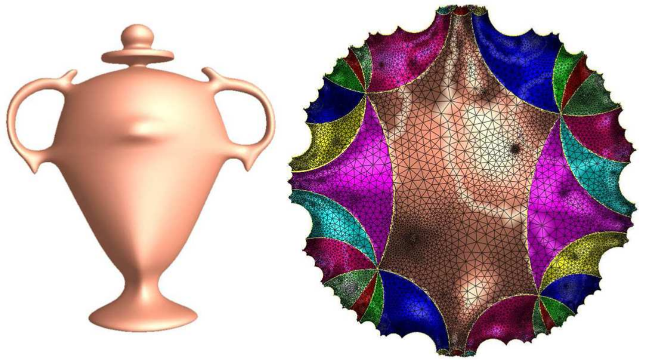
- Energy

$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$

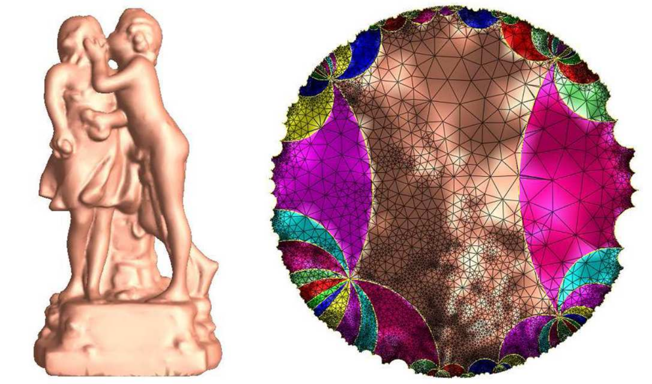
- Hessian of E denoted as Δ,

$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Genus Two Example



Genus Three Example



Computational Algorithms

Topological Quadrilateral

Topological Quadrilateral

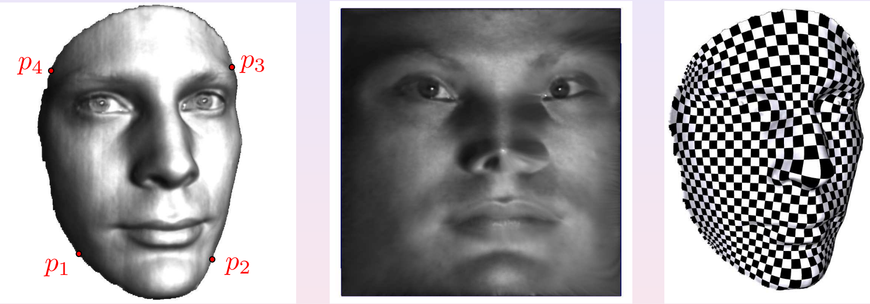


Figure: Topological quadrilateral.

Topological Quadrilateral

Definition (Topological Quadrilateral)

Suppose S is a surface of genus zero with a single boundary, and four marked boundary points $\{p_1, p_2, p_3, p_4\}$ sorted counter-clock-wisely. Then S is called a topological quadrilateral, and denoted as $Q(p1, p2, p3, p4)$.

Theorem

Suppose $Q(p1, p2, p3, p4)$ is a topological quadrilateral with a Riemannian metric $\mathbf{g}$, then there exists a unique conformal map $\phi : S \to \mathbb{C}$, such that ϕ maps Q to a rectangle, $\phi(p_1) = 0$, $\phi(p_2) = 1$. The height of the image rectangle is the conformal module of the surface.

Algorithm: Topological Quadrilateral

Input: A topological quadrilateral M
Output: Conformal mapping from M to a planar rectangle $\phi : M \to \mathbb{D}$

1. Set the target curvatures at corners $\{p_0, p_1, p_2, p_3\}$ to be $\frac{\pi}{2}$,
2. Set the target curvatures to be 0 everywhere else,
3. Run ricci flow to get the target conformal metric $\bar{\mathbf{u}}$,
4. Isometrically embed the surface using the target metric.

Topological Annulus

Topological Annulus

Figure: Topological annulus.

Definition (Topological Annulus)

Suppose S is a surface of genus zero with two boundaries, the S is called a topological annulus.

Theorem

Suppose S is a topological annulus with a Riemannian metric $\mathbf{g}$, the boundary of S are two loops $\partial S = \gamma_1 - \gamma_2$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical annulus, $\phi(\gamma_1)$ is the unit circle, $\phi(\gamma_2)$ is another concentric circle with radius γ. Then $-\log\gamma$ is the conformal module of S. The mapping ϕ is unique up to a planar rotation.

Algorithm: Topological Annulus

Input: A topological annulus M, $\partial M = \gamma_0 - \gamma_1$
Output: a conformal mapping from the surface to a planar annulus $\phi : M \to \mathbb{A}$

1. Set the target curvature to be 0 every where,
2. Run Ricci flow to get the target metric,
3. Find the shortest path γ_2 connecting γ_0 and γ_1, slice M along γ_2 to obtain $\bar{M}$,
4. Isometrically embed $\bar{M}$ to the plane, further transform it to a flat annulus
$$\{z | r \leq Re(z) \leq 0\} / \{z \to z + 2k\sqrt{-1}\pi\}$$
by planar translation and scaling,
5. Compute the exponential map $z \to \exp(z)$, maps the flat annulus to a canonical annulus.

Riemann Mapping

Simply Connected Domains

Topological Disk

Definition (Topological Disk)

Suppose S is a surface of genus zero with one boundary, the S is called a topological disk.

Theorem

Suppose S is a topological disk with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the canonical disk. The mapping ϕ is unique up to a Möbius transformation,

$$z \to e^{i\theta}\frac{z - z_0}{1 - \bar{z}_0 z}.$$

Algorithm: Topological Disk

Input: A topological disk M, an interior point $p \in M$
Output: Riemann mapping $\phi : M \to mathbbD$, maps M to the unit disk and p to the origin

1. Punch a small hole at p in the mesh M,
2. Use the algorithm for topological annulus to compute the conformal mapping.

Multiply connected domains

Multiply-Connected Annulus

Definition (Multiply-Connected Annulus)

Suppose S is a surface of genus zero with multiple boundaries, then S is called a multiply connected annulus.

Theorem

Suppose S is a multiply connected annulus with a Riemannian metric $\mathbf{g}$, then there exists a conformal mapping $\phi : S \to \mathbb{C}$, which maps S to the unit disk with circular holes. The radii and the centers of the inner circles are the conformal module of S. Such kind of conformal mapping are unique up to Möbius transformations.

Algorithm: Multiply-Connected Annulus

Input: A multiply-connected annulus M,

$$\partial M = \gamma_0 - \gamma_1, \cdots, \gamma_n.$$

Output: a conformal mapping $\phi : M \to \mathbb{A}$, $\mathbb{A}$ is a circle domain.

1. Fill all the interior holes γ_1 to γ_n
2. Punch a hole at γ_k, $1 \leq k \leq n$
3. Conformally map the annulus to a planar canonical annulus
4. Fill the inner circular hole of the canonical annulus
5. Repeat step 2 through 4, each round choose different interior boundary γ_k. The holes become rounder and rounder, and converge to canonical circles.

Koebe's Iteration - I

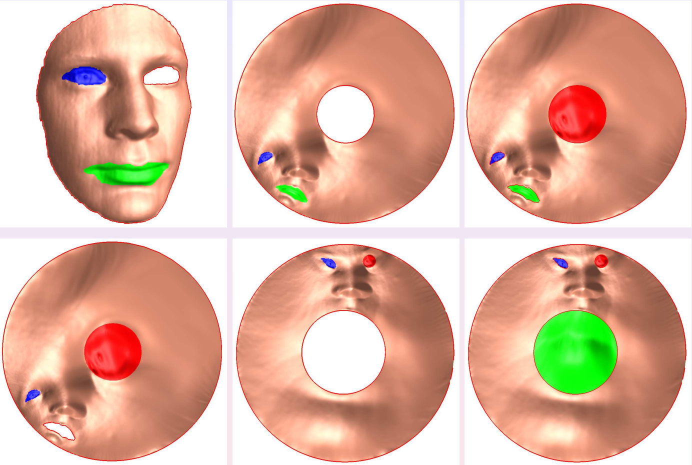


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Koebe's Iteration - II

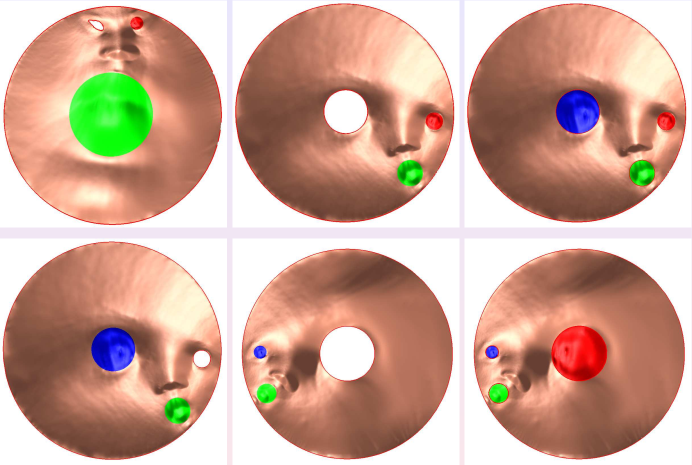


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Koebe's Iteration - III

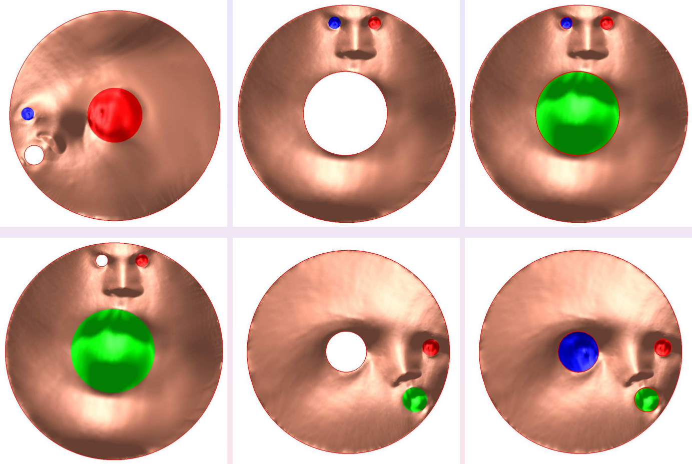


Figure: Koebe's iteration for computing conformal maps for multiply connected domains.

Convergence Analysis

Theorem (Gu and Luo 2009)

Suppose genus zero surface has n boundaries, then there exists constants $C_1 > 0$ and $0 < C_2 < 1$, for step k, for all $z \in \mathbb{C}$,

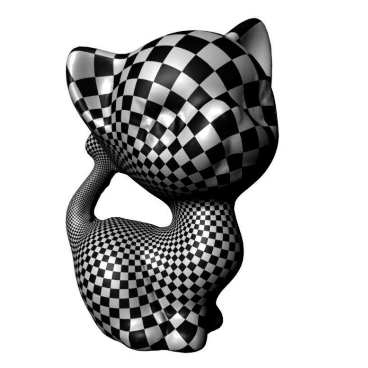
$$|f_k \circ f^{-1}(z) - z| < C_1 C_2^{2[\frac{k}{n}]},$$

where f is the desired conformal mapping.

W. Zeng, X. Yin, M. Zhang, F. Luo and X. Gu, ”Generalized Koebe's method for conformal mapping multiply connected domains”, Proceeding SPM'09 SIAM/ACM Joint Conference on Geometric and Physical Modeling, Pages 89-100.

Topological Torus

Topological torus



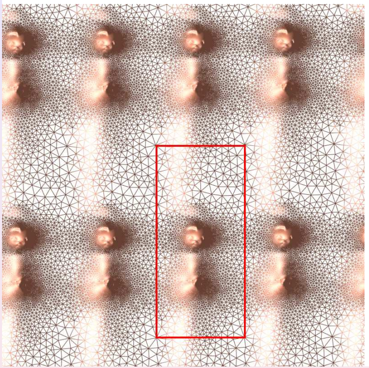
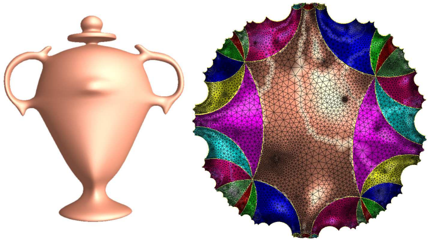


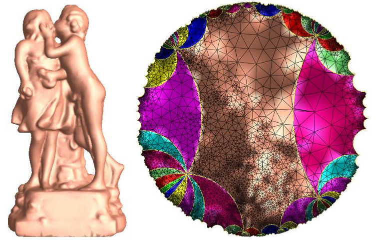
Figure: Genus one closed surface.

Input: A topological torus M
Output: A conformal mapping which maps M to a flat torus $\mathbb{C}/\{m+n\alpha|m,n\mathbb{Z}\}$

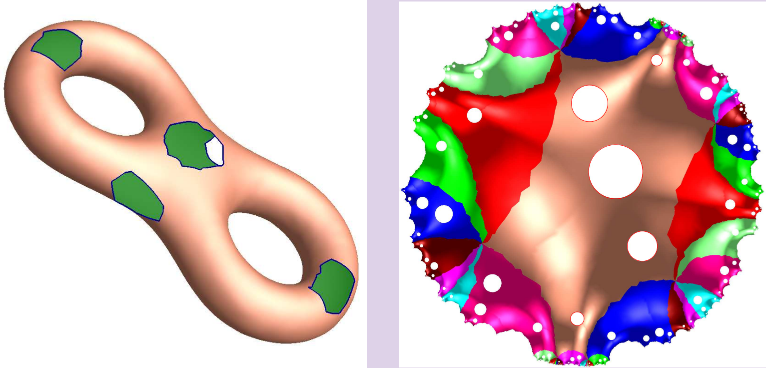
1. Compute a basis for the fundamental group $\pi_1(M)$, $\{\gamma_1,\gamma_2\}$.
2. Slice the surface along γ_1,γ_2 to get a fundamental domain $\bar{M}$,
3. Set the target curvature to be 0 everywhere
4. Run Ricci flow to get the flat metric
5. Isometrically embed $\tilde{S}$ to the plane

Computational results for genus 2 and genus 3 surfaces.





Hyperbolic Koebe's Iteration



M. Zhang, Y. Li, W. Zeng and X. Gu. "Canonical conformal mapping for high genus surfaces with boundaries", Computer and Graphics, Vol 36, Issue 5, Pages 417-426, August 2012.