Surface Differential Geometry, Movable Frame Method

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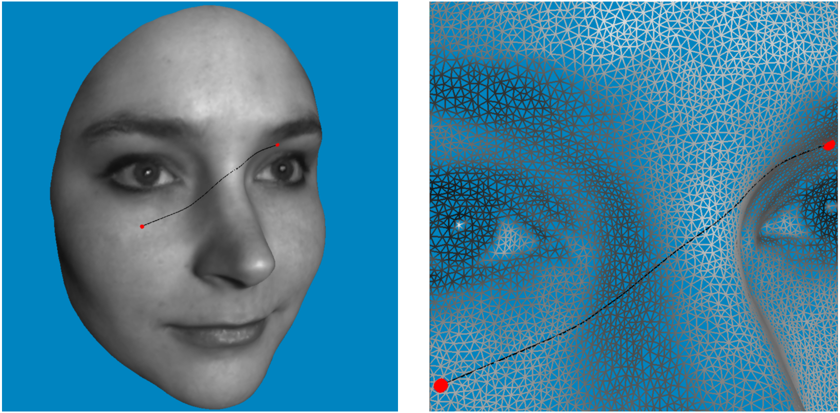


Figure: Geodesic on polyhedral surfaces.

Geodesic on a surface $\gamma : [0,1] \to (S, \mathbf{g})$:

$$D_{\dot{\gamma}}\dot{\gamma} \equiv 0.$$

Compute Geodesics

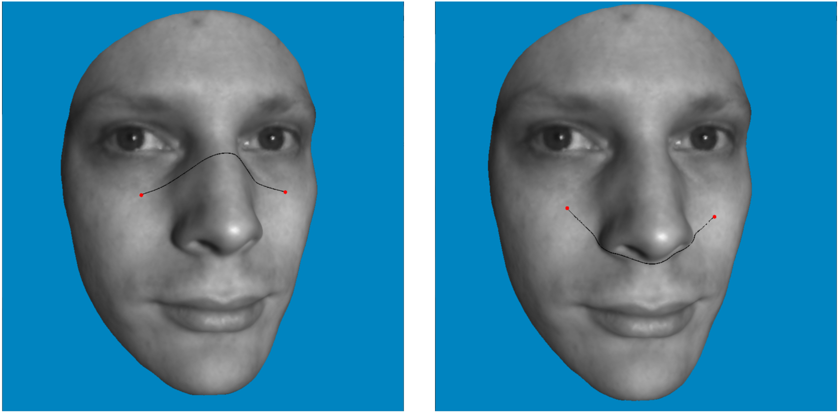
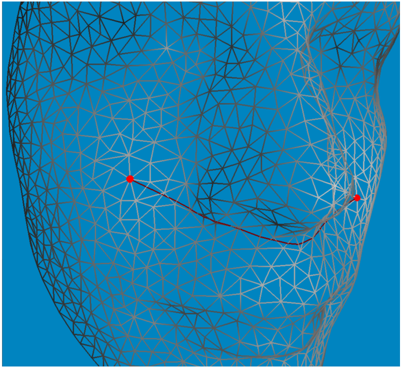


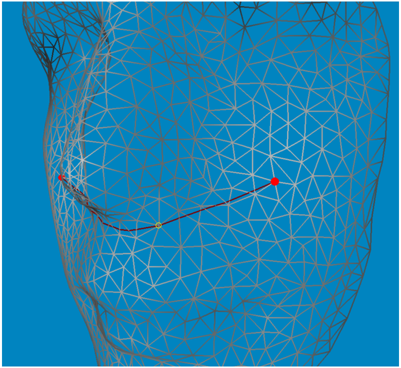
Figure: Conjugate point of geodesics.

Geodesic on a surface $\gamma : [0,1] \to (S, \mathbf{g})$:

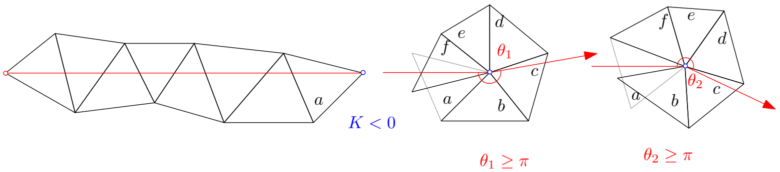
$$D_{\dot{\gamma}}\dot{\gamma} \equiv 0.$$

Discrete Geodesics





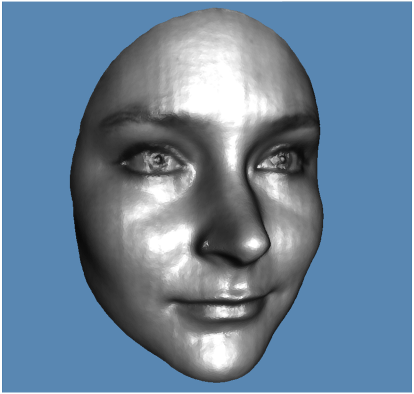
Discrete Geodesics



Suppose γ is a discrete geodesic:

1. isometrically flatten the strip of curve γ onto the plane;
2. when the γ crosses an edge, it is straight;
3. γ never crosses any convex vertex;
4. when γ crosses a concave vertex, if we flatten the neighborhood from right, then $\theta_1 \geq \pi$; flatten from left, $\theta_2 \geq \pi$.

Discrete Harmonic Map



Smooth surface harmonic map $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$, $\Delta_{\mathbf{g}}\varphi \equiv 0$, with Dirichlet boundary condition $\varphi|_{\partial}S = f$. A discrete harmonic ma satisfies $\sum_{v_i \sim v_j} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0$, $\forall v_i \notin \partial M$.

Compute Minimal Surface

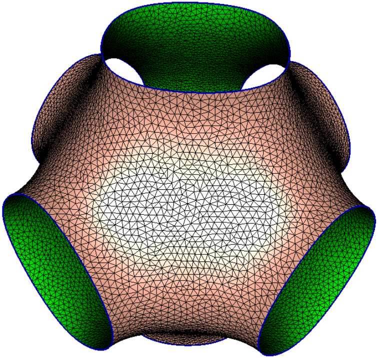
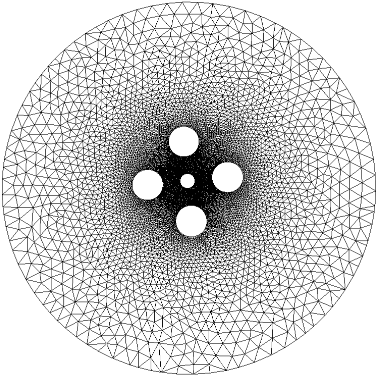
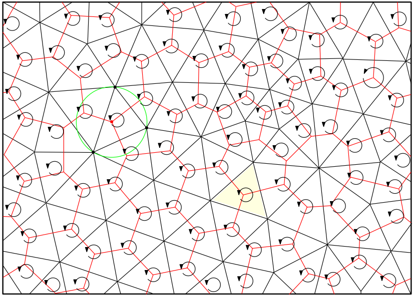
 

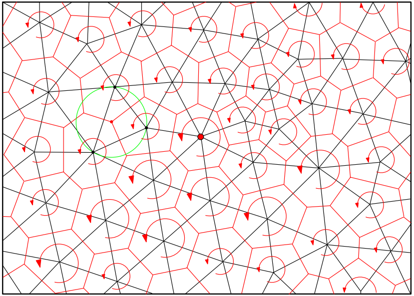
Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$.

Discarete Harmonic One-Form



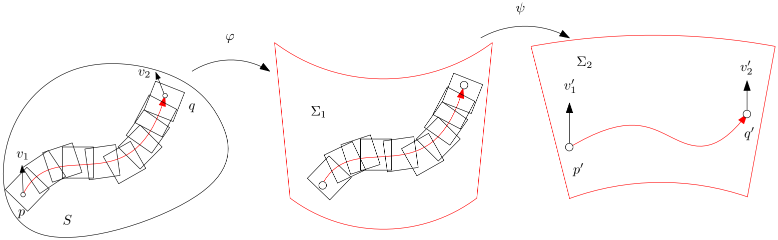
$d\omega = 0$



$\delta\omega = 0$

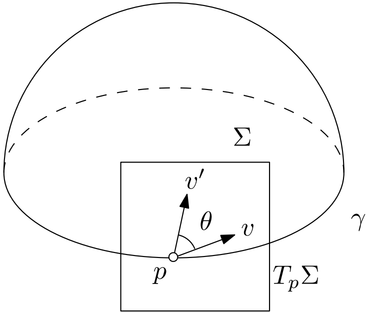
Harmonic map $\varphi : M \to \mathbb{D}^2$; minimal surface $\varphi : M \to \mathbb{R}^3$.

Parallel Transport



Given $\gamma \subset S$, find an envelope surface Σ_1 of all the tangent planes along γ, $\varphi : \gamma \to \Sigma_1$ isometrically maps γ to Σ_1. Σ_1 is developable, flatten Σ_1 to obtain a planar domain Σ_2, $\psi : \Sigma_1 \to \Sigma_2$. The composition $\psi \circ \varphi$ maps $p, q, v_1 \in T_pS, v_2 \in T_pS$ to p', q', v'_1, v'_2. On the plane, translate a tangent vector v'_1 from starting point p to the ending point q to get v'_2, maps back v'_2, $v_2 = (\psi \circ \varphi)^{-1}(v'_2)$. Then v_1 is parallelly transported along γ to get v_2.

Gaussian Curvature



Parallel transport v along $\partial\Sigma$, to get v' when returned to the original point p, then the angle difference between v and v' equals to the total Gaussian curvature,

$$\theta = \int_\Sigma K dA.$$

Gaussian Curvature

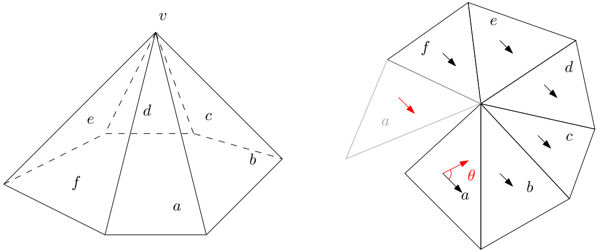


Figure: Discrete parallel transport, $K(v) = \theta$.

Parallel transport a vector, when return to the original position, the difference angle equals to the discrete Gaussian curvature of the interior vertices.

Gaussian Curvature

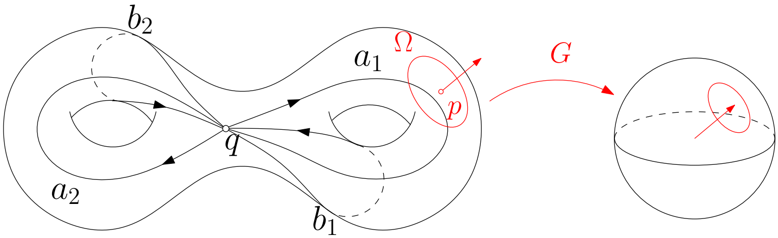


Figure: Gaussian curvature.

Gauss map: $\mathbf{r}(p) \mapsto \mathbf{n}(p)$,

$$K(p) := \lim_{\Omega \to \{p\}} \frac{|G(\Omega)|}{|\Omega|}$$

Gaussian Curvature

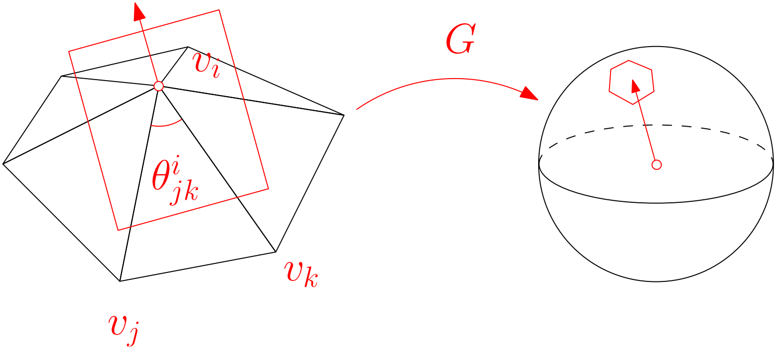


Figure: Discrete Gaussian curvature.

$$G(v_i) := \{\mathbf{n} \in \mathbb{S}^2 | \exists \text{Support plane with normal } \mathbf{n}\}.$$

Gaussian Curvature

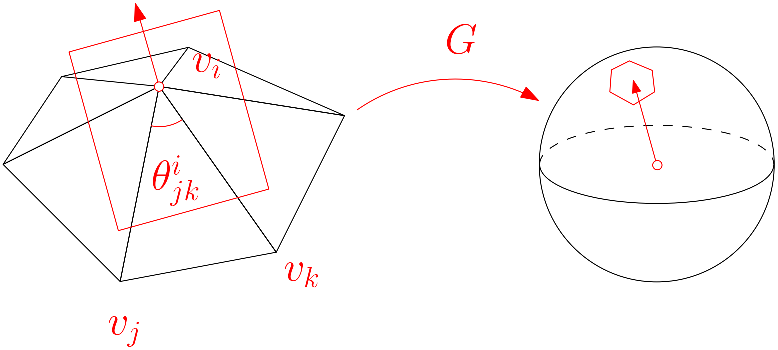


Figure: Discrete Gaussian curvature for convex vertex.

$$K(v_i) := |G(v_i)| = 2\pi - \sum_{jk} \theta^i_{jk}.$$

Gauss-Bonnet

For a closed oriented metric surface $(S, \mathbf{g})$,

$$\int_S K dA = 2\pi\chi(S).$$

For a closed oriented discrete polygonal surface M,

$$\sum_{v_i} K(v_i) = 2\pi\chi(M).$$

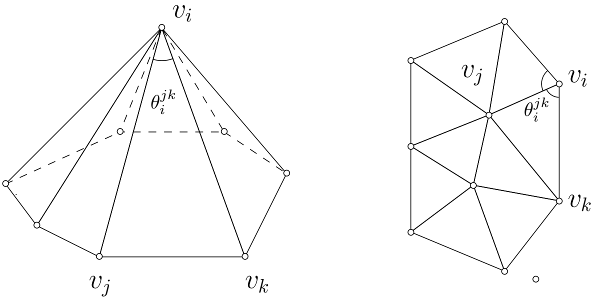


Figure: Discrete Gaussian curvature.

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \quad (1)$$

Gauss-Bonnet

Theorem (Discrete Gauss-Bonnet Theorem)

Given polyhedral surface $(S, V, \mathbf{d})$, *the total discrete curvature is*

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(S),$$

where $\chi(S)$ *is the Euler characteristic number of* S.

Proof.

We denote the polyhedral surface $M = (V, E, F)$, if M is closed, then

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jK} = 2\pi|V| - \pi|F|.$$

Since M is closed, $3|F| = 2|E|$,
$\chi(S) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F| = |V| - \frac{1}{2}|F|.$ □

Discrete Guass-Bonnet

continued.

Assume M has bounary ∂M. Assume the interior vertex set is V_0, boundary vertex set is V_1, then $|V| = |V_0| + |V_1|$; assume interior edge set is E_0, boundary edge set is E_1, then $|E| = |E_0| + |E_1|$. Furthermore, all boundaries are closed loops, hence boundry vertex number equals to the boundary edge number, $|V_1| = |E_1|$. Every interior edge is adjacent to two faces, every boundary edge is adjacent to one face, we have $3|F| = 2|E_0| + |E_1| = 2|E_0| + |v_1|$. We compute the Euler number

$$\chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1| = |V_0| + |F| - |E_0|,$$

by $|E_0| = 1/2(3|F| - |V_1|)$

$$\chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|$$

Discrete Guass-Bonnet

continued.

we have:

$$
\begin{aligned}
\sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) &= \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk} \right) \\
&= 2\pi|V_0| + \pi|V_1| - \pi|F| \\
&= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right) \\
&= 2\pi\chi(M).
\end{aligned}
\tag{2}
$$

□.

Movable Frame

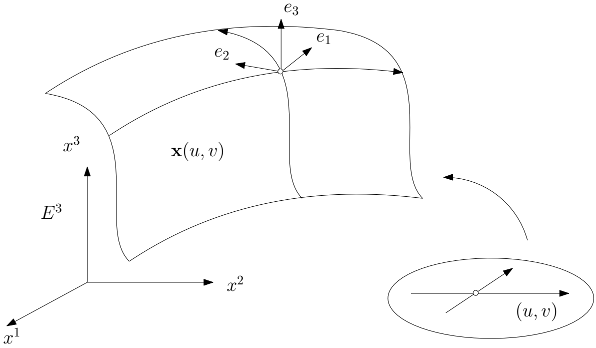


Figure: A parametric surface.

Orthonormal Movable frame

Movable Frame

Suppose a regular surface S is embedded in $\mathbb{R}^3$, a parametric representation is $\mathbf{r}(u,v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let $\mathbf{e}_3$ be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orhonormal frame field* of the surface.

Orthonormal Movalbe frame

Tangent Vector

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$. $d\mathbf{r}$ is orthogonal to the normal vector $\mathbf{e}_3$.

Motion Equation

$$d\mathbf{e}_i = \omega_{i1}\mathbf{e}_1 + \omega_{i2}\mathbf{e}_2 + \omega_{i3}\mathbf{e}_3,$$

where $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$. Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \quad 0 = d\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle + \langle \mathbf{e}_i, d\mathbf{e}_j \rangle$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

Motion Equation

Motion Equation

$$d\mathbf{r} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1\omega_1 + \omega_2\omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1\omega_{31} - \omega_2\omega_{32} = \omega_1\omega_{13} + \omega_2\omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \to \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \to d\mathbf{e}_3, \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 \to \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K\omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gaussian curvature

Weigarten Mapping

$\{\omega_1, \omega_2\}$ form the basis of the cotangent space, therefore ω_{13}, ω_{23} can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so $K = h_{11}h_{22} - h_{12}h_{21}$, the mean curvature $H = \frac{1}{2}(h_{11} + h_{22})$.

Gauss's theorem Egregium

Theorem (Gauss' Theorem Egregium)

The Gaussian curvature is intrinsic, solely determined by the first fundamental form.

Proof.

$$
\begin{aligned}
0 =& d^2\mathbf{e}_1 \\
=& d(\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) \\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge d\mathbf{e}_2 + d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge d\mathbf{e}_3 \\
=& d\omega_{12}\mathbf{e}_2 - \omega_{12} \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) + \\
& d\omega_{13}\mathbf{e}_3 - \omega_{13} \wedge (\omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2) \\
=& (d\omega_{12} - \omega_{13} \wedge \omega_{32})\mathbf{e}_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23})\mathbf{e}_3
\end{aligned}
$$

therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{23} = -K\omega_1 \wedge \omega_2.$$

Gauss's theorem Egregium

Lemma

$$\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2$$

Proof.

$$\begin{aligned}
0 &= d^2\mathbf{r} \\
&= d(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2\mathbf{e}_2 - \omega_w \wedge d\mathbf{e}_2 \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge (\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) + \\
& \quad d\omega_2\mathbf{e}_2 - \omega_2 \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) \\
&= (d\omega_1 - \omega_2 \wedge \omega_{21})\mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12})\mathbf{e}_2 + \\
& \quad -(\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23})\mathbf{e}_3.
\end{aligned}$$

Therefore $d\omega_1 = \omega_2 \wedge \omega_{21}$,$d\omega_2 = \omega_1 \wedge \omega_{12}$ and $h_{12} = h_{21}$. □

Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvautre is given by

$$K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u.$$

Proof.

Let $(S, \mathbf{g})$ be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then

$$\begin{cases} \omega_1 &= e^u dx \\ \omega_2 &= e^u dy \end{cases} \quad \begin{cases} \mathbf{e}_1 &= e^{-u}\frac{\partial}{\partial x} \\ \mathbf{e}_2 &= e^{-u}\frac{\partial}{\partial y} \end{cases}$$

□

Continued.

By direct computation,

$$
\begin{aligned}
d\omega_1 &= de^u \wedge dx & d\omega_2 &= de^u \wedge dy\\
&= e^u(u_x dx + u_y dy) \wedge dx & &= e^u(u_x dx + u_y dy) \wedge dy\\
&= e^u u_y dy \wedge dx & &= e^u u_x dx \wedge dy.
\end{aligned}
$$

therefore

$$
\begin{aligned}
\omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2\\
&= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy\\
\omega_{12} &= -u_y dx + u_x dy.
\end{aligned}
$$

Gaussian Curvature

Continued.

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge_\omega 2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u} dx \wedge dy} = -\frac{1}{e^{2u}} \Delta u.$$

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$

□

Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1-x^2-y^2) - 2x(-2x)}{(1-x^2-y^2)^2} = \frac{2+2x^2-2y^2}{(1-x^2-y^2)^2}$$

similarly

$$u_{yy} = \frac{2+2y^2-2x^2}{(1-x^2-y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1-x^2-y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx}+u_{yy}) = -1.$$

□

Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, *then*

$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}
\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda) \\
&= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\
&= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).
\end{aligned}$$

Theorem (Gauss-Bonnet)

Suppose M is a closed orientable C^2 surface, then

$$\int_M K dA = 2\pi\chi(M),$$

where dA is the area element of hte surface, $\chi(M)$ is the Euler characteristic number of M.

Proof.

Construct a smooth vector field v, with isolated zeros $\{p_1, p_2, \cdots, p_n\}$. Choose a small disk $D(p_i, \varepsilon)$. On the surface

$$\bar{M} = M \setminus \bigcup_{i=1}^{n} D(p_i, \varepsilon)$$

□

Gauss-Bonnet Theorem

Proof.

construct orthonormal frame $\{p, e_1, e_2, e_3\}$, where

$$e_1(p) = \frac{v(p)}{|v(p)|}, \quad e_3(p) = n(p).$$

The integration

$$\int_{\bar{M}} KdA = \int_{\bar{M}} K\omega_1 \wedge \omega_2 = -\int_{\bar{M}} d\omega_{12}$$

by Stokes theorem and Poincarère-Hopf theorem, we obtain

$$-\sum_{i=1}^{n} \int_{\partial D(p_i,\varepsilon)} \omega_{12} = 2\pi \sum_{i=1}^{n} \text{Index}(p_i, v) = 2\pi\chi(M).$$

Here by $\omega_{12} = \langle de_1, e_2 \rangle$, ω_{12} is the rotation speed of e_1. Let $\varepsilon \to 0$, the equation holds. □

Computing Geodesics

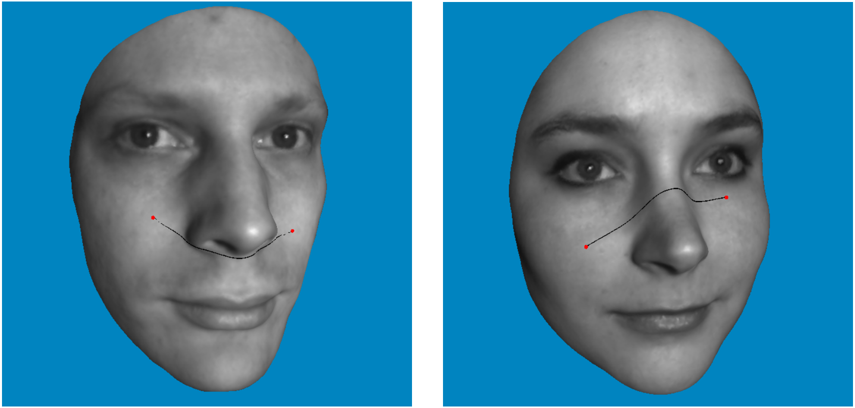


Figure: Geodesics.

Covariant Differential

Definition (Covariant Differentiation)

Covariant differentiation is the generalization of directional derivatives, satisfies the following properties: assume v and w are tangent vector fields on a surface, $f : S \to \mathbb{R}$ is a C^1 function, then

1. $D(\mathbf{v}+\mathbf{w}) = D(\mathbf{v}) + D(\mathbf{w})$,
2. $D(f\mathbf{v}) = df\ \mathbf{v} + fD\mathbf{v}$,
3. $D\langle \mathbf{v},\mathbf{w}\rangle = \langle D\mathbf{v},\mathbf{w}\rangle + \langle \mathbf{v}, D\mathbf{w}\rangle$.

By movable framework, the motion equation of the surface is

$$d\mathbf{e_1} = \omega_{12}\mathbf{e_2} + \omega_{13}\mathbf{e_3}, \quad d\mathbf{e_2} = \omega_{21}\mathbf{e_1} + \omega_{23}\mathbf{e_3},$$

We only keep tangential component, and delete the normal part to obtain covariant differential

$$D\mathbf{e_1} = \omega_{12}\mathbf{e_1}, \quad D\mathbf{e_2} = \omega_{21}\mathbf{e_1}.$$

Covariant Differential

Definition (Parallel transport)

Suppose S is a metric surface, $\gamma : [0,1] \to S$ is a smooth curve, $v(t)$ is a vector field along γ, if

$$\frac{Dv}{dt} \equiv 0,$$

then we say the vector field $v(t)$ is parallel transportation along γ.

Given a tangent vector field $v = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, then

$$\begin{aligned} Dv &= df_1\mathbf{e_1} + f_1 D\mathbf{e_1} + df_2\mathbf{e_2} + f_2 D\mathbf{e_2} \\ &= (df_1 - f_2\omega_{12})\mathbf{e_1} + (df_2 + f_1\omega_{12})\mathbf{e_2}. \end{aligned}$$

and

$$\frac{Dv}{dt} = \left(\frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt}\right)\mathbf{e_1} + \left(\frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt}\right)\mathbf{e_2}.$$

where $\frac{\omega_{12}}{dt} = \langle \omega_{12}, \dot{\gamma} \rangle$. If $\omega_{12} = \alpha dx + \beta dy$, then $\frac{\omega_{12}}{dt} = \alpha\dot{x} + \beta\dot{y}$.

Parallel Transport

Parallel Transport Equation

Therefore parallel vector field satisfies the ODE

$$\begin{cases} \frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt} & = & 0 \\ \frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt} & = & 0 \end{cases}$$

Given an intial condition $v(0)$, the solution uniquely exists.

Suppose the geodesic has local representation $\gamma(t) = (x(t), y(t))$, then $d\gamma = \dot{x}\partial_x + \dot{y}\partial_y = e^u\dot{x}\mathbf{e_1} + e^u\dot{y}\mathbf{e_2}$, $\omega_{12}/dt = -u_y\dot{x} + u_x\dot{y}$,

$$e^u(\ddot{x} + \dot{u} - \dot{y}(-u_y\dot{x} + u_x\dot{y})) = 0$$
$$e^u(\ddot{y} + \dot{u} + \dot{x}(-u_y\dot{x} + u_x\dot{y})) = 0$$

$$\begin{cases} \ddot{x} + \dot{u} + u_y\dot{x}\dot{y} - u_x\dot{y}^2 = 0 \\ \ddot{y} + \dot{u} + u_x\dot{x}\dot{y} - u_y\dot{x}^2 = 0 \end{cases}$$

Levy-Civita Connection

Definition (Levy-Civita Connection)

The connection D is the Levy-Civita connection with respect to the Riemannianmetic $\mathbf{g}$, it it satisfies:

1. compatible with the metric

$$\mathbf{x}\langle \mathbf{y}, \mathbf{z}\rangle_{\mathbf{g}} = \langle D_{\mathbf{x}}\mathbf{y}, \mathbf{z}\rangle_{\mathbf{g}} + \langle \mathbf{y}, D_{\mathbf{x}}\mathbf{z}\rangle_{\mathbf{g}}$$

2. free of torsion

$$D_{\mathbf{v}}\mathbf{w} - D_{\mathbf{w}}\mathbf{v} = [\mathbf{v}, \mathbf{w}]$$

Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vector fields parallel along γ, then

$$\frac{d}{dt}\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{g}} = \dot{\gamma}\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{g}} = \langle D_{\dot{\gamma}}\mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{v}, D_{\dot{\gamma}}\mathbf{w}\rangle \equiv 0.$$

Namely, parallel transportation preserves inner product.

Geodesic Curvature

Definition (Geodesic Curvature)

Assume $\gamma : [0,1] \to S$ is a C^2 curve on a surface S, s is the arc length parameter. Construct orthonormal frame field along the curve $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e}_3\}$, where $\mathbf{e_1}$ is the tangent vector field of γ, $\mathbf{e_3}$ is the normal field of the surface,

$$k_g := \frac{D\mathbf{e_1}}{ds} = k_g \mathbf{e_2}$$

is called geodesic curvature vector,

$$k_g = \langle \frac{D\mathbf{e_1}}{ds}, \mathbf{e_2} \rangle = \frac{\omega_{12}}{ds}$$

is called geodesic curvature.

Geodesic Curvature

Geodesic curvature, normal curvature

Given a spacial curve, its curvature vector satisfies

$$\frac{d^2\gamma}{ds^2} = k_g \mathbf{e_2} + k_n \mathbf{e_3},$$

where k_n is the normal curvature of the curve. The curvature of the curve, geodesic curvature and normal curvature satisfy

$$k^2 = k_g^2 + k_n^2.$$

Geodesic curvature k_g only depends on the Riemannian metric of the surface, is independent of the 2nd fundamental form. Therefore k_g is intrinsic, k_n is extrinsic.

Gauss-Bonnet

Theorem

Suppose $(S, \mathbf{g})$ *is an oriented metric surface with boundaries, then*

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S).$$

Proof.

Construct a vector field with isolated zeros $\{p_i\}$, $\mathbf{e}_1$ is tangent to ∂S, small disks $D(p_i, \varepsilon)$. Define $\bar{S} := S \setminus \bigcup_i D(p_i, \varepsilon)$,

$$\begin{aligned}
\int_{\bar{S}} K dA &= -\int_{\bar{S}} \frac{d\omega_{12}}{\omega_1 \wedge \omega_2} dA = -\int_{\bar{S}} d\omega_{12} = -\int_{\partial \bar{S}} \omega_{12} \\
&= -\int_{\partial S - \bigcup_i \partial D(p_i, \varepsilon)} \omega_{12} = -\int_{\partial S} \frac{\omega_{12}}{ds} ds + \sum_i \int_{\partial D(p_i, \varepsilon)} \omega_{12} \\
&= -\int_{\partial S} k_g ds + 2\pi \sum_i \text{Index}(p_i) = -\int_{\partial S} k_g ds + 2\pi\chi(S).
\end{aligned}$$

Geodesic Curvature

We use isothermal parameter (u, v) of $(S, \mathbf{g})$, given a curve $\gamma(s)$ with arc length parameter s. Construct orthonormal frame $\{p; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\mathbf{e}_3$ is the normal field of S. The tangent vector of γ is $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$ is orthogonal to $\bar{\mathbf{e}}_1$ everywhere. The angle between $\bar{\mathbf{e}}_1$ and $\mathbf{e}_1$ is $\theta(s)$,

$$\left\{ \begin{array}{rcr} \bar{\mathbf{e}}_1 & = & \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2 \\ \bar{\mathbf{e}}_2 & = & -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2 \end{array} \right.$$

Direct computation

$$\begin{aligned} D\bar{\mathbf{e}}_1 &= D(\cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2) = d\cos\theta\mathbf{e}_1 + \cos\theta_1 D\mathbf{e}_1 + d\sin\theta\mathbf{e}_2 + \sin\theta D\mathbf{e}_2 \\ &= -\sin\theta d\theta\mathbf{e}_1 + \cos\theta\omega_{12}\mathbf{e}_2 + \cos\theta d\theta\mathbf{e}_2 - \sin\theta\omega_{12}\mathbf{e}_1 \\ &= -\sin\theta(d\theta + \omega_{12})\mathbf{e}_1 + \cos\theta(\omega_{12} + d\theta)\mathbf{e}_2 \end{aligned}$$

$$k_g = \langle \frac{D\bar{\mathbf{e}}_1}{ds}, \bar{\mathbf{e}}_2 \rangle = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

Geodesic Curvature

Under the isothermal coordinates, we have $\omega_{12} = -u_y dx + u_x dy$. Suppose on the parameter domain, the planar curve arc length is dt, then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} \\ &= \frac{d\theta}{dt}\frac{dt}{ds} + \frac{-u_y dx + u_x dy}{dt}\frac{dt}{ds} \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_{\mathbf{n}} u) \end{aligned}$$

where k is the curvature of the planar curve, n is the normal to the planar curve.

Geodesic Curvature

Lemma

Given a metric surface $(S, \mathbf{g})$*, under conformal deformation,* $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$*, the geodesic curvature satisfies*

$$k_{\bar{\mathbf{g}}} = e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda).$$

Proof.

$$\begin{aligned} k_{\mathbf{g}} &= e^{-(u+\lambda)}(k - \partial_{\mathbf{n}}(u+\lambda)) \\ &= e^{-\lambda}(e^{-u}(k - \partial_{\mathbf{n}}u) - e^{-u}\partial_{\mathbf{n}}\lambda) \\ &= e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda) \end{aligned}$$

□

Definition (geodesic)

Given a metric surface $(S, \mathbf{g})$, a curve $\gamma : [0, 1] \to S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

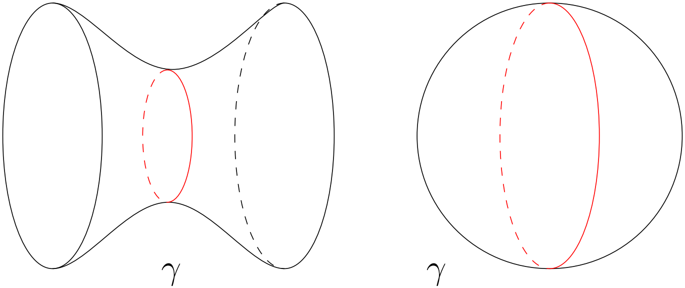


Figure: Stable and unstable geodesics.

Lemma (geodesic)

If γ is the shortest curve connecting p and q, then γ is a geodesic.

Proof.

Consider a family of curves, $\Gamma : (-\varepsilon, \varepsilon) \to S$, such that $\Gamma(0,t) = \gamma(t)$, and

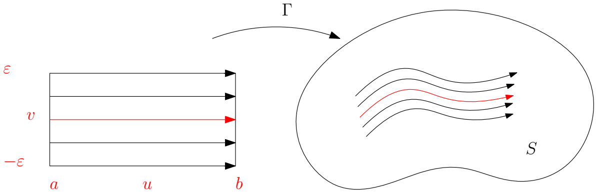
$$\Gamma(s,0) = p, \Gamma(s,1) = q, \frac{\partial \Gamma(s,t)}{\partial s} = \varphi(t)\mathbf{e_2}(t),$$

where $\varphi : [0,1] \to \mathbb{R}$, $\varphi(0) = \varphi(1) = 0$. Fix parameter s, curve $\gamma_s := \Gamma(s,\cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right| dt, \quad \frac{\partial L(s)}{\partial s} = -\int_0^1 \varphi k_{\mathbf{g}}(\tau) d\tau.$$

□

First Variation of arc length



Let $\gamma_v : [a, b] \to M$, where $v \in (-\varepsilon, \varepsilon) \in \mathbb{R}$ be a 1-parameter family of paths. We define the map $\Gamma : [a, b] \times [0, 1] \to M$ by

$$\Gamma(u, v) := \gamma_v(u).$$

Define the vector fields **u** and **v** along γ_v by

$$\mathbf{u} := \frac{\partial \Gamma}{\partial u} = \Gamma_*(\partial_u), \quad \text{and} \quad \mathbf{v} := \frac{\partial \Gamma}{\partial v} = \Gamma_*(\partial_v),$$

We call **u** the *tangent vector field* and **v** the *variation vector field*.

First Variation of arc length

Lemma (First variation of arc length)

If The length of γ_v *is given by*

$$L(\gamma_v) := \int_a^b |\mathbf{u}(\gamma_v(u))| du.$$

γ_0 *is parameterized by arc length, that is,* $|\mathbf{u}(\gamma_0(u))| \equiv 1$, *then*

$$\frac{d}{dv}\Big|_{v=0} L(\gamma_v) = -\int_a^b \langle D_{\mathbf{u}}\mathbf{u}, \mathbf{v}\rangle du + \langle \mathbf{u}, \mathbf{v}\rangle\Big|_a^b.$$

If we choose $\mathbf{u} = \mathbf{e_1}$, the tangent vector of γ, $\mathbf{v} = \mathbf{e_2}$ orthogonal to $\mathbf{e_1}$, and fix the starting and ending points of paths, then

$$\frac{d}{dv} L(\gamma_v) = -\int_a^b k_g ds.$$

First variation of arc length

Proof.

Fixing $u \in [a, b]$, we may consider $\mathbf{u}$ and $\mathbf{v}$ as vector fields along the path $v \mapsto \gamma_v(u)$. Then

$$\begin{aligned}\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))| &= \frac{\partial}{\partial v}\sqrt{|\mathbf{u}(\gamma_v(u))|^2} \\ &= \frac{1}{2|\mathbf{u}(\gamma_v(u))|}\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|^2 \\ &= \frac{1}{2|\mathbf{u}|}\mathbf{v}|\mathbf{u}|^2 = |\mathbf{u}|^{-1}\langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} = \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}}\end{aligned}$$

□

First variation of arc length

Proof.

$$\frac{d}{dv}L(\gamma_v) = \int_a^b \frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|du = \int_a^b \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} du$$

Since $D_{\mathbf{v}}\mathbf{u} - D_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$, and $[\mathbf{v}, \mathbf{u}] = \Gamma_*([\partial_v, \partial_u]) = 0$,

$$\begin{aligned}
\frac{d}{dv}L(\gamma_v) &= \int_a^b \langle D_{\mathbf{u}}\mathbf{v}, \mathbf{u}\rangle_{\mathbf{g}} du \\
&= \int_a^b \left(\frac{d}{du}\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}} - \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} \right) du \\
&= \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}}\Big|_a^b - \int_a^b \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} du.
\end{aligned}$$

□

Geodesics

The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.

Geodesics

Lemma (Uniqueness of geodesics)

Suppose $(S, \mathbf{g})$ *is a closed oriented metric surface,* $\mathbf{g}$ *induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.*

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ, by Gauss-Bonnet,

$$\int_\Sigma K dA + \int_{\partial\Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. □

Algorithm: Homotopy Detection

Input: A high genus closed mesh M, two loops γ_1 and γ_2;
Output: Whether $\gamma_1 \sim \gamma_2$;

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ_k to geodesics, $k = 1, 2$;
3. if two geodesics coincide, return true; otherwise, return false;

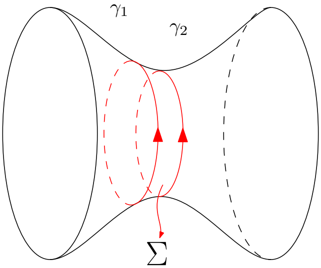


Figure: Geodesics uniqueness.

Algorithm: Shortest Word

Input: A high genus closed mesh M, one loop γ

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ to a geodesic;
3. Compute a set of canonical fundamental group basis;
4. Embed a finite portion of the universal covering space onto the Poincaré disk;
5. Lift γ to the universal covering space $\tilde{\gamma}$. If $\tilde{\gamma}$ crosses $b_i^{\pm}$, append $a_i^{\pm}$; crosses $a_i^{\pm}$, append $b_i^{\mp}$.

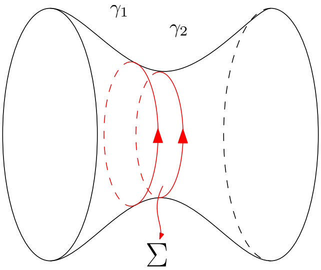


Figure: Geodesics uniqueness

Compute Minimal Surface

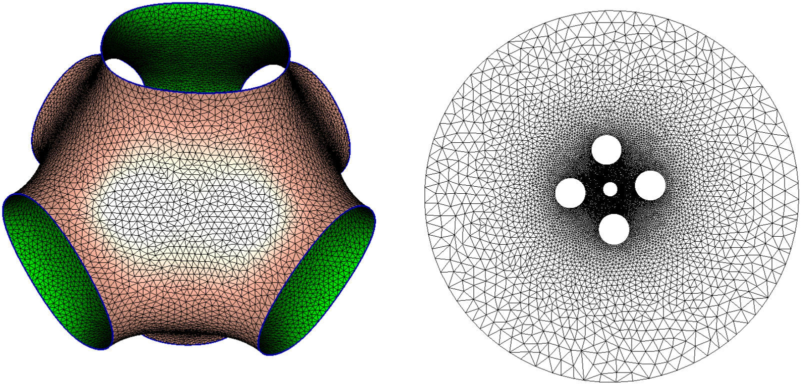


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$.

Minimal Surface

Lemma

Given a metric surface $(S, \mathbf{g})$ *embedded in* $\mathbb{R}^3$, *then* $\Delta_{\mathbf{g}}\mathbf{r} = 2H(p)\mathbf{n}$, *where* $\mathbf{r}$, $\mathbf{n}$ *are the position and normal vectors.*

Proof.

We choose isothermal coordinates (x, y). Then $\mathbf{g} =^{2\lambda}$,
$\omega_{12} = -\lambda_y dx + \lambda_x dy$, $\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$, $\omega_{23} = h_{12}\omega_1 + h_{22}\omega_2$,
$\omega_1 = e^\lambda dx$, $\omega_2 = e^\lambda dy$,

$$
\begin{aligned}
\frac{\partial}{\partial x}\mathbf{r_x} &= \frac{\partial}{\partial x} e^\lambda \mathbf{e_1} = e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \frac{\partial}{\partial x}\mathbf{e_1} \\
&= e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \langle d\mathbf{e}_1, \frac{\partial}{\partial x}\rangle = e^\lambda \lambda_x \mathbf{e_1} + e^\lambda \langle \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e_3}, \partial_x \rangle \\
&= e^\lambda \lambda_x \mathbf{e_1} + e^\lambda(-\lambda_y)\mathbf{e_2} + e^\lambda \mathbf{e_3}\langle h_{11}\omega_1, \partial_x\rangle \\
&= e^\lambda \lambda_x \mathbf{e_1} - e^\lambda \lambda_y \mathbf{e_2} + e^{2\lambda} h_{11}\mathbf{e_3}
\end{aligned}
$$

Minimal Surface

Proof.

Similarly,

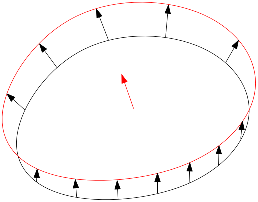
$$\begin{aligned}\frac{\partial}{\partial y}\mathbf{r_y} &= \frac{\partial}{\partial y}e^{\lambda}\mathbf{e_2} = e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\frac{\partial}{\partial y}\mathbf{e_2}\\ &= e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\langle d\mathbf{e}_2, \frac{\partial}{\partial y}\rangle = e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}\langle \omega_{21}\mathbf{e_1} + \omega_{23}\mathbf{e_3}, \partial_y\rangle\\ &= e^{\lambda}\lambda_y\mathbf{e_2} + e^{\lambda}(-\lambda_y)\mathbf{e_2} + e^{\lambda}\mathbf{e_3}\langle h_{22}\omega_2, \partial_y\rangle\\ &= e^{\lambda}\lambda_y\mathbf{e_2} - e^{\lambda}\lambda_x\mathbf{e_1} + e^{2\lambda}h_{22}\mathbf{e_3}\end{aligned}$$

Therefore

$$\Delta_{\mathbf{g}}\mathbf{r} = \frac{1}{e^{2\lambda}}(\mathbf{r}_{xx} + \mathbf{r}_{yy}) = (h_{11} + h_{22})\mathbf{e_3} = 2H\mathbf{e_3}.$$

□

Surface Area Variation



Lemma

Given a surface S with position vector $\mathbf{r}(x,y)$, perturb the surface along the normal direction

$$\mathbf{r}_{\varepsilon,\varphi}(x,y)=\mathbf{r}(x,y)+\varepsilon\varphi(x,y)\mathbf{n}(x,y),$$

the area variation is given by

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}Area(\mathbf{r}_{\varepsilon,\varphi})=\int_S 2\varphi(x,y)He^{2u(x,y)}dxdy=\int_S 2\varphi HdA.$$

Surface Area Variation

Proof.

We use isothermal coordinate, the first fundamental form:

$$E = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_x + \varepsilon \mathbf{n}_x \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \varepsilon^2 |\mathbf{n}_x|^2$$
$$G = \langle \mathbf{r}_y + \varepsilon \mathbf{n}_y, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_y, \mathbf{n}_y \rangle + \varepsilon^2 |\mathbf{n}_y|^2$$
$$F = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = \varepsilon \langle \mathbf{r}_x, \mathbf{n}_y \rangle + \varepsilon \langle \mathbf{r}_y, \mathbf{n}_x \rangle + \varepsilon^2 \langle \mathbf{n}_x, \mathbf{n}_y \rangle$$

$$EG - F^2 = e^{4u} + 2\varepsilon e^{2u}(\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle) + O(\varepsilon^2)$$
$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sqrt{EG - F^2} = \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle = 2He^{2u}$$

where we use the mean curvature formula

$$2H = \mathrm{Tr}\left(-\frac{II}{I}\right) = -e^{-2u}(\langle \mathbf{r}_{xx}, \mathbf{n} \rangle + \langle \mathbf{r}_{yy}, \mathbf{n} \rangle) = e^{-2u}(\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle r_y, \mathbf{n}_x \rangle)$$

$$\frac{d}{d\varepsilon} Area(\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_S \sqrt{EG - F^2} dxdy = \int_S 2He^{2u} dxdy.$$

Minimal Surface

Lemma

A surface M, $\mathbf{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$, with isothermal coordinates is minimal if and only if x_1, x_2, and x_3 are all harmonic.

Proof.

If M is minimal, then $H = 0$, $\Delta \mathbf{x} = (2H)e^{2\lambda}\mathbf{n} = 0$, therefore x_1, x_2, x_3 are harmonic.
If x_1, x_2, x_3 are harmonic, then $\Delta x = 0$, $(2H)e^{2\lambda}\mathbf{n} = 0$. Now $\mathbf{n}$ is the unit normal vector, so $\mathbf{n} \neq 0$ and $e^{2\lambda} = \langle x_u, x_u \rangle = |x_u|^2 \neq 0$. So $H = 0$, M is minimal. □

Weierstrass-Ennerper Representation

Lemma

Let $z = u + \sqrt{-1}v$, $\frac{\partial x^j}{\partial z} = \frac{1}{2}(x_u^j - \sqrt{-1}x_v^j)$, *define*

$$\varphi = \frac{\partial \mathbf{x}}{\partial z} = (x_z^1, x_z^2, x_z^3)$$
$$(\varphi)^2 = (x_z^1)^2 + (x_z^2)^2 + (x_z^3)^2$$

if $\mathbf{x}$ *is isothermal, then* $(\varphi)^2 = 0$.

Proof.

$(\varphi^j)^2 = (x_z^j)^2 = \frac{1}{4}((x_j^j)^2 - (x_v^j)^2 - 2ix_u^j x_v^j)$, so
$(\varphi)^2 = \frac{1}{4}(|\mathbf{x}_u|^2 - |\mathbf{x}_v|^2 - 2i\mathbf{x}_u \cdot \mathbf{x}_v)$. If $\mathbf{x}$ is isothermal, then $(\varphi)^2 = 0$. □

Weierstrass-Ennerper Representation

Theorem

Suppose M is a surface with position $\mathbf{x}$. Let $\varphi = \frac{\partial \mathbf{x}}{\partial z}$ and suppose $(\varphi)^2 = 0$. Then M is minimal if and only if φ^j is holomorphic.

Proof.

M is minimal, then x^j is harmonic, therefore $\Delta \mathbf{x} = 0$, therefore

$$\frac{\partial}{\partial \bar{z}} \left(\frac{\partial \mathbf{x}}{\partial z} \right) = \frac{\partial \varphi}{\partial \bar{z}} = 0$$

If φ^j is holomorphic, then $\frac{\partial \varphi}{\partial \bar{z}} = 0$, then $\Delta \mathbf{x} = 0$, x^j is harmonic, hence M is minimal. □

Lemma

$$x^j(z,\bar{z}) = c_j + \Re\left(\int \varphi^j dz\right).$$

Proof.

$$\varphi^j dz + \bar{\varphi}^j d\bar{z}^j = x^j_u du + x^j_v dv = dx^j.$$

hence

$$x^j = c_j + \int dx^j = c_j + \Re\left(\int \varphi^j dz\right).$$

□

Let f be a holomorphic function and g be a meromorphic function, such that fg^2 is holomorphic,

$$\varphi^1 = \frac{1}{2}f(1-g^2), \varphi^2 = \frac{i}{2}f(1+g^2), \varphi^3 = fg,$$

then

$$(\varphi)^2 = \frac{1}{4}f^2(1-g^2)^2 - \frac{1}{4}f^2(1+g^2)^2 + f^2g^2 = 0.$$

Weierstrass-Ennerper Representation

Theorem (Weierstrass-Ennerper)

If f is holomorphic on a domain Ω, g is meromorphic in Ω, and fg^2 is holomorphic on Ω, then a minimal surface is defined by $\mathbf{x}(z,\bar{z}) = (x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z}))$, *where*

$$x^1(z,\bar{z}) = \Re\left(\int f(1-g^2)dz\right)$$

$$x^2(z,\bar{z}) = \Re\left(\int \sqrt{-1}f(1+g^2)dz\right)$$

$$x^3(z,\bar{z}) = \Re\left(\int 2fgdz\right)$$