Surface Matching and Registration Using Symmetric Conformal Mapping

Wei Zeng
Computer Science Department
Stony Brook University
Stony Brook, New York 11794
zengwei@cs.sunysb.edu

Xianfeng Gu
Computer Science Department
Stony Brook University
Stony Brook, New York 11794
gu@cs.sunysb.edu

Abstract

Non-rigid surface matching and registration play fundamental roles in computer graphics, computer vision and Computer Aided Geometric Design. Recently, various conformal geometric methods have been proposed for non-rigid surface registration in these fields. The major disadvantage for conformal geometric method is that they are too sensitive to the boundary conditions. In reality, due to partial occlusion, cluttering, surfaces acquired from real life are always with inconsistent boundaries. Many surfaces in practice are symmetric, such as human faces, surfaces of furniture. Symmetry can be utilized to overcome the inconsistency of boundaries.

This work propose to improve the robustness of conformal geometric methods to the boundaries by incorporating the symmetric information of the input surface. The key idea is to preserve the symmetry during the conformal mapping, such that eventually, the image on the parameter domain is symmetric, the area distortion factor on the parameter image is also symmetric.

We developed novel algorithms based on different conformal geometric tools, one is based on solving Riemann-Cauchy equation, one is based on curvature flow. We tested our algorithms on geometric data sets acquired from real life, and make thorough comparison with most existing techniques. Experimental results demonstrate that the symmetric conformal mapping is insensitive to the boundary occlusions. The method outperforms all the others in terms of robustness. The method has the potential to be generalized to high genus surfaces using hyperbolic curvature flow.

1. Introduction

In recent decades, there has been a lot of research into surface representations for 3D surface analysis, which is a fundamental issue for many applications in computer graphics, computer vision and geometric modeling, such as 3D shape registration, partial scan alignment, 3D object reconstruction, 3D object recognition, and classification [1], [2], [3], [4]. In particular, as 3D scanning technologies improve, large databases of 3D scans require automated methods for matching and registration. However, matching surfaces undergoing non-rigid deformation is still a challenging problem, especially when data is noisy and with complicated topology. Different approaches include curvature-based representations [5], [6], regional point representations [3], [7], spherical harmonic representations [8], [9], shape distributions [10], multi-dimensional scaling[11], [12], local isometric mapping [13], summation invariants [14], landmark-sliding [15], physics-based deformable models [16], Free-Form Deformation (FFD) [17], and Level-Set based methods [18]. However, many surface representations that use local geometric invariants can not guarantee a global convergence and might suffer from local minima in the presence of non-rigid deformations. To address this issue, many global parameterization methods have been developed recently based on conformal geometric maps [19], [20], [21], [22], [23], [24]. Although the previous methods have met with a great deal of success in both computer vision and graphics, there are major shortcomings in conformal maps when applied to matching of real discrete data such as the output of 3D scanners: *inconsistent boundaries*. In this paper we will address the above issue by introducing intrinsic symmetry to the registration.

There are four categories of conformal geometric methods. The first category is based on *harmonic maps*, which has been applied for surface matching in [19] high resolution tracking of non-rigid 3D motion of densely sampled data [20] and conformal brain mapping [21]. The second rows in figure 1 and 4 show the harmonic maps. The second category is based on *solving Riemann-Cauchy equation*, such as least square conformal maps introduced in [22]. This method has been applied for 3D shape matching, recognition, and stitching[24], automatic non-rigid registration of 3D dynamic data for facial expression synthesis and transfer [25]. The third rows in figure 1 and 4 illustrate the result using this method. The third category is based on *holomorphic differentials*, which induces different conformal mappings, by combining them, better results can be achieved for surface matching and registration [26]. The third rows in figure 1 and 4 show the Riemann mapping results using holomorhic differentials. The last category is using *Ricci flow*, Euclidean Ricci flow has been applied for shape

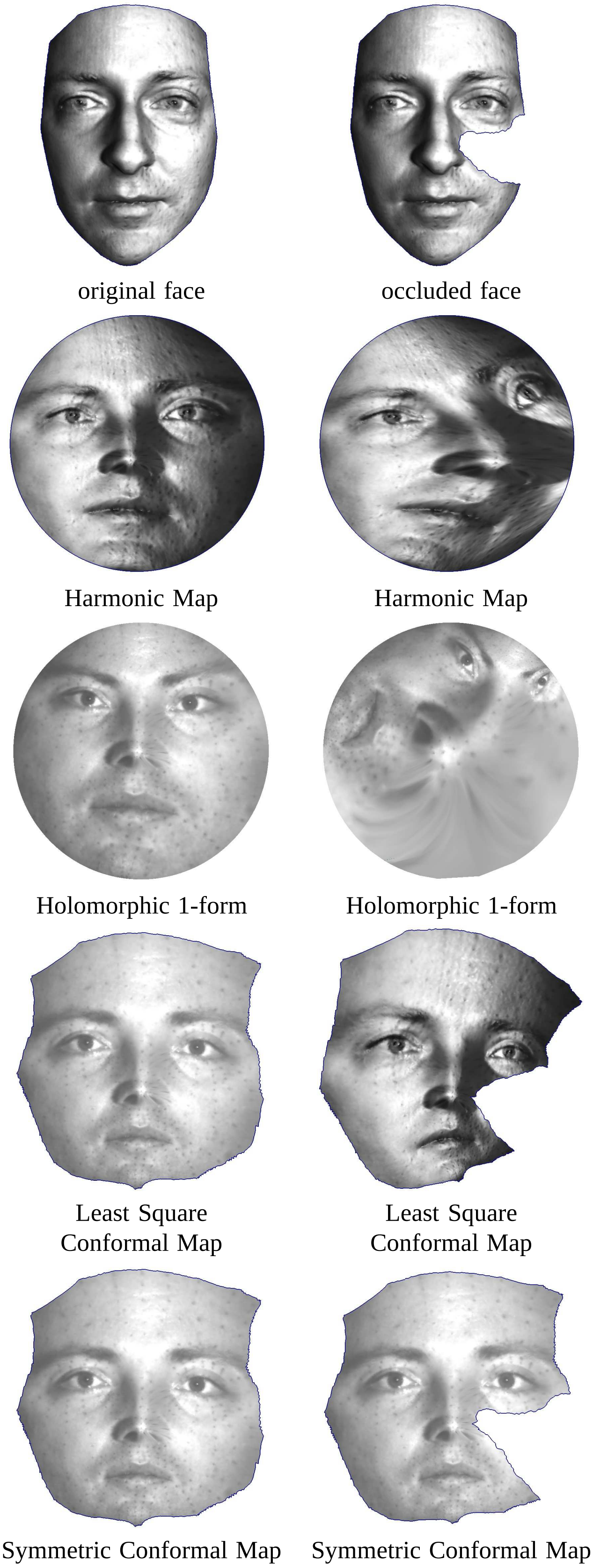

Figure 1. Comparision among different conformal mapping methods. Symmetric conformal map is the most robust to boundary occlusion.

analysis in [27], and hyperbolic Ricci flow has been used for face matching and registration in [28]. Recently, discrete surface Yamabe flow has been introduced by Luo in [32], it has been reintroduced in [33]. A similar method is applied for conformal parameterization in [34]. The fourth row in figure 4 shows the conformal mapping using Euclidean Ricci flow.

In general, harmonic maps, least square conformal maps are linear methods, but can only handle surfaces with simple topologies, such as topological disks. Holomorphic differentials can handle multiply connected domains and high genus surfaces, but it introduces singularities. Ricci flow method is very general and has no topological limitations, but it is a nonlinear optimization. All of them are very sensitive to the boundaries. In reality, due to partial occlusion, noises, arbitrary surface patch acquired by a single scan by a camera-based 3D scanner, eg. face frontal scan, cloth, machine parts etc, is a genus zero surface with arbitrary number of holes. The boundaries are in general inconsistent. This affects the mapping quality. As shown in figure 1 and 4, inconsistent boundary conditions produce drastically different conformal mappings and lead to the failure for partial matching and registration.

In real applications in graphics and CAD, many categories of surfaces of interests are symmetric, such as human faces, human bodies, most furniture, buildings, automobiles etc. To address this critical issue, we propose to incorporate the symmetry of the input surface to the conformal mapping, such that the conformal mapping preserves the intrinsic symmetry of the surface and is more robust to the inconsistency of the boundaries. The conformal mapping preserves the symmetry in the following ways: first the image of the mapping is still symmetric; second, the area distortion factor on the image is symmetric as well. The bottom rows in figures 1 and 4 are the conformal mappings preserving the symmetry, which are much more robust to the boundary occlusions and inconsistency.

We make the following contributions in our paper:

- A conformal mapping method based on solving Riemann-Cauchy equation is introduced, which preserves the symmetry of the input surface.
- A conformal mapping method based on discrete curvature flow (Yamabe Flow) is introduced, which preserves the symmetry of the input surface.
- A robust method for non-rigid surface matching and registration based on symmetric conformal mapping is introduced, which is very robust to boundary occlusion and clutter.

Although the work focuses on topological disks, it can be generalized to surfaces with more complicated topologies, such as multiply connected domains or high genus surfaces, as long as the surface has intrinsic symmetry.

2. Mathematical Background

All surfaces embedded in $\mathbb{R}^3$ have the induced Euclidean metric $\mathbf{g}$. A *conformal structure* is an atlas, such that on each local chart, the metric can be represented as

$$\mathbf{g} = e^{2u}(dx^2 + dy^2).$$

we can use complex parameter to represent it $z = x + iy$, which is called *isothermal coordinates*. Suppose two charts have overlapping region on the surface, then the chart transition function is an analytic function. A surface with a conformal structure is a *Riemann surface*, therefore, all surfaces in $\mathbb{R}^3$ are Riemann surfaces.

A complex valued function $f : \mathbb{C} \to \mathbb{C}$ is *holomorphic*, if it satisfies the following Riemann-Cauchy equation, $f : z \to w$, where $z = x + iy$ and $w = u + iv$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad (1)$$

A mapping between two Riemann surfaces $f : S_1 \to S_2$ between two surfaces is *conformal*, if it satisfies the following condition: Arbitrarily choosing a local isothermal coordinates of S_1, (U_α, ϕ_α), a local isothermal coordinates of S_2, (V_β, ϕ_β), then the local presentation of f is

$$\phi_\beta \circ f \circ \phi_\alpha^{-1}$$

is holomorphic. In this work, S_1 is a genus zero surface with a single boundary, S_2 is a planar domain.

There are many ways to compute conformal mappings. There are mainly four categories.

2.1. Harmonic Maps

Let $f : S \to D$ be a mapping between two surfaces, then the *harmonic energy* of f is defined as

$$E(f) = \int_S |\nabla f|^2 dA,$$

where ∇f is the gradient of f, dA is the area element on S. The harmonic map is the critical point of the harmonic energy, which satisfies the following Laplace equation

$$\Delta f = 0.$$

The harmonic map can be achieved using the heat flow method

$$\frac{df}{dt} = -\Delta f,$$

where Δ is the Laplace-Beltrami operator on S. In general, if the target domain is convex, the boundary mapping $f : \partial S \to \partial D$ is a homeomorphism, then the harmonic map is a diffeomorphism. Especially, if D is a genus zero closed surface, then the hamronic map is also a conformal map. Figure 1 (c) and (d) are computed using harmonic maps.

2.2. Solving Riemann-Cauchy Equation

Conformal maps satisfy the Riemann-Cauchy equation 1. Therefore by solving Riemann-Cauchy equation with boundary conditions, a conformal map can be obtained. in practice, one can solve the equation by minimizing the following energy,

$$E(f) = \int_S (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})^2 + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})^2 dxdy.$$

Figure 1 (g) and (h) are computed by minimizing the above energy using the method described in [22].

2.3. Holomorphic 1-form Method

Let ω be a complex-valued differential form on the Riemann surface S, such that on each local chart (U_α, ϕ_α) with isothermal coordinates z_α, ω has local representation

$$\omega = g_\alpha(z_\alpha)dz_\alpha,$$

where g_α is holomorhpic, then ω is called a *holomorphic 1-form*. On another local chart (U_β, ϕ_β) with isothermal coordinates z_β, ω has local representation

$$\omega = g_\beta(z_\beta)dz_\beta$$

where

$$g_\alpha \frac{dz_\alpha}{dz_\beta} = g_\beta,$$

where $\frac{dz_\alpha}{dz_\beta}$ is a holomorphic function. All the holomorphic 1-form form a group, which is isomorphic to the first cohomology group of the surface.

The holomorphic 1-form group basis can be computed using the following method: first we compute the homology group basis of the surface, the the dual cohomology group basis, then we use Hodge theory to get the unique harmonic 1-form for each cohomologous class. Finally, by using Hodge star, we can compute the conjugate harmonic 1-forms, each pair of harmonic 1-form and its conjugate form a holomorphic 1-form. This method has been introduced in [35].

By using holomorhpic 1-forms, the surface can be conformally mapped to the canonical planar domains. For example, multiply connected domains can be mapped to rectangles with parallel slits. Figure 1 (e) and (f) are computed using holomorphic 1-forms.

2.4. Ricci Curvature Flow

Let S be a surface embedded in $\mathbb{R}^3$. S has a Riemannian metric induced from the Euclidean metric of $\mathbb{R}^3$, denoted by $\mathbf{g}$. Suppose $u : S \to \mathbb{R}$ is a scalar function defined on S. It can be verified that $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$ is also a Riemannian metric on S. Furthermore, angles measured by $\mathbf{g}$ are equal to those measured by $\bar{\mathbf{g}}$. Therefore, we say $\bar{\mathbf{g}}$ is a *conformal* to $\mathbf{g}$, e^{2u} is called the *conformal factor*.

When the Riemannian metric is conformally deformed, Gaussian curvatures will also be changed accordingly. The Gaussian curvature will become

$$\bar{K} = e^{-2u}(-\Delta_{\mathbf{g}} u + K), \tag{2}$$

where $\Delta_{\mathbf{g}}$ is the Laplacian-Beltrami operator under the original metric $\mathbf{g}$. The above equation is called the *Yamabe equation*. By solving the Yamabe equation, one can design a conformal metric $e^{2u}\mathbf{g}$ by a prescribed curvature $\bar{K}$.

Yamabe equation can be solved using *Ricci flow* method. The Ricci flow deforms the metric $\mathbf{g}(t)$ according to the Gaussian curvature $K(t)$ (induced by itself), where t is the time parameter

$$\frac{dg_{ij}(t)}{dt} = 2(\bar{K} - K(t))g_{ij}(t). \tag{3}$$

Ricci flow method can be applied to design Riemannian metric with prescribed Gaussian curvature. If the target curvature is zero on every interior point, then the surface can be flattened onto a planar domain with the resulting metric.

Surface Ricci flow has been generalized to the discrete setting by Luo and Chow in [36]. In surface case, Ricci flow is equivalent to Yamabe flow. Discrete Yamabe flow was first introduced by Luo in [32].

3. Symmetric Conformal Mapping Method

All the existing conformal mapping methods are sensitive to boundary conditions. Surface registration algorithms based on conformal geometric methods are susceptible to occluded boundaries, clutters and inconsistent boundaries. We propose to improve the robustness of conformal mapping methods by utilizing the symmetry of the input surface.

Suppose the input surface S has some symmetries. For example, suppose τ is a plane in $\mathbb{R}^3$, R_τ is the reflection about τ. If S is symmetric about τ, then $R_\tau(S) = S$. Let γ be the intersection curvature of the surface and the symmetric plane, $\gamma = S \cap \tau$, $\phi : S \to \mathbb{C}$ is a conformal mapping of the surface to the complex plane. We say the conformal mapping preserves symmetry, if

$$\phi(R_\tau(p)) = -\phi(p),$$

where $\phi(p)$ means the conjugate of $\phi(p)$. Namely, ϕ maps γ to the imaginary axis, the images of the symmetric points p and $R_\tau(p)$ are symmetric about the imaginary axis. This can be accomplished by adding symmetric constraints during the optimization process.

In practice, surfaces are approximated by triangle meshes, conformal mappings are approximated by piecewise linear maps.

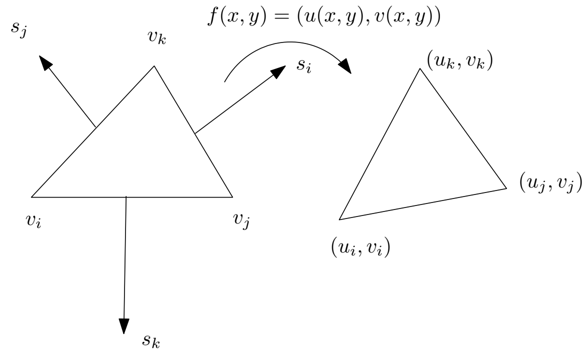

Figure 3. Discrete approximation of Riemann-Cauchy euqation.

3.1. Riemann-Cauchy Equation Method

This method is a direct generalization of LSCM in [22] by adding symmetric constraints. Let $[v_i, v_j, v_k]$ be a face on the mesh. The images of them under the linear map $f : [v_i, v_j, v_k] \to \mathbb{R}^2$ are $(u_i, v_i), (u_j, v_j), (u_k, v_k)$. Let $\mathbf{s}_i = \mathbf{n} \times (v_k - v_j), \mathbf{s}_j = \mathbf{n} \times (v_i - v_k), \mathbf{s}_k = \mathbf{n} \times (v_j - v_i)$, $\mathbf{n}$ is the normal vector of the face, then

$$\begin{aligned} -\nabla u &= u_i \mathbf{s}_i + u_j \mathbf{s}_j + u_k \mathbf{s}_k, \\ -\nabla v &= v_i \mathbf{s}_i + v_j \mathbf{s}_j + v_k \mathbf{s}_k, \end{aligned}$$

Riemann-Cauchy energy on face $[v_i, v_j, v_k]$ can be approximated by

$$E([v_i, v_j, v_k]) = |\nabla v - \mathbf{n} \times \nabla u|^2.$$

The energy 1 can be approximated as

$$\sum_{[v_i, v_j, v_k] \in M} E([v_i, v_j, v_k]) A([v_i, v_j, v_k]), \tag{4}$$

where $A([v_i, v_j, v_k])$ represents the area of the face.

The symmetric constraints can be inserted naturally during the optimization of the above energy. Suppose v_i, v_j are symmetric vertices of the mesh, $R_\tau(v_i) = v_j$, then we add constraint

$$u_i = -u_j, v_i = v_j.$$

3.2. Yamabe Flow Method

Symmetry constraints can also be added to the curvature flow method naturally. Here we use Yamabe flow method introduced in [32]. On a triangle mesh, the *discrete metric* is the edge length function $\ell : E \to \mathbb{R}^+$ satisfying triangle inequality. The *vertex discrete curvature* is defined as angle deficiency,

$$K_i = \begin{cases} 2\pi - \sum_{[v_i, v_j, v_k] \in F} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{[v_i, v_j, v_k] \in F} \theta_i^{jk} & v_i \in \partial M \end{cases}$$

Figure 2. Conformal mapping preserving symmetry. γ is the intersection curve between the surface and the symmetric plane. p and $R_\tau(p)$ are symmetric points. The symmetry is preserved on the image of the conformal mapping ϕ.

where θ_i^{jk} is the corner angle at v_i in the face $[v_i, v_j, v_k]$, ∂M is the boundary of M. Let $\mathbf{u} : V \to \mathbb{R}$ be the discrete conformal factor. The edge length of $[v_i, v_j]$ is defined as

$$\ell_{ij} := \exp(u_i)\exp(u_j)\ell_{ij}^0,$$

where ℓ_{ij}^0 is the original edge length in $\mathbb{R}^3$. The discrete Yamabe flow is defined as

$$\frac{du_i}{dt} = \bar{K}_i - K_i,$$

with the constraint $\sum_i u_i = 0$. The discrete Yamabe flow converges, and the final discrete metric induces the prescribed curvature; a detailed proof can be found in [32].

During the Yamabe flow, we can enforce the symmetry in the following way. Assume v_i and v_j are two symmetric interior vertices, $R_\tau(v_i) = v_j$, $v_i, v_j \notin \partial M$, therefore their target curvatures are the same $\bar{K}_i = \bar{K}_j$, then during the Yamabe flow, we always ensure $u_i = u_j$.

4. Computational Algorithm

The computational algorithm for symmetric conformal mapping is straight forward. It includes the following steps.

4.1. Finding the symmetric plane

Assume the input surface has a reflective symmetric plane τ, this step aims at find the plane. Although there are rich literature on finding symmetry of images, we focus on finding the symmetry of a 3D surface. The generalized Hough transformation has been introduced in [29] for finding the symmetry plane of 3D point clouds. We adapt the method to locate the symmetry plane for our dense point clouds of human face surfaces.

4.2. Finding Feature Points

The scanned data sets have both texture information and geometric information. In current work, we only utilize the texture information for locating feature points. We apply conventional SIFT method [30] on the texture image to find major feature points, such as eye corners, mouth corners etc. The symmetry of feature points can be computed by the method in [31]. Then we project back the feature points from the texture image to the 3D surfaces.

4.3. Cross Registration

$$\begin{tikzcd}
S_1 \arrow[r, "f"] \arrow[d, "\phi_1"'] & S_2 \arrow[d, "\phi_2"] \\
D_1 \arrow[r, "g"'] & D_2
\end{tikzcd}$$

Given two 3D face surfaces S_1 and S_2 of the same person with different expressions and different boundaries, we want to register them using symmetric conformal mapping. First, we compute symmetric conformal maps $\phi_1 : S_1 \to \mathbb{D}_1$, $\phi_2 : S_2 \to \mathbb{D}_2$, using the symmetric information obtained in the first step. Then we compute a constrained harmonic map $g : \mathbb{D}_1 \to \mathbb{D}_2$, such that g align the major corresponding features and also preserves symmetry. The correspondence between the major features are specified by the user. $g = (g_1, g_2)$ minimizes the harmonic energy

$$E(g) = \int_{\mathbb{D}_1} (\frac{\partial g_1}{\partial x} + \frac{\partial g_1}{\partial y})^2 + (\frac{\partial g_2}{\partial x} + \frac{\partial g_2}{\partial y})^2 dxdy,$$

such that $g_1(-x, y) = -g_1(x, y)$, $g_2(-x, y) = g_2(x, y)$. Then the registration is given by

$$f = \phi_2^{-1} \circ g \circ \phi_1 : S_1 \to S_2.$$

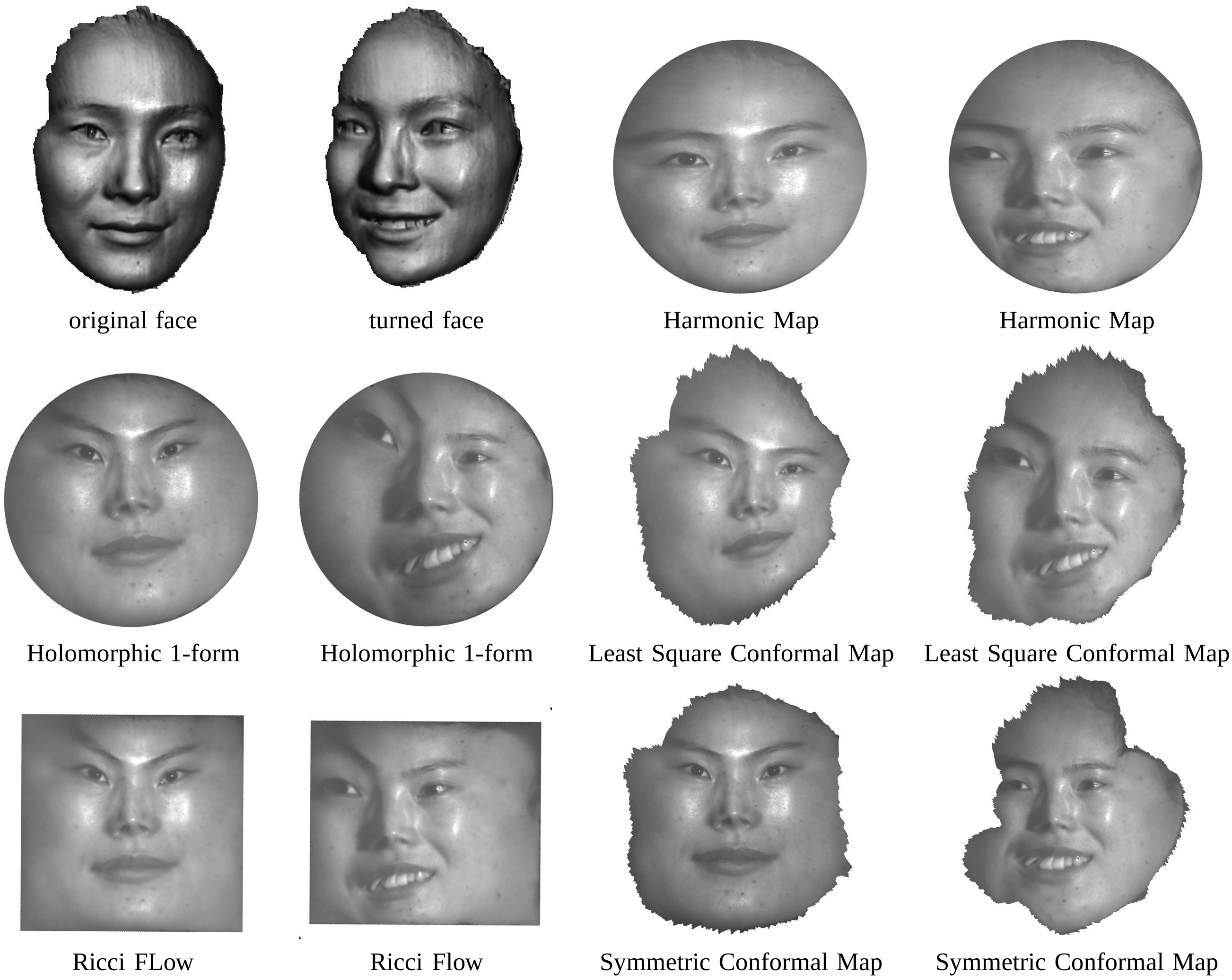

Figure 4. Comparison among different conformal mapping methods. Symmetric conformal map is most robust to inconsistent boundaries.

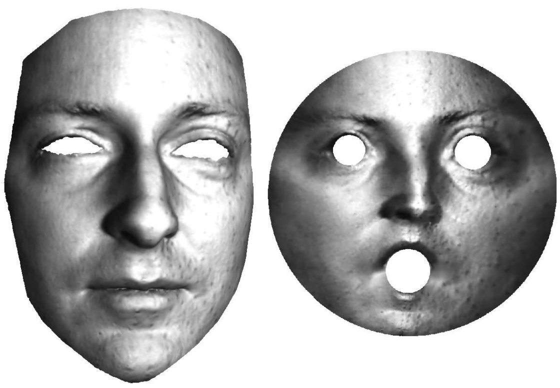

Figure 5. Symmetric Conformal Mapping using Yamabe flow.

5. Experimental Results

We implemented our algorithm using generic C++ on Windows XP and used conjugate gradient optimization for acceleration. The human face data sets are acquired using high speed 3D scanner based on phase-shifting method in [20]. The scanning speed is 30 frames per second, the resolution for each frame is 640×480. The experiments are conducted on a HP xw4600 Workstation with Intel Core 2Duo CPU 2.33GHz, 3.98 GB of RAM. The running time is reported in the following table.

Table 1. Computational time for symmetric conformal maps.

Name	Luke	Anna 1	Anna 2	David 1	David 2
Faces	50,000	156,401	147,430	148,305	147,038
Verts	25,246	78,773	74,281	74,699	74,063
Time (s)	8	17	16	28	14

The symmetric conformal mapping for various human face surfaces are illustrated in figures 1, 4 and 7. The (partial) registration results for face surfaces with different expressions and postures are illustrated in figure 8. Although the boundaries are significantly different, and the registrations are performed on the relatively small overlapping regions, the texture pattern on the overlapping regions among the four frames are very consistent. This demonstrates the robustness of our method.

Figure 5 demonstrates the symmetric Yamabe flow method as described in previous section. The target curvatures are set to preserve the symmetry. During the flow, the conformal factors u are constrained to be symmetric. The final conformal mapping image is also symmetric. This example shows the flexibility of our method, that can handle surfaces with complicated topologies.

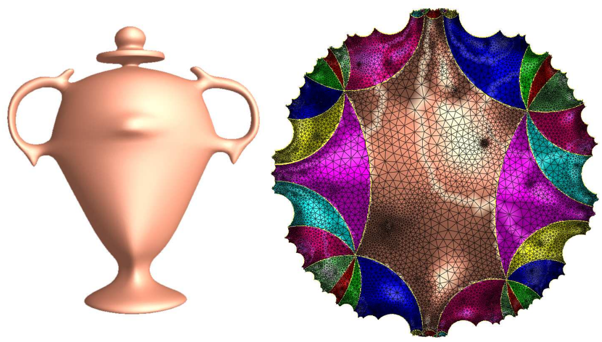

Figure 6. Symmetric hyperbolic Yamabe flow.

6. Conclusion and Future Works

Conventional conformal mapping methods are susceptible to inconsistent boundaries. This work proposes to improve the robustness of conformal geometric methods by incorporating the symmetric information into the mapping process. Novel conformal mapping algorithms based on solving Riemann-Cauchy equation and curvature flow are developed, which preserves the symmetry of the input surface. Experimental results demonstrate the symmetric conformal mapping is insensitive to the boundary occlusions.

Although current work focuses on genus zero surfaces, it can be directly generalized to high genus surfaces as well. Figure 6 demonstrates such an example, a genus two surface is conformally mapped to the hyperbolic space periodically using hyperbolic Yamabe flow method. In the future, we will continue the exploration for high genus surfaces. Furthermore, we will investigate to generalize the method for surfaces with symmetries other than mirror reflection and incorporate more geometric structural characteristics to conformal mappings to improve the robustness and accuracy.

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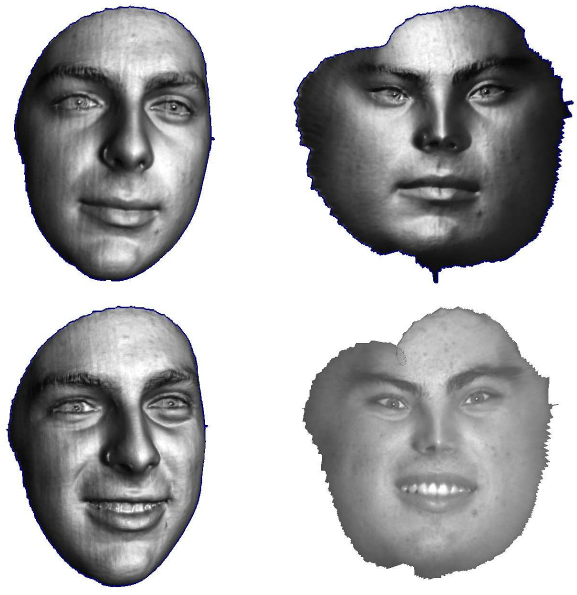

Figure 7. Symmetric Conformal Mapping for David

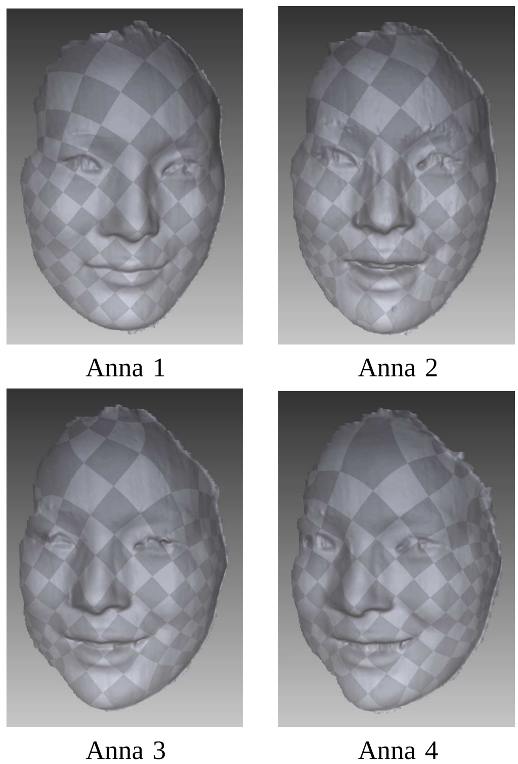

Figure 8. Registration for a sequence of Anna's face surfaces with different expressions and postures.

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