Variational Method on Discrete Ricci Flow

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Conformal geometry is in the core of pure mathematics. It is more flexible than Riemaniann metric but more rigid than topology. Conformal geometric methods have played important roles in engineering fields.

This work introduces a theoretically rigorous and practically efficient method for computing Riemannian metrics with prescribed Gaussian curvatures on discrete surfaces – discrete surface Ricci flow, whose continuous counter part has been used in the proof of Poincaré conjecture. Continuous Ricci flow conformally deforms a Riemannian metric on a smooth surface such that the Gaussian curvature evolves like a heat diffusion process. Eventually, the Gaussian curvature becomes constant and the limiting Riemannian metric is conformal to the original one.

In the discrete case, surfaces are represented as piecewise linear triangle meshes. Since the Riemannian metric and the Gaussian curvature are discretized as the edge lengths and the angle deficits, the discrete Ricci flow can be defined as the deformation of edge lengths driven by the discrete curvature. We invented numerical algorithms to compute Riemannian metrics with prescribed Gaussian curvatures using discrete Ricci flow.

We also showed broad applications using discrete Ricci flow in graphics, geometric modeling, and medical imaging, such as surface parameterization, surface matching, manifold splines, and construction of geometric structures on general surfaces.

Keywords: Conformal geometry, Discrete Ricci flow, Riemannian metric, Gaussian curvature, Global conformal parametrization

1. Introduction

Conformal geometry offers rigorous and powerful theoretic tools for practical engineering applications. Ricci flow is a novel curvature flow method in computational conformal geometry, which will play important roles in practice because of its universality and flexibility.

1.1. *Conformal Geometry*

Conformal geometry is in the core of pure mathematics, which is the intersection of complex analysis, algebraic topology, differential geometry, algebraic geometry, and many other fields in mathematics.

With the development of $3D$ acquisition technologies and computational power, conformal geometry plays more and more important roles in engineering fields. For example, conformal geometry has been broadly applied in computer graphics, computer vision, geometric modeling and medical imaging. The theoretic foundation for computational conformal geometry is developing rapidly and many practical algorithms converting classical theories in conformal geometry have been invented.

So far, the computational methodologies in conformal geometry for general surfaces are mainly in the following categories: harmonic maps, holomorphic differentials, and newly invented method in geometric analysis, *Ricci flow*. Ricci flow is very powerful and flexible, which will cause great impact in engineering field. In this work, we mainly focus on the introduction to the theories and algorithms of discrete surface Ricci flow.

1.2. *Ricci Flow*

Recently, the term of *Ricci flow* becomes popular, due to the fact that it has been applied for the proof of the Poincaré conjecture on 3-manifolds [14–16]. Richard Hamilton introduced the Ricci flow for Riemannian manifolds of any dimension in his seminal work [10] in 1982. Intuitively, a surface Ricci flow is the process to deform the Riemannian metric of the surface. The deformation is proportional to Gaussian curvatures, such that the curvature evolves like the heat diffusion.

It has been considered as a powerful tool for computing the conformal Riemannian metrics with prescribed Gaussian curvatures. For many engineering applications, it is also highly desirable to compute Riemannian metrics on surfaces with prescribed Gaussian curvatures, such as parameterization in graphics, spline construction in geometric modeling, conformal brain mapping in medical images, and so on.

Surface parameterization refers to the process of mapping a surface onto a planar domain. If such a parameterization is known, any functions or signals (e.g., texture) on the flat parametric domain can be easily pulled back to the surface, such that complicated processing on surfaces can be transferred to easy computing on the flat parametric domain. Therefore it is a key ingredient for digital geometry processing, such as texturing [12], deformation [2], and resampling [1]. The process for parameterizing surfaces is quadrivalent to finding a special flat Riemannian metric, with *zero* Gaussian curvatures everywhere.

Constructing splines whose parametric domain is an arbitrarily topological manifold is an important issue for computer-aided geometric modeling [5]. In order to define such parameters and knots of the spline, a special atlas of the surface is required such that all local coordinate transition maps are affine [6]. One way to construct such an atlas is to find a flat metric of the surface first, then locate a collection of patches covering the whole surface, and flatten each patch using the flat metric to form an atlas. Here again, the key step is to obtain the flat metric of the given surface.

In medical imaging field, it is important to deform the human brain cortex surface to the unit sphere in order to easily compare and register several different brain cortexes on a canonical domain [7]. This is equivalent to find a Riemannian metric on the cortex surface, such that the Gaussian curvature induced by the metric equals to *one* everywhere.

Comparing to existing methods, which can only handle a subproblem in the scope of Euclidean parametrization, Ricci flow can handle arbitrary topologies and find arbitrary conformal mappings, which include not just Euclidean, but also hyperbolic and spherical parameterizations.

The discrete Ricci flow on piecewise linear surfaces was introduced in [4]. The existence and convergence of the discrete Ricci flow for surfaces were established. However, the discrete Ricci flow is not a very efficient algorithm for practical use due to the gradient nature of the flow. Recently, we improve the gradient descent method by the Newton's method and drastically speed up the search for the limiting metric by the order of magnitudes. Furthermore, we generalize the results from constant discrete curvature to arbitrarily prescribed discrete curvature, from the metric induced by the combinatorial structure of the mesh to the induced Euclidean metric. We have developed an effective and complete system to compute Riemannian metrics with prescribed Gaussian curvatures on generally topological surfaces has been developed in this paper based on discrete Euclidean Ricci flow, discrete hyperbolic Ricci flow, and discrete spherical Ricci flow.

1.3. *Outline*

In Sec. 2, theoretic background in differential geometry and Riemannian surface are introduced; in Sec. 3, the theories and algorithms of discrete surface Ricci flow are given; practical applications are presented in Sec. 4; in the conclusion Sec. 5, future directions are pointed out.

2. Theoretical Background

In this section, we introduce several important concepts of differential geometry and Riemannian surface, which are directly related with our algorithms.

2.1. *Riemannian Metric*

Suppose S is a C^2 smooth surface embedded in $\mathbb{R}^3$ with local parameter $S(u_1, u_2)$. Let $\mathbf{r}(u_1, u_2)$ be a point on S and $d\mathbf{r} = \mathbf{r}_1 du_1 + \mathbf{r}_2 du_2$ be the tangent vector defined at that point, where $\mathbf{r}_1, \mathbf{r}_2$ are the partial derivatives of $\mathbf{r}$ with respect to u_1 and u_2, respectively. We call the length of the tangent vector as the *Riemannian metric*, and it is calculated using the *first fundamental form* as follows:

$$< d\mathbf{r}, d\mathbf{r} >=< \mathbf{r}_u, \mathbf{r}_u > du^2 + 2 < \mathbf{r}_u, \mathbf{r}_v > dudv + < \mathbf{r}_v, \mathbf{r}_v > dv^2.$$

The first fundamental form is an *intrinsic* property since it is independent of the choice of the surface parameterization as well as the rigid motion of the given surface S.

In this discrete setting, the edge lengths of a mesh Σ are sufficient to define the Riemannian metric on Σ,

$$l : E \to \mathbb{R}^+,$$

as long as for any single face f_{ijk}, the edge lengths satisfy the triangle inequality: $l_{ij} + l_{jk} > l_{ki}$.

2.2. *Gaussian Curvature*

The Gaussian curvature $K(p)$ of a point p on a surface S is defined as the ratio between the infinitesimal area on the unit sphere and the infinitesimal area around p on S. The area on the unit sphere is defined by the bunch of normals in the infinitesimal area around p. It determines whether a surface is locally plane(when it is zero), locally convex (when it is positive) or locally saddle (when it is negative). But from Gauss's *remarkable theorem* [9], Gaussian curvature is also an intrinsic property of the surface, meaning it does not depend on the particular embedding of the surface, only depending on the Riemannian metric of the surface.

The discrete Gaussian curvature on a mesh can be computed from the angle deficit [13],

$$K_i = \begin{cases} 2\pi - \sum_{f_{ijk} \in F} \theta_i^{jk}, & \textit{interior vertex} \\ \pi - \sum_{f_{ijk} \in F} \theta_i^{jk}, & \textit{boundary vertex} \end{cases} \tag{1}$$

where θ_i^{jk} represents the inner angle of vertex v_i in the face f_{ijk}.

2.3. *Gauss-Bonnet Theorem*

Gauss-Bonnet theorem [9] explains the connection between the total Gaussian curvatures on a surface S (the integration of Gaussian curvatures over the surface) and the topology of S.

$$\int_S KdA + \int_{\partial S} k_g ds = 2\pi\chi(S), \tag{2}$$

where K, k_g, and $\chi(S)$ are the Gaussian curvature on S, the geodesic curvature along boundaries of S, and the Euler number of S, respectively.

In the discrete setting, the Gauss-Bonnet theorem (Eq. 2) still holds on meshes as follows.

$$\sum_{v_i \in V} K_i + \lambda \sum_{f_i \in F} A_i = 2\pi\chi(M),$$

where A_i denotes the area of face f_i, and λ represents the constant curvature for the canonical geometry; $+1$ for the sphere, 0 for the plane, and -1 for the hyperbolic space.

2.4. *Conformal Deformation*

Suppose a surface S is embedded in $\mathbb{R}^3$, then it has a Riemannian metric, induced from the Euclidean metric of $\mathbb{R}^3$, denoted by $\mathbf{g}$. Suppose $u : S \to \mathbb{R}$ is a scalar function defined on S It can be verified that $e^{2u}\mathbf{g}$ is another Riemannian metric on S, denoted by $\bar{\mathbf{g}}$. It can also be proven that angles measured by $\mathbf{g}$ are equal to those measured by $\bar{\mathbf{g}}$, which means $\bar{\mathbf{g}}$ is conformal to $\mathbf{g}$. It maps infinitesimal circles to infinitesimal circles and preserves the intersection angles among the circles, which calls conformal map. In Fig. 1, the bunny surface is conformally mapped to plane, and its coordinates in plane is used as texture coordinates to pull the plane textures(check-board and tangent circles) back to the surface. As we can see, those right angles and tangency property of circles are well preserved by the mapping.

When the Riemannian metric is conformally deformed, curvatures will also be changed accordingly. Suppose $\mathbf{g}$ is changed to $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$. Then the Gaussian curvature will become

$$\bar{K} = e^{-2u}(-\Delta u + K), \tag{3}$$

where Δ is the Laplacian–Beltrami operator under the original metric $\mathbf{g}$. The geodesic curvature will become

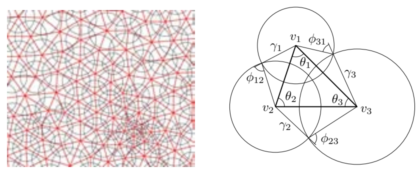
$$\bar{k} = e^{-u}(\partial_{\mathbf{r}} u + k), \tag{4}$$

where $\mathbf{r}$ is the tangent vector orthogonal to the boundary. According to Gauss-Bonnet theorem, the total curvature is still:

$$\int_S KdA + \int_{\partial S} kds = \int_{\bar{S}} \bar{K}d\bar{A} + \int_{\partial\bar{S}} \bar{k}d\bar{s} = 2\pi\chi(S), \tag{5}$$

where $\chi(S)$ is the Euler characteristic number of S and ∂S is the boundary of S.

Fig. 1. The bunny surface is conformally mapped to plane, and its coordinates in plane is used as texture coordinates to pull the plane textures(check-board and tangent circles) back to the surface. Conformal mapping preserves those right angles in the left one, also maps infinitesimal circles to infinitesimal circles and keeps tangency in the right one.



(a) (b)

Fig. 2. Circle Packing Metric: (a) Flat circle packing metric (b) Circle packing metric on a triangle.

In order to approximate conformal deformation of metrics in discrete setting, the circle packing metric is introduced in [17,18]. Let us denote Γ as a function which assigns a radius γ_i to each vertex v_i,

$$\Gamma : V \to \mathbb{R}^+.$$

Similarly, let a *weight* a function on the mesh,

$$\Phi : E \to [0, \frac{\pi}{2}],$$

an acute angle $\Phi(e_{ij})$ to each edge e_{ij}. The pair of vertex radius and edge weight function on a mesh Σ, (Γ, Φ), is called a *circle packing metric* of Σ. If all the intersection angles are acute, then the edge lengths induced by a circle packing satisfy triangle inequality.

Fig 2. illustrates the circle packing metrics. Each vertex v_i has a circle whose radius is γ_i. For each edge e_{ij}, the intersection angle ϕ_{ij} is defined by the two circles of v_i and v_j, which either intersect or are tangent. Two circle packing metrics (Γ_1, Φ_1) and (Γ_2, Φ_2) on a same mesh are *conformally equivalent*, if $\Phi_1 \equiv \Phi_2$. Therefore, a conformal deformation of a circle packing metric only modifies the vertex radii.

3. Discrete Surface Ricci Flow

3.1. *Smooth Surface Ricci Flow*

Suppose S is a smooth surface with Riemannian metric $\mathbf{g}$. The Ricci flow deforms the metric $\mathbf{g}(t)$ according to its induced Gaussian curvature $K(t)$, where t is the time parameter

$$\frac{dg_{ij}(t)}{dt} = -2K(t)g_{ij}(t). \tag{6}$$

Suppose $T(t)$ is a temperature field on the surface. The heat diffusion equation is $dT(t)/dt = -\Delta T(t)$, where Δ is the Laplace-Beltrami operator induced by the surface metric. The temperature field becomes more and more uniform with the increase of t, and it will become constant eventually.

In a physical sense, the curvature evolution induced by the Ricci flow is exactly the same as heat diffusion on the surface, as follows:

$$\frac{dK(t)}{dt} = -\Delta_{\mathbf{g}(t)}K(t), \tag{7}$$

where $\Delta_{\mathbf{g}(t)}$ is the Laplace-Beltrami operator induced by the metric $\mathbf{g}(t)$. We can simplify the Ricci flow in Eq. 6 with $g(t) = e^{2u(t)}g(0)$, then the Ricci flow is

$$\frac{du(t)}{dt} = -2K(t). \tag{8}$$

The Ricci flow defined in Eq. 6. is convergent and leads to a conformal metric [3,10].

3.2. *Theory of Discrete Surface Ricci Flow*

Suppose Σ is a discrete surface with an initial circle packing metric. Let u_i be

$$u_i = \begin{cases} \log \gamma_i & \mathbb{E}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases} \tag{9}$$

then the discrete Ricci flow is defined as follows.

$$\frac{du_i(t)}{dt} = (\bar{K}_i - K_i) \tag{10}$$

Similarly, in the discrete case, the circle packing metric determines the discrete Gaussian curvature, and the discrete Ricci flow conformally deforms the circle packing metric with respect to the Gaussian curvature (Eqs. 9 and 10).

Discrete Ricci flow can be formulated in the variational setting, since it is a negative gradient flow of one special energy form. Let Σ a triangle mesh which admits spherical (Euclidean or hyperbolic) geometry. For arbitrary two vertices v_i, v_j, the following symmetric relation holds

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}.$$

Let $\omega = \sum_{i=1}^{n} K_i du_i$ be a differential one-form [19]. The symmetric relation guarantees that the one-form is closed (curl free) in the simply connected u domain.

$$d\omega = \sum_{i,j} (\frac{\partial K_i}{\partial u_j} - \frac{\partial K_j}{\partial u_i}) du_i \wedge du_j = 0.$$

By Stokes theorem, the following integration is path independent,

$$f(\mathbf{u}) = \int_{\mathbf{u_0}}^{\mathbf{u}} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i, \tag{11}$$

where $\mathbf{u_0}$ is an arbitrary initial metric. Therefore, the above integration is well defined, and called the *Ricci energy*. The discrete Ricci flow is the negative gradient flow of the discrete Ricci energy. The discrete metric which induces $\bar{K}$ is the minimizer of the energy.

Computing the desired metric with constant or user-defined curvature $\bar{K}$ is equivalent to minimizing the discrete Ricci energy. For the Euclidean (or hyperbolic) case, the discrete Ricci energy is strictly convex (namely, its Hessian is positive definite). The global minimum uniquely exists, corresponding to the metric $\bar{\mathbf{u}}$, which induces $\bar{\mathbf{k}}$. The discrete Ricci flow converges to this global minimum [4].

Theorem 1 (Chow & Luo: Euclidean Ricci Energy. The Euclidean Ricci energy $f(\mathbf{u})$ on the space of normalized metric $\sum u_i = 0$ is strictly convex.

Theorem 2 (Chow & Luo: Hyperbolic Ricci Energy). The hyperbolic Ricci energy is strictly convex.

3.3. *Discrete Euclidean Ricci Flow for Genus One Surface*

Discrete Euclidean Ricci flow method computes special metrics of the surface conformal to the original metric with prescribed target curvature. For genus one closed surfaces, we set the target Gaussian curvature to be zero everywhere. The universal covering space of the surface can be isometrically embedded on the plane. Fig. 3(b) shows one example. The kitten surface is of genus one, the universal covering space is embedded on the plane. The rectangle is a fundamental polygon.

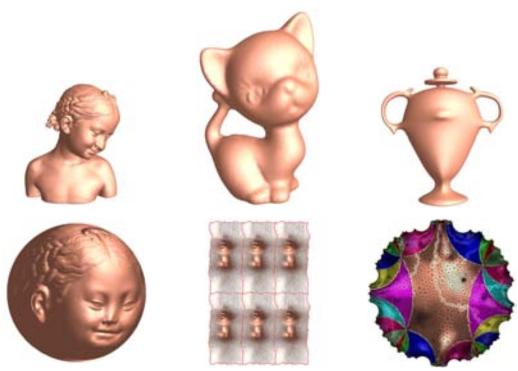
3.4. *Discrete Hyperbolic Ricci Flow for High Genus Surface*

For high genus surfaces, there exists a unique Riemannian metric, which is conformal to the original Riemannian metric, and induces constant Gaussian curvature everywhere, the constant is -1. Such kind of metric can be computed using discrete hyperbolic Ricci flow. The universal covering space of the surface can be isometrically embedded on the hyperbolic space. Fig. (c) demonstrates the embedding of the universal covering space of a genus two surface on the Poincaré model of hyperbolic space.

3.5. *Discrete Spherical Ricci Flow for Genus Zero Surface*

For genus zero surfaces, there exists a unique Riemannian metric, which is conformal to the original Riemannian metric, and induces constant Gaussian curvature everywhere, the constant is $+1$. Such kind of metric can be computed using discrete spherical Ricci flow. The universal covering space of the surface is itself. Fig. 3(a) demonstrates the embedding of the universal covering space of a genus zero surface on the unit sphere.

The conformal metrics and the curvatures of a surface are essentially of one-to-one correspondence. The conformal metric can be computed using a prescribed curvature on the surface using Euclidean Ricci flow method. Fig. 4 shows one example. The input surface is a topological disk. It is mapped to the planar domains specified by curvature on the boundaries. The curvature of interior points are zero everywhere. The conformal mapping induced by the metric is fully controlled by the prescribed boundary curvatures.



(a) $\chi > 0$ (b) $\chi = 0$ (c) $\chi < 0$

Fig. 3. (a) Spherical metric of a genus zero surface, computed using discrete Spherical Ricci flow. (b) Conformal flat metric of a genus one surface, computed using discrete Euclidean Ricci flow. (c) Hyperbolic metric of a genus two surface, computed using discrete hyperbolic Ricci flow.

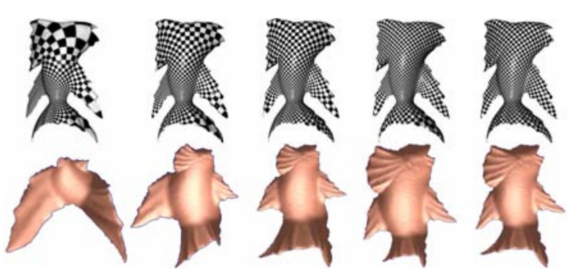


Fig. 4. Conformal flat metrics are designed by the target curvature.

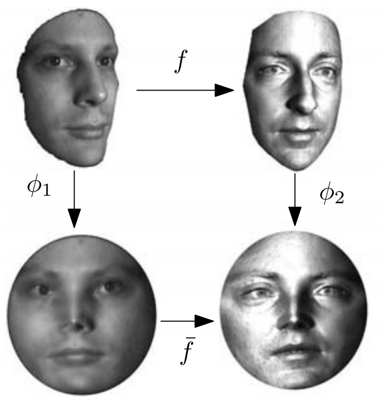


Fig. 5. Surface matching using conformal mapping.

4. Applications of Discrete Ricci Flow

4.1. *Surface Matching*

Surface matching is a fundamental task for computer vision, graphics and medical imaging. Fig.5 shows the basic idea of using conformal mappings to convert $3D$ matching problems to $2D$ ones. Suppose S_1 and S_2 are two surfaces in $\mathbb{R}^3$. $\phi_1 : S_1 \to \mathbb{D}$ and $\phi_2 : S_2 \to \mathbb{D}$ are conformal mappings to map surfaces to the canonical planar domain. $\bar{f} : \mathbb{D} \to \mathbb{D}$ is a a map from D to itself, this is a $2D$ matching process. Then

$$f = \phi_2^{-1} \circ \bar{f} \circ \phi_1, S_1 \to S_2,$$

is the desired $3D$ matching.

4.2. *Shape Space*

Surfaces can be classified by conformal equivalence. For closed surfaces with one handle (genus one), all conformal classes form a 2 dimensional space, namely, each conformal class can be represented by 2 real parameters. For genus $g > 1$ closed surfaces, all the conformal equivalent classes form a $6g - 6$ dimensional space, which is called the Teichmüller space. The *Teichmüller Coordinates* of a surface can be explicitly computed and as the fingerprint of the shape, which can be applied to geometric database indexing and the shape comparison purposes.

Given a pair of topological pants (a topological annulus with two holes), we can compute a unique hyperbolic metric and embed it in the hyperbolic space. The Teichmüller coordinates of it are the geodesic lengths of three boundaries under the hyperbolic metric. Fig. 6 shows three such kind of surfaces and their embedding in the hyperbolic space. If the fundamental polygons are congruent in the hyperbolic space, the corresponding surfaces are conformally equivalent. It is easy to see that every surface in the figure is not conformal equivalent to any other one.

4.3. *Manifold Splines*

Manifold spline is introduced in [6], which generalizes planar splines to general surface domains. The key idea is that most planar splines are based on polar forms, which are *parametric affine invariant*. That is, if we change the parameters by an affine transformation, and rebuild the spline surface, then the shape of the spline surface doesn't change. Therefore, if

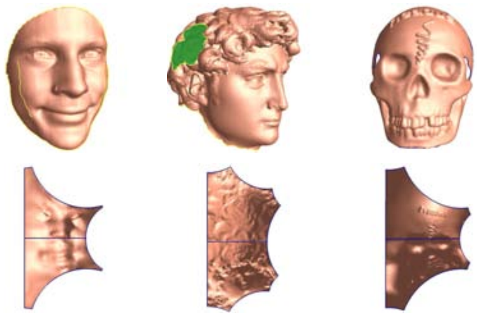


Fig. 6. The coordinates of topological annuli with 2 holes in the shape space are the lengths of their boundaries under the hyperbolic metric. The shape of their fundamental polygon indicates the shape space coordinates.

(a) Parametric domain (b) The central chart (c) Manifold splines (d) Spline patchwork (e) Control point distribution.

Fig. 7. Examples of manifold triangular *B*-spline. The affine atlas are computed using discrete Euclidean Ricci flow, and The transition function is a combination of translation and rotation.

we can cover a surface domain by a set of coordinate charts, such that the coordinate transition functions are affine, then we can construct splines defined on the surface domains. Details can be found in [6].

It is very challenging to construct such kind of affine atlas. and a set of affine atlas can be constructed only on closed genus one surfaces or open surfaces. Splines defined on manifold domains with complicated topologies are plagued by the existence of topology-dependent singularity points, where the surface is not continuous any more. The previous method in [6] uses differential forms borrowed from [8] and [11] to construct affine atlas, which has to produce $2g - 2$ singularities for a genus g surface.

While using discrete Euclidean Ricci flow, all curvatures of a closed manifold with complicated topology can be put to only one singular vertex, with the position of this singularity controllable, which produces an everywhere flat metric of the manifold domain except at one singular point. This metric induces an affine atlas covering the whole manifold except for the one singular point, such that a high degree continuous manifold spline is constructed over this affine atlas except this point filled out with minimal surface.

5. Future Works

Computational conformal geometry is an emerging field. There are a lot of challenging open problems both in theory and in practice. Establishing the convergence of discrete conformal mapping to the smooth solution and estimating the error bounds are widely open. Designing algorithms to compute extremals quasi-conformal maps, designing data structures for holomorphic quadratics are under investigation. Applying computation conformal geometric methods for broader applications and adapt them to real systems is also developing.

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