Surface and Volume Based Techniques for Shape Modeling and Analysis

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SIGGRAPH Asia 2013 Course

Overview

Collaborators

The work is collaborated with Shing-Tung Yau, Feng Luo, Ronald Lok Ming Lui, Paul M. Thompson, Tony F. Chan, Arie Kaufman, Hong Qin, Dimitris Samaras, Jie Gao and many other mathematicians, computer scientists and doctors.

Introduction

Fundamental Problems

Shapes

How to model the space of all shapes?

Mapping

How to model the space of all mappings between two shapes?

Outline

Main Topics

1. Discrete Surface Ricci flow
2. Discrete Optimal Mass Transportation
3. Quasi-Conformal Geometry

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

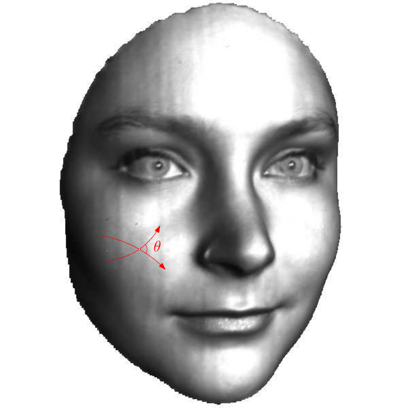
- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

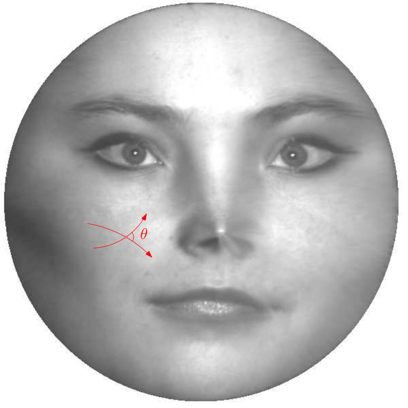
Transformation Groups

Suppose a mapping $\varphi : (S_1, \mathbf{g}_1) \rightarrow (S_2, \mathbf{g}_2)$ is given,

1. Homeomorphism: φ is continuous, bijective, φ^{-1} is also continuous.
2. Conformal: angle preserving
3. Area preserving mapping
4. Isometry: length preserving
5. Rigid motion: rotation and translation in $\mathbb{R}^3$.

Angle Preserving Mapping





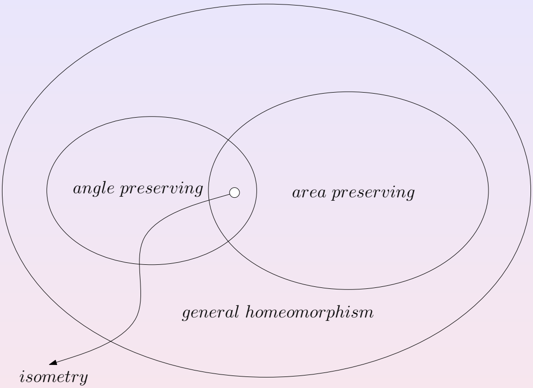
The angle between γ_1 and γ_2 equals to that between $\phi(\gamma_1)$ and $\phi(\gamma_2)$.

Area Preserving Mapping

For any Borel set $\Omega \subset S_1$, the area of Ω equals to that of $\phi(\Omega)$.

General Diffeomorphisms

Mapping Space



- angle preserving mapping - surface Ricci flow
- area preserving mapping - optimal mass transport
- general mapping - quasi-conformal mapping

Hierarchy

The transformation groups have the relation:

$$\{\textit{rigid motion}\} \lhd \{\textit{isometry}\} \lhd \{\textit{conformal}\} \lhd \{\textit{homeomorphism}\}$$

The corresponding shape spaces

$$\mathscr{S}/\{\textit{rigid motion}\} \rhd \mathscr{S}/\{\textit{isometry}\} \rhd \mathscr{S}/\{\textit{conformal}\} \rhd \mathscr{S}/\{\textit{homeomo}$$

where
$\mathscr{S} = \{$ *compact orientatable metric surfaces embedded in* $\mathbb{E}^3\}$.

Definition (Topologically Equivalence)

Two surfaces are topologically equivalent, if there exists a homeomorphism between them.

Definition (Topological Invariants)

Orientability, genus, number of boundaries. Fundamental group, homology group, cohomology group.

Conformal Geometry

Definition (Conformal Equivalence)

Two surfaces are conformal equivalent, if there exists a conformal mapping between them.

Definition (Conformal Invariants)

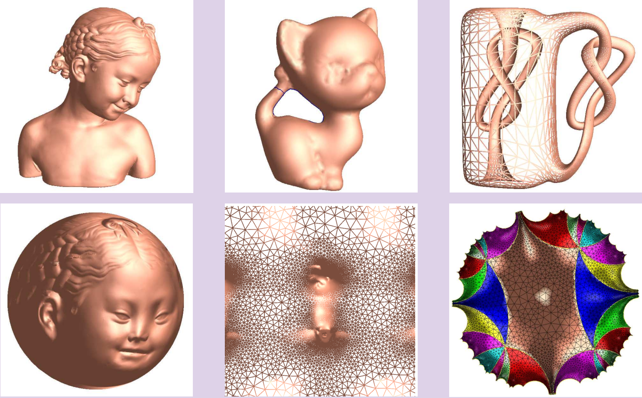
Conformal module, uniformization domain:

$$S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i),$$

1. S is a constant curvature space, the unit sphere $\mathbb{S}^2$, the Euclidean plane $\mathbb{E}^2$ and the hyperbolic plane $\mathbb{H}^2$.
2. Γ is a fixed point free subgroup of the rigid motion group of S.
3. $C(c_i, r_i)$ is a geodesic circle on S/Γ with center c_i and radius r_i.

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Conformal Canonical Representations

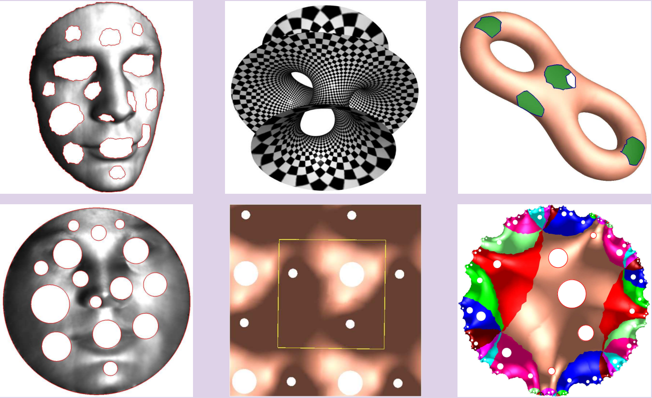
Definition (Circle Domain in a Riemann Surface)

A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

Theorem

Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

Uniformization of Open Surfaces



Spherical Euclidean Hyperbolic

Definition (Isometric Equivalence)

Two surfaces are isometric equivalent, if there exists an isometric mapping between them.

Definition (Isometric Invariants)

Suppose the surface $(M,\mathbf{g})$ has the canonical conformal representation $S/\Gamma - \cup_{i=1}^{n} C(c_i,r_i)$, the Riemannnian metric of M is given by

$$\mathbf{g} = e^{2\lambda}\mathbf{g}_S,$$

where λ is the conformal factor, $\mathbf{g}_S$ is the spherical, Euclidean, or hyperbolic metric.

Therefore, a compact, orienatable metric surface has the representation

$$(S/\Gamma - \cup_{i=1}^{n} C(c_i,r_i),\lambda)$$

Suppose two compact surfaces embedded in $\mathbb{E}^3$, $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ differ by a rigid motion, if and only if they share the same

1. conformal representation $S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i)$,
2. conformal factor λ,
3. mean curvature H.
4. conformal factor and mean curvature satisfies Gauss-Codazzi equations

$$(\log \lambda)_{z\bar{z}} = \frac{\mu\bar{\mu}}{\lambda^2} - \frac{\lambda^2}{4} H^2,$$

$$\mu_{\bar{z}} = \frac{\lambda^2}{2} H_z,$$

$$\mu_{z\bar{z}} = \frac{1}{2}\lambda(2\lambda_z H_z + \lambda H_{zz}).$$

Canonical Representation

Suppose $(M,\mathbf{g})$ is a compact, orientable, metric surface embedded in $\mathbb{E}^3$, then its representation is a triple

$$(S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i), \lambda, H).$$

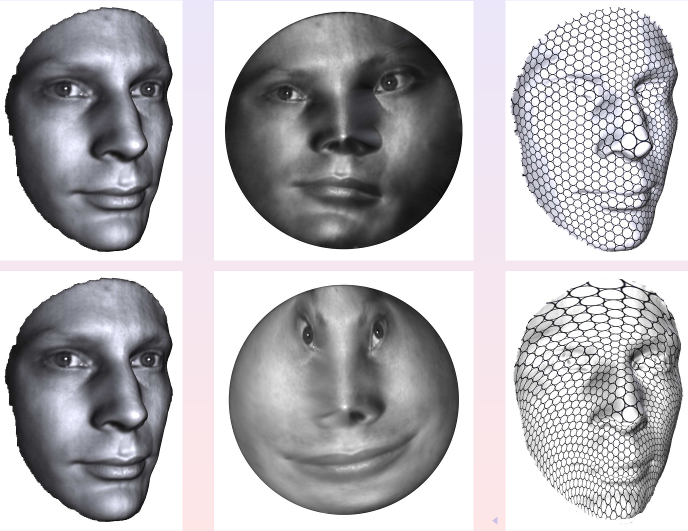
where S is one of three canonical spaces $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$, Γ is a subgroup of isometries of the canonical space, λ the conformal factor, H the mean curvature, furthermore λ and H satisfy Gauss-Codazzi equations.

Mappings Space

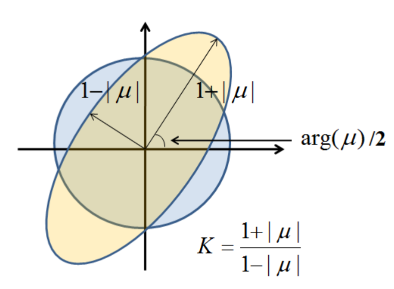
Intuition

1. All diffeomorphisms between two compact Riemann surfaces are quasi-conformal.
2. Each quasi-conformal mapping corresponds to a unique Beltrami differential.
3. The space of diffeomorphisms equals to the space of all Beltrami differentials.
4. Variational calculus can be carried out on the space of diffeomorphisms.

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.



Beltrami-Equation



Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

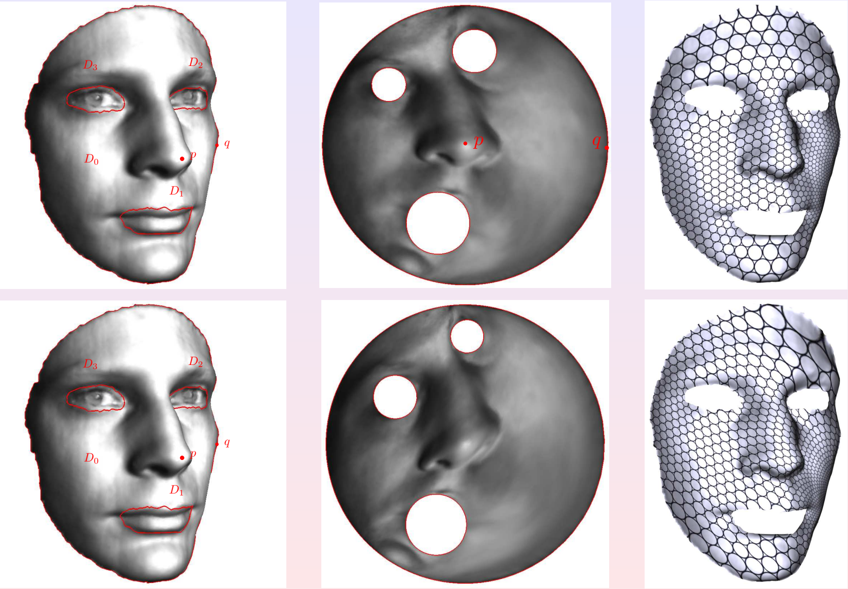
Mapping Space

Mapping Representation

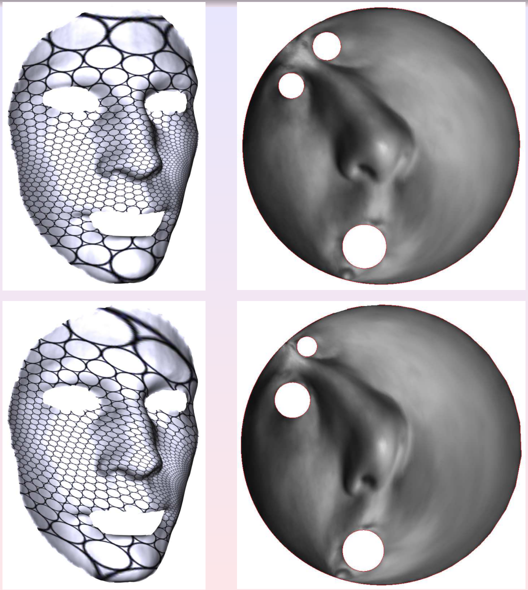
Given two genus zero metric surface with a single boundary,

$$\{Diffeomorphisms\} \cong \frac{\{Beltrami\ Coefficient\}}{\{Mobius\}}.$$

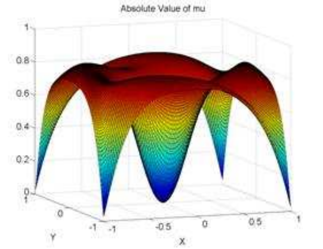
Quasi-Conformal Map Examples



Quasi-Conformal Map Examples



Solving Beltrami Equation



Direct Applications

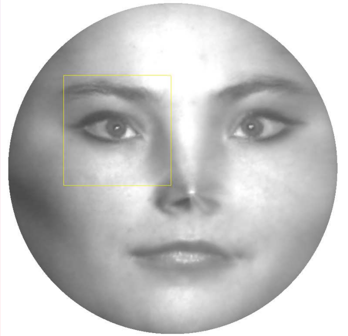
Geometric Approximation

Meshing

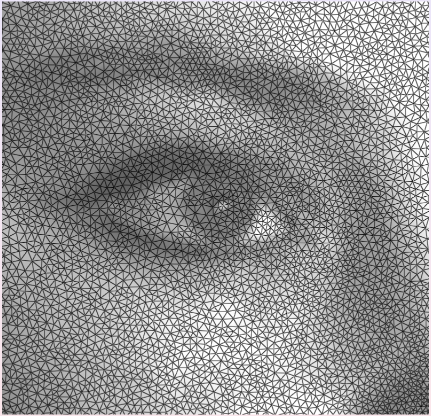
Theorem

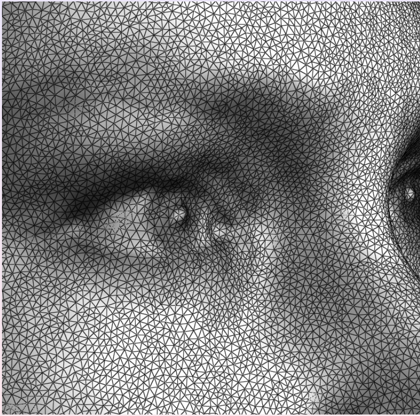
Suppose S is a surface with a Riemannian metric. Then there exist meshing method which ensures the convergence of curvatures.

Key idea: Delaunay triangulations on uniformization domains. Angles are bounded, areas are bounded.



Meshing





Theorem

Let M be a compact Riemannian surface embedded in $\mathbb{E}^3$ with the induced Euclidean metric, T the triangulation generated by Delaunay refinement on conformal uniformization domain, with circumradius bound ε. If B is the relative interior of a union of triangles of T, then

$$\begin{aligned} |\phi_T^G(B) - \phi_M^G(\pi(B))| &\leq K\varepsilon \\ |\phi_T^H(B) - \phi_M^H(\pi(B))| &\leq K\varepsilon \end{aligned}$$

where $\pi : T \rightarrow M$ is the closest point projection, ϕ^H, ϕ^G are the mean and Gaussian curvature measures, where

$$K = O(area(B)) + O(length(\partial B)).$$

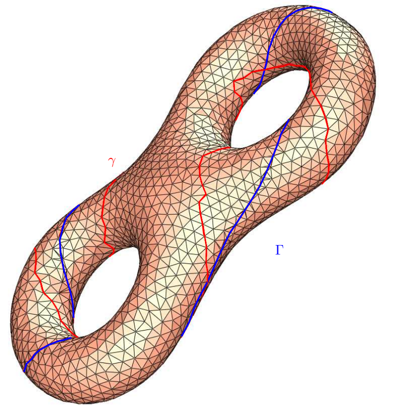
H. Li, W. Zeng, J. Morvan, L. Chen and X. Gu, "Surface Meshing with Curvature Convergence" IEEE TVCG 2013.

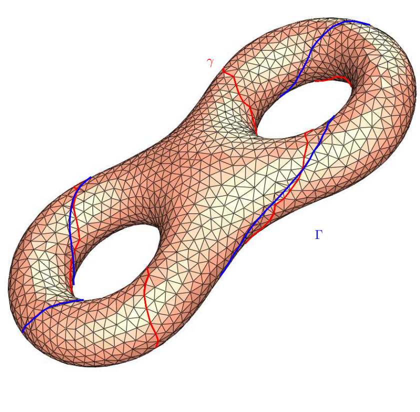
Computational Topology

Computational Topology Application

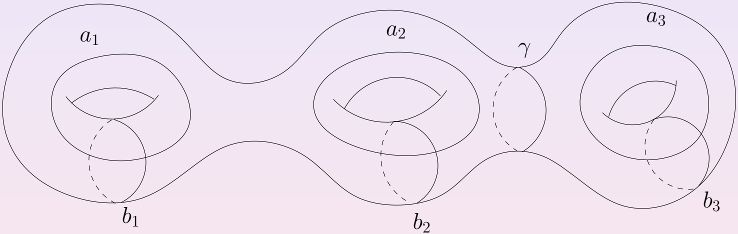
Canonical Homotopy Class Representative

Under hyperbolic metric, each homotopy class has a unique geodesic, which is the representative of the homotopy class.

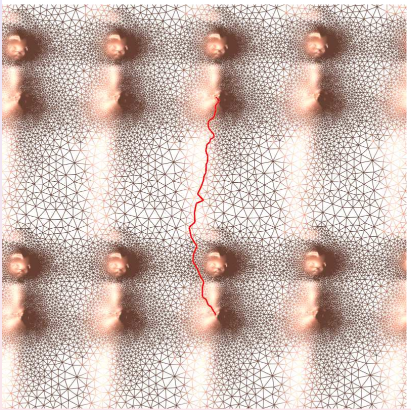


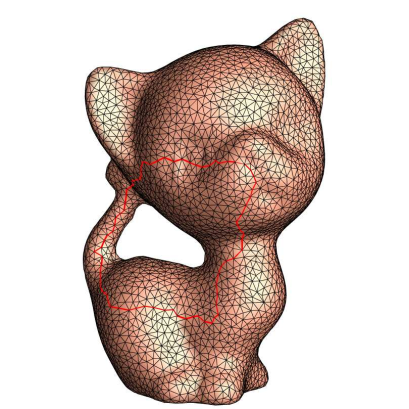


Shortest word Problem (NP Hard):

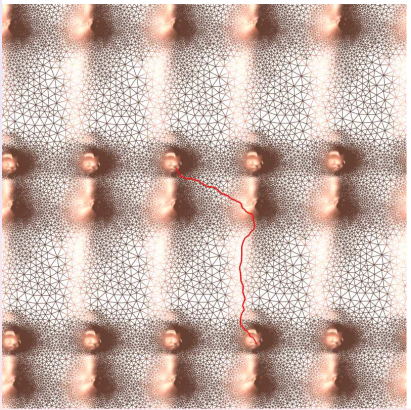


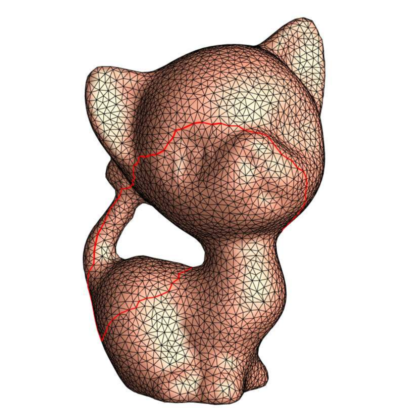
$$\gamma = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = (a_3 b_3 a_3^{-1} b_3^{-1})^{-1}$$



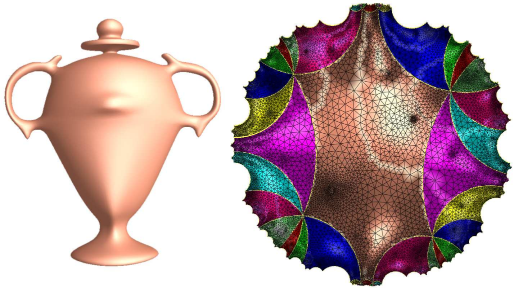


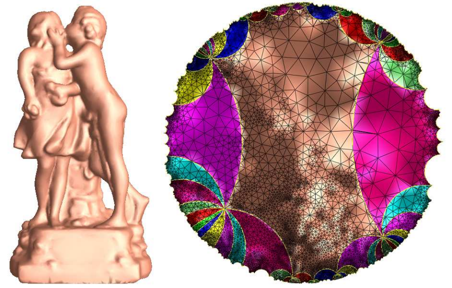
Loop Lifting





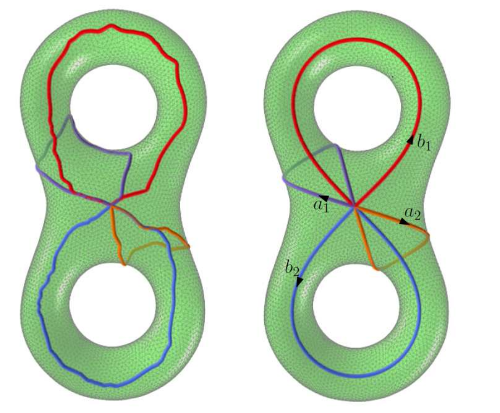
Hyperbolic Ricci Flow





Birkoff Curve Shorting

Birkoff curve shortening deforms a loop to a geodesic.



Computing Shortest Word

Solving Shortest Word

1. Compute the uniformization metric using Ricci flow.
2. Compute the geodesic loop by Birkoff curve shortening.
3. Lift the geodesic loop to the universal covering space.
4. Trace the lifted loop to compute the word.

X. Yin, Y. Li, W. Han, F. Luo, X. Gu and S.-T. Yau, “Computing Shortest Words via Shortest Loops on Hyperbolic Surfaces”, Computer-Aided Design (CAD), 43(11), 2011.

Shape Analysis

Theorem

Discrete heat kernel determines the discrete Riemannian metric.

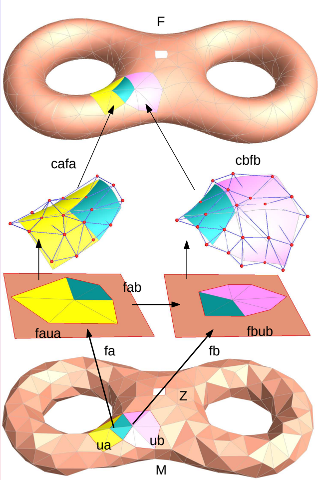
W. Zeng, R. Guo, F. Luo and X. Gu, “Discrete Heat Kernel Determines Discrete Reimannian Metric”, Graphical Models, 2012.

Geometric Modeling

Geometric Modeling Application: Manifold Spline

Manifold Spline

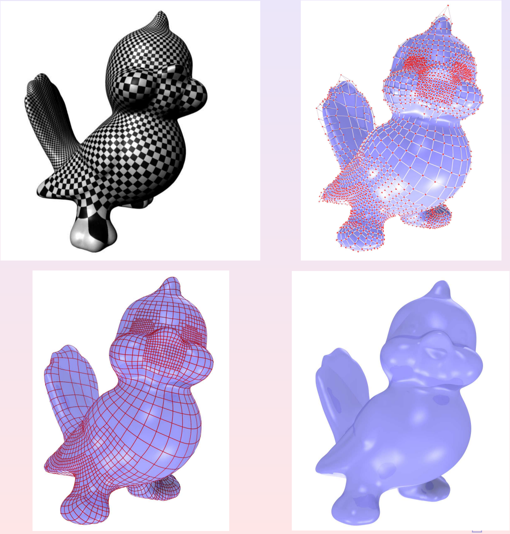
- Convert scanned polygonal surfaces to smooth spline surfaces.
- Conventional spline scheme is based on affine geometry. This requires us to define affine geometry on arbitrary surfaces.
- This can be achieved by designing a metric, which is flat everywhere except at several singularities (extraordinary points).
- The position and indices of extraordinary points can be fully controlled.



Y. He, X. Gu, Y. He, and H. Qin, “Manifold splines”. Graphical Models, 68(3):237-254, 2006.

Manifold Spline

Converting scanned data to spline surfaces, the control points, knot structure are shown.



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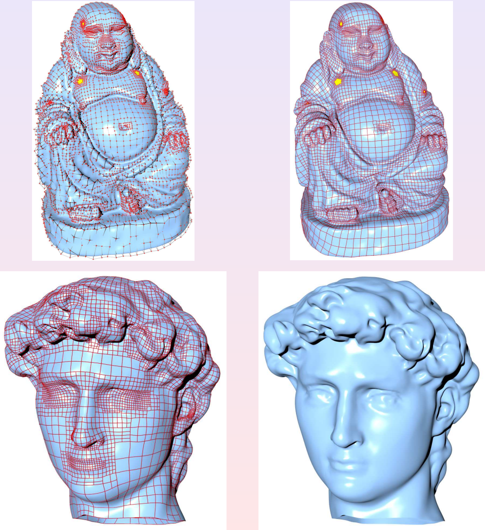
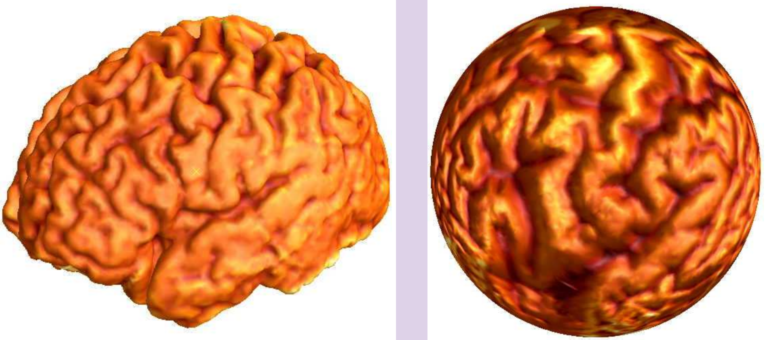


Medical Imaging

Brain Cortex Surface Spherical Mapping

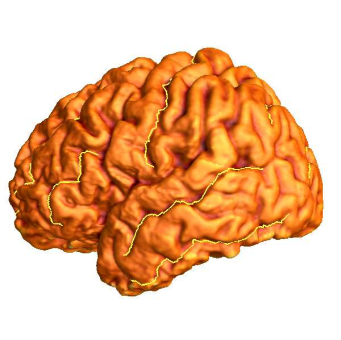
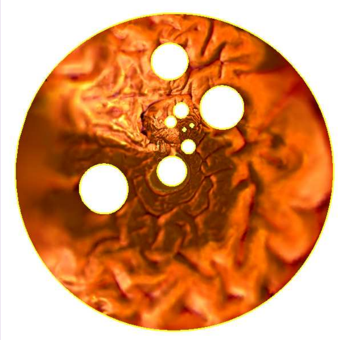


X. Gu, Y. Wang, T. F. Chan, P. M. Thompson and S.-T. Yau, "Genus Zero Surface Conformal Mapping and Its Application to Brain Surface Mapping", IEEE TMI, 23(8):949-958, 2004.

Conformal Brain Mapping

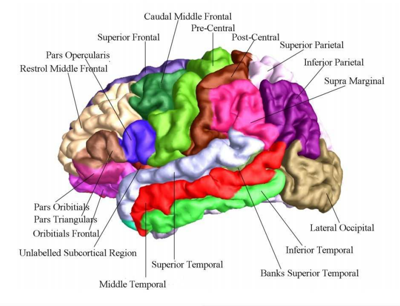
Using conformal module to analyze shape abnormalities.

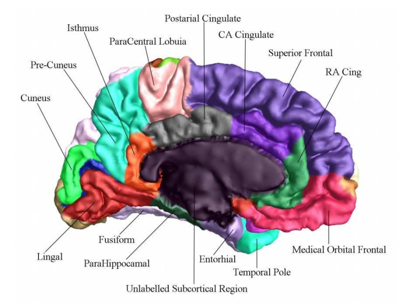
Brain Cortex Surface

Y. Wang, L. M. Lui, X. Gu, K. Hayashi, T. F. Chan, A. W. Toga, P. M. Thompson and S.-T. Yau, “Brain Surface Conformal Parameterization using Riemann Surface Structure”, IEEE TMI, 26(6):853-865, June 2007.

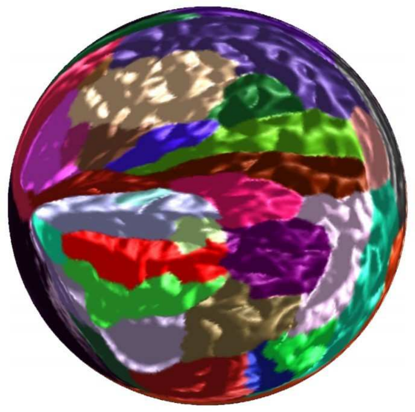
Alzheimer Study





Alzheimer Study

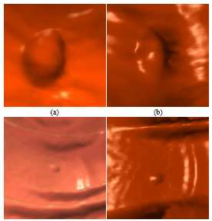
Conformal Mapping

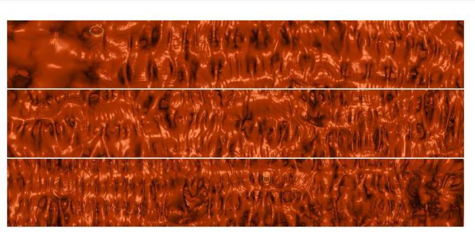


Optimal Transportation Map

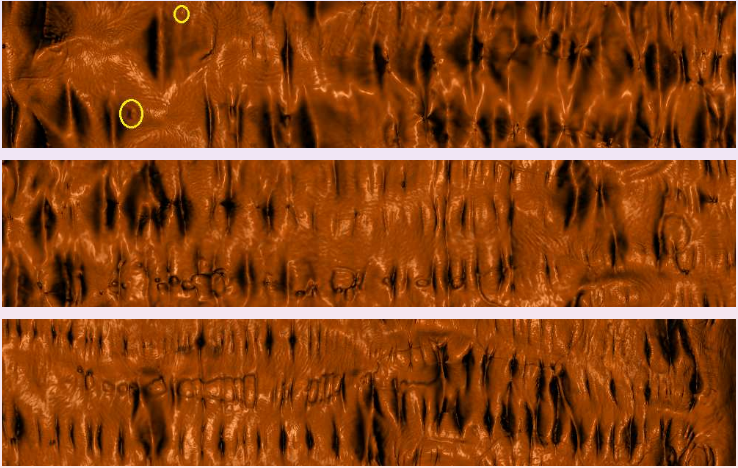
Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.



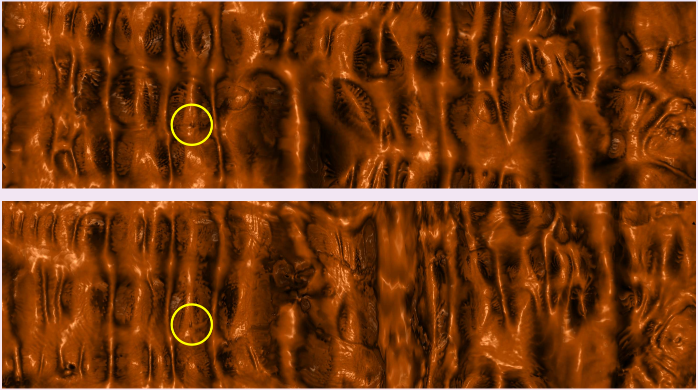


Colon Flattening



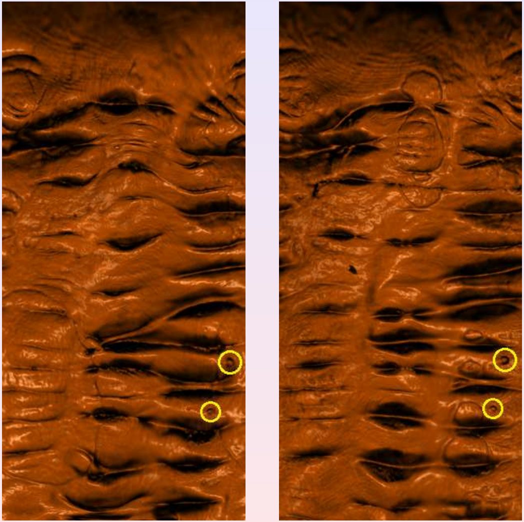
Virtual Colonoscopy

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.



W. Zeng, J. Marino, K. C. Gurijala, X. Gu and A. Kaufman, "Supine and Prone Colon Registration Using Quasi-Conformal Mapping", IEEE TVCG, 16(6): 1348-1357, 2010.

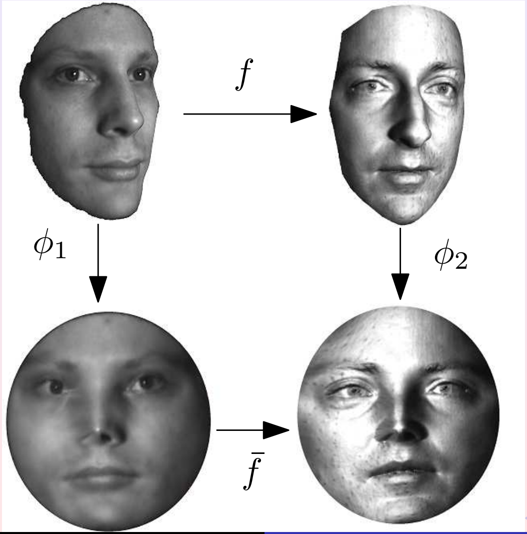
Colon Registration



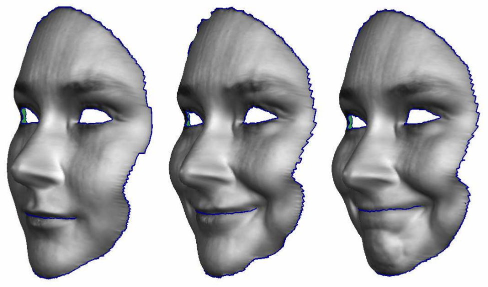
Computer Vision

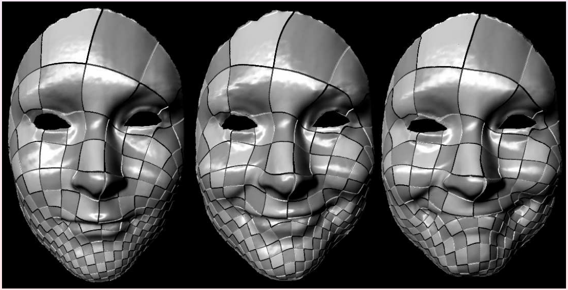
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

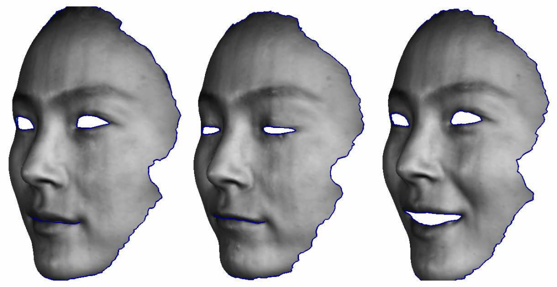


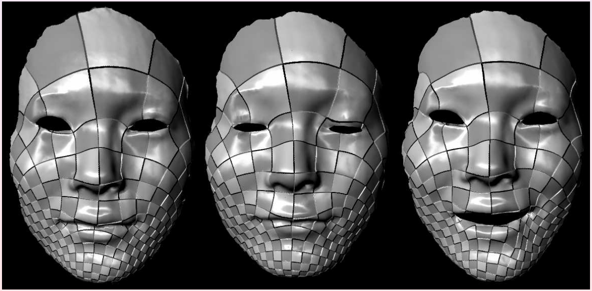
Face Surfaces with Different Expressions are Matched

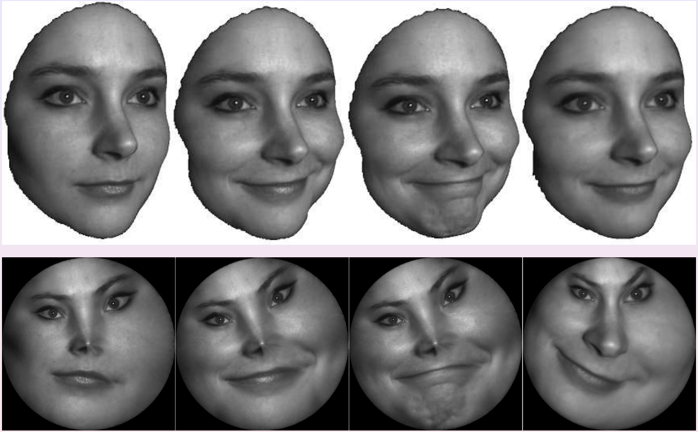




Face Surfaces with Different Expressions are Matched

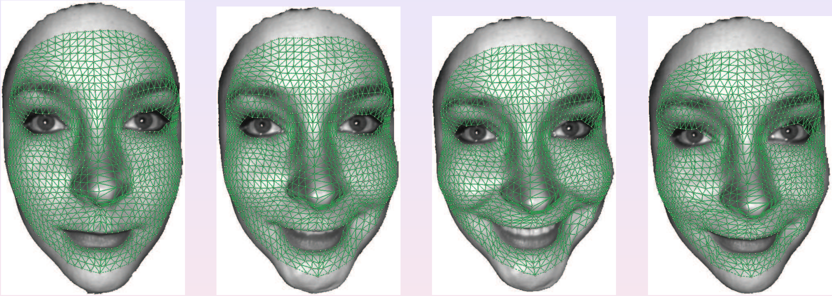






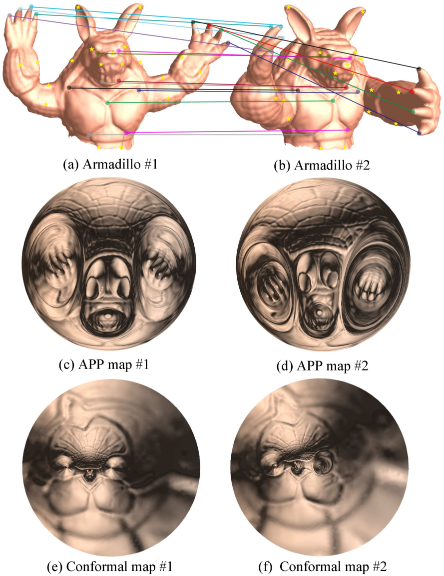
W. Zeng, D. Samaras and X. Gu, “Ricci Flow for 3D Shape Analysis”. IEEE TPAMI, 32(4): 662-677, 2010.

Face Expression Tracking



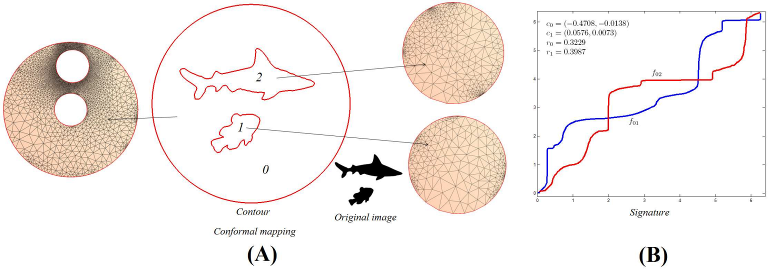
Y.Wang, M. Gupta, S.Zhang, S. Wang, Xianfeng Gu, Dimitris Samaras, and P. Huang, "High Resolution Tracking of Non-Rigid Motion of Densely Sampled 3D Data Using Harmonic Maps", IJCV, 76(3),2007.

Surface Registration



2D Shape Space-Conformal Welding

$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$

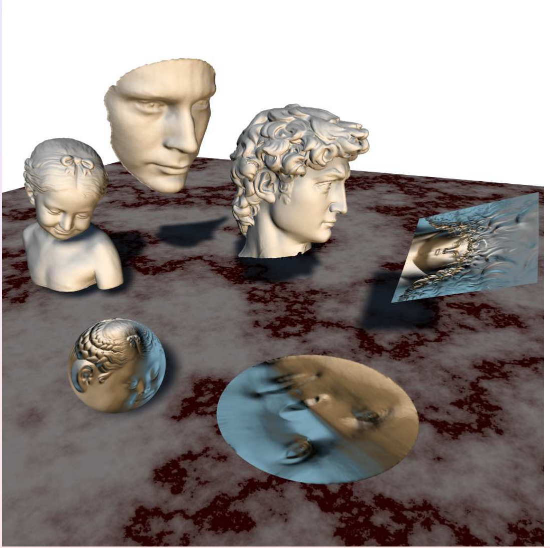


L. M. Lui, W. Zeng, S.-T. Yau and X. Gu, "Shape Analysis of Planar Multiply-connected Objects using Conformal Welding", IEEE TPAMI 2013.

Computer Graphics

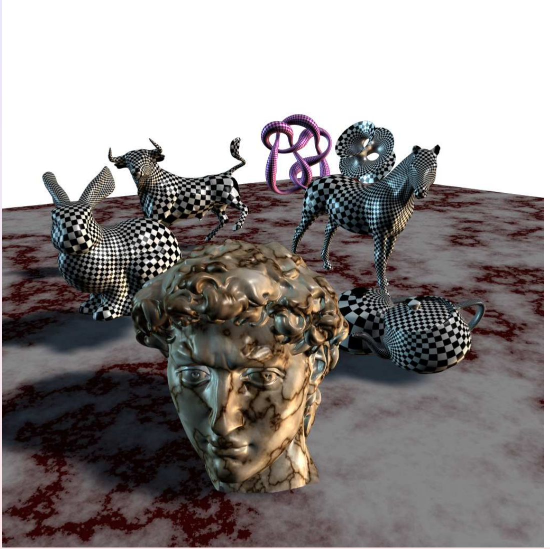
Surface Parameterization

Map the surfaces onto canonical parameter domains

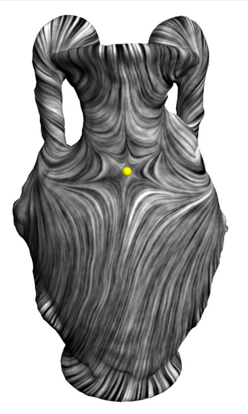


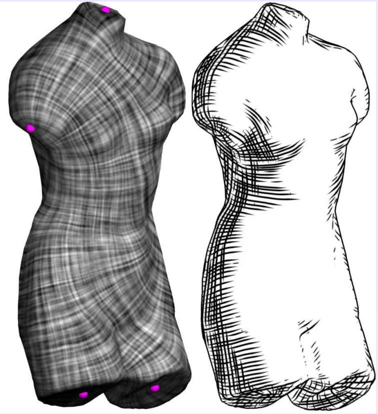
Surface Parameterization

Applied for texture mapping.



Non-photo-realistic rendering

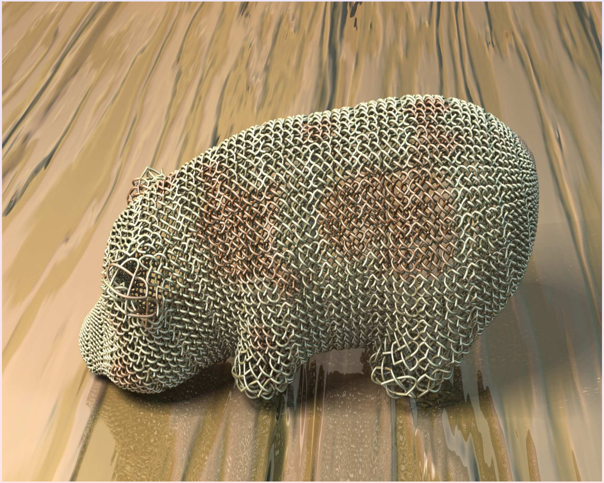




Y. Lai, M. Jin, X. Xie, Y. He, J. Palacios, E. Zhang, S.-M.Hu and X. Gu, “Metric Driven RoSy Field Design and Remeshing”, IEEE TVCG, 16(1):95-108, 2009.

n-Rosy Field Design

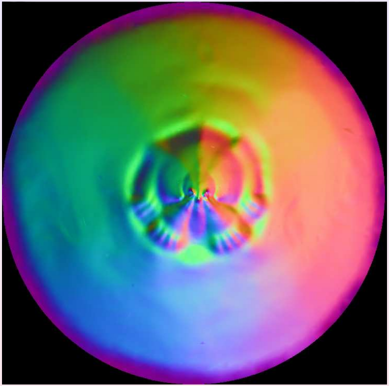
Convert the surface to knot structure using smooth vector fields.



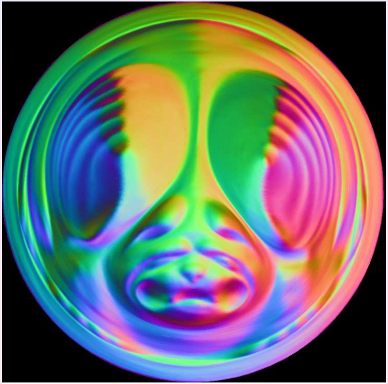
Visualization

Normal Map

Visualization

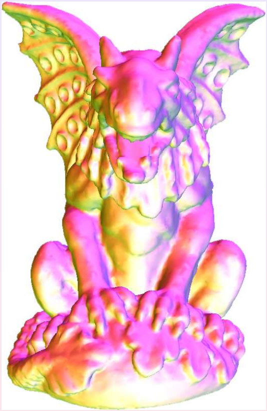


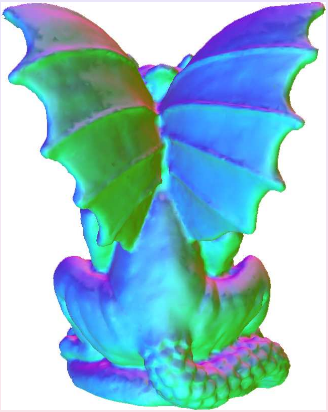
Conformal mapping



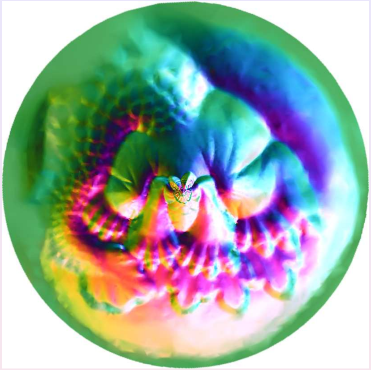
Area-preserving mapping

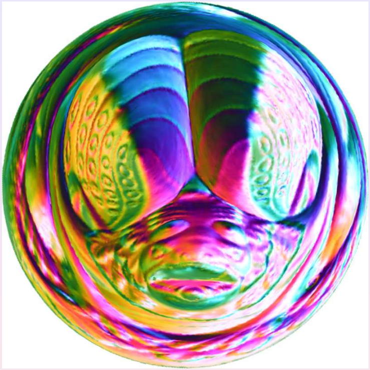
Visualization





Visualization

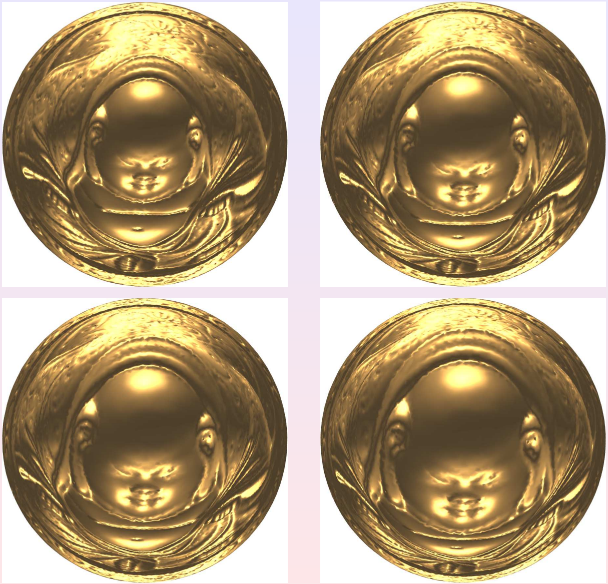




X. Zhao, Z. Su, X. Gu, A. Kaufman, J. Sun, J. Gao, F. Luo, "Area-preservation Mapping using Optimal Mass Transport", IEEE TVCG, 2013.

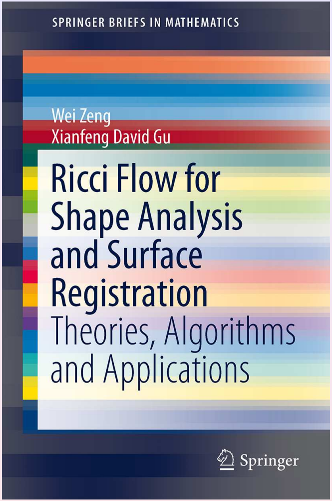
Visualization

Visualization



Reading Materials

Ricci Flow for Shape Analysis and Surface Registration



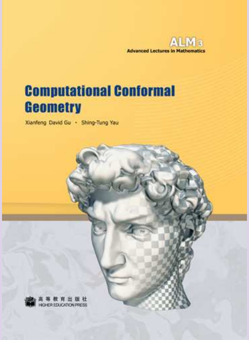
Book Cover

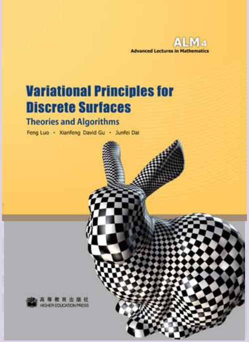
eBook

Book

Books

The theory, algorithms and sample code can be found in the following books.





Resources

1. Detailed lecture notes can be found at:

   http://www.cs.stonybrook.edu/˜gu/lectures/index.html

2. Source code, demos and data sets can be found at:

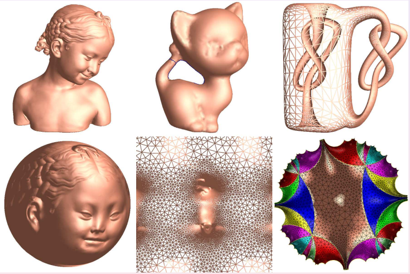
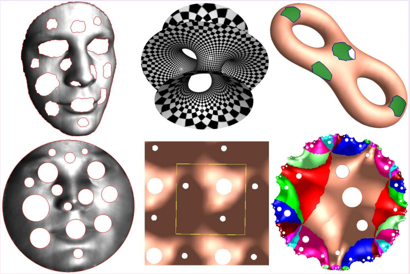
   http://www.cs.stonybrook.edu/˜gu/software/index.html

3. Talk slides

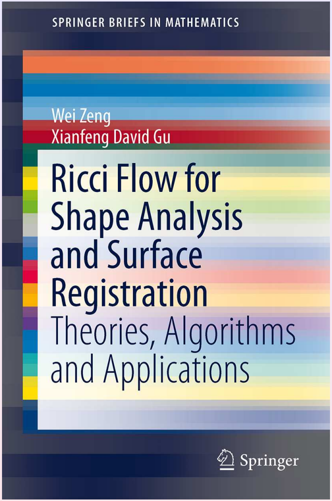
   http://www.cs.stonybrook.edu/˜gu/talks/index.html

4. Talk slides
   http://saturno.ge.imati.cnr.it/ima/personal/patane/PersonalPage/Patanes_Home_Page/Courses/Entries/2013/11/17_Surface-_and_volume-based_techniques_for_shape_modeling_and_analysis.html

Please email me gu@cs.stonybrook.edu for updated code library on computational conformal geometry.

Ricci Flow for Shape Analysis and Surface Registration



Book Cover

eBook

Book