Presheaves & Sheaves

. Presheaf $\mathcal{F}$ on X consists of:

- $\forall U \subset X$ open, $\mathcal{F}(U)$ an Abelian group

  — the group of sections of $\mathcal{F}$ over U.

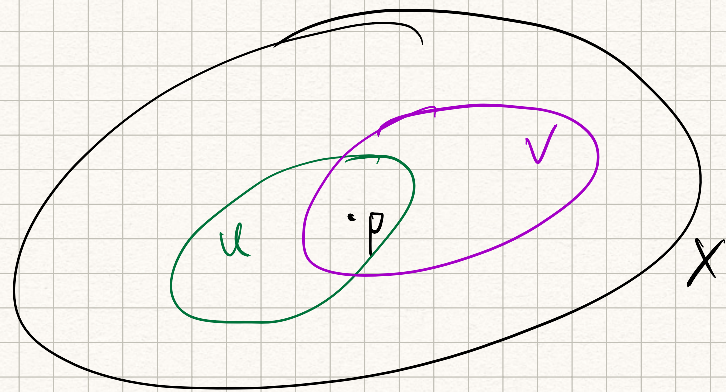
- $\forall V \subset U$ open sets, $res_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$.

such that:

- $res_{U,U} = id_U$, $\forall U \subset X$

- $U \subset V \subset W$ inclusions of open sets, then:

$$res_{W,U} = res_{V,U} \circ res_{W,V}$$



- germs at p:

$$\{(f, U) \mid p \in U,\ f \in \mathcal{F}(U)\} \Big/ \left\{ \begin{array}{l} (f, U) \sim (g, V) \text{ if } \\ \exists W \subset U \cap V, \text{ s.t. } \\ f|_W = g|_W \end{array} \right\}$$

- stalk at p: $\mathcal{F}_p$

  the set of all germs at p.

- Sheaf = presheaf +

- <u>Identity Axiom</u>: If $\{U_i\}$ is a cover of U, and $f_1, f_2 \in \mathcal{F}(U)$. Then $f_1|_{U_i} = f_2|_{U_i}$ for $\forall\, U_i$ implies $f_1 \equiv f_2$ on U.

- <u>Gluability</u>: $\{U_i\}$ open covers of U and $f_i \in \mathcal{F}(U_i)$. If $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ $\forall\, i, j$, then $\exists\, f \in \mathcal{F}(U)$ s.t. $f|_{U_i} = f_i$, $f|_{U_j} = f_j$.

· skyscraper sheaf

$p \in X$, $\quad i_p : p \hookrightarrow X$, $\quad A$: abelian group

$$i_{p,*}(A) := \begin{cases} A, & p \in U \\ \{0\}, & \text{otherwise} \end{cases}$$

⚠ May consider as the sheaf having only non-trivial stalk at p.

· Constant sheaf :

A sheaf such that all its stalks equal to an Abelian group.

- Module over a ring $\rightsquigarrow$ vector space over a field

$\mathcal{O}_X$ – sheaf of rings on X

$\mathcal{F}$ – sheaf of Abelian groups on X.

$\rightsquigarrow$
$$\begin{tikzcd}
\mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r, "\text{action}"] \arrow[d, "res_{V,U} \times res_{V,U}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(V) \arrow[d, "res_{V,U}"] \\
\mathcal{O}_X(U) \times \mathcal{F}(U) \arrow[r, "\text{action}"] & \mathcal{F}(U)
\end{tikzcd}$$

- X : differentiable manifold.

$\mathcal{O}_X$: sheaf of differentiable functions.

$q: V \longrightarrow X$ vector bundle, the sheaf of differentiable sections $s: X \longrightarrow V$ is an $\mathcal{O}_X$-module.

sheaf reflects local properties

$\longleftarrow$ $\boxed{\text{properties can be checked on level of stalks.}}$

- In particular, morphisms are determined by stalks.

$$\mathcal{F}(U) \hookrightarrow \prod_{p \in U} \mathcal{F}_p$$

germ map.

-
$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\varphi_1, \varphi_2"] \arrow[d] & \mathcal{G}(U) \arrow[d, hook] \\
\prod_p \mathcal{F}_p \arrow[r] & \prod_p \mathcal{G}_p
\end{tikzcd}$$

$\mathcal{F}$: presheaf

$\mathcal{G}$: sheaf

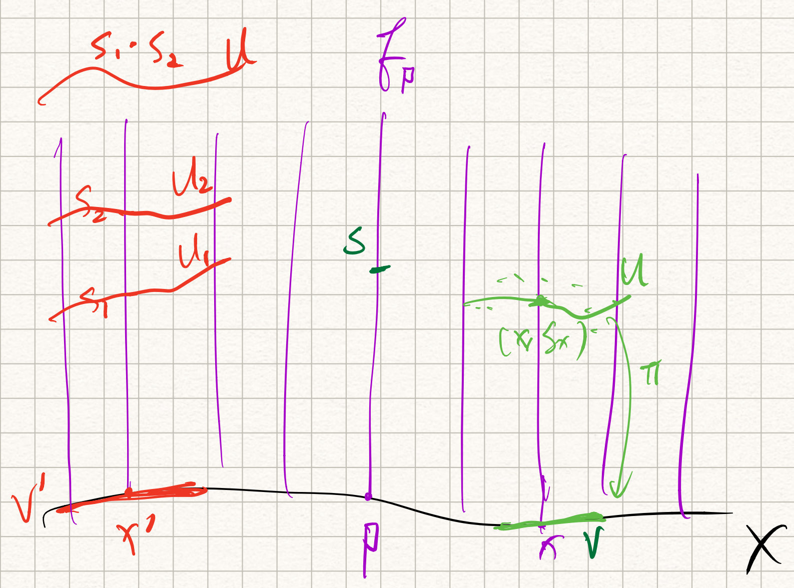
if φ_1, φ_2 agree on stalks $\Rightarrow$ $\varphi_1 = \varphi_2$.

Sheafification:

$\mathcal{F}$ a presheaf. $\mathcal{F}^{sh}$ is the associated sheaf

$$\mathcal{F}^{sh}(U) = \left\{ (f_p \in \mathcal{F}_p)_{p \in U} \;\middle|\; \begin{array}{l} \exists\, p \in V \subset U \text{ and } s \in \mathcal{F}(V) \\ \text{s.t. } s_q = f_q,\ \forall q \in V \end{array} \right\}$$

What is $\mathcal{F}(U) \longrightarrow \mathcal{F}^{sh}(U)$?



- $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ a morphism of sheaves.

- We say φ is injective (monomorphism) if
  - $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is injective, $\forall\, p$

    $\Updownarrow$
  - $\varphi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective, $\forall\, U$

- We say φ is surjective (epimorphism) if

  $\varphi_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ is surjective.

$\rightsquigarrow$ notion of sub/quotient sheaf def on stalks.

> $\triangledown$ surjectivity of sheaves cannot be checked on open sets.

Homological Algebra

- $\mathcal{F}$ a sheaf of Abelian groups on X.

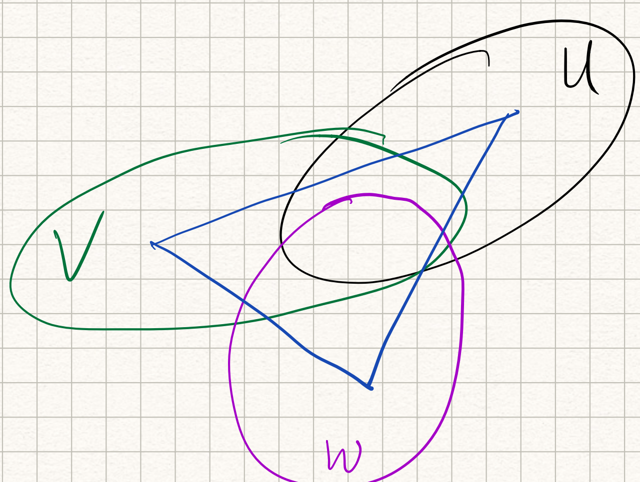
  $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering. We define a p-cochain w.r.t $\mathcal{U}$ as linear combination of the form $h_{0,1,\dots,p} \in \Gamma(U_{0,1,\dots,p}, \mathcal{F})$

  where $U_{0,1,\dots,p} := U_0 \cap U_1 \cap \cdots \cap U_p \neq \phi$.

- Group of p-cochains : $\check{C}^p(\mathcal{U}, \mathcal{F})$.

- $\delta : \check{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^{p+1}(\mathcal{U}, \mathcal{F})$

  s.t $\delta(h)_{0,\dots,p+1} = \sum_{i=0}^{p+1} (-1)^i h_{0,1,\dots,\hat{i},\dots,p+1}$



- verify that $\delta^2 = 0$!

- $\check{Z}^p(\mathcal{U}, \mathcal{F}) := \ker\left(\check{C}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^{p+1}(\mathcal{U}, \mathcal{F}) \right)$

$\check{B}^p(\mathcal{U}, \mathcal{F}) := \operatorname{Im}\left(\check{C}^{p-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^p(\mathcal{U}, \mathcal{F}) \right)$

$\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) := \check{Z}^p(\mathcal{U}, \mathcal{F}) / \check{B}^p(\mathcal{U}, \mathcal{F})$

Lemma: $\check{H}^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.

$$0 \longrightarrow \prod_{|U_I|=1} \mathcal{F}(U_I) \xrightarrow{\delta^0} \prod_{|U_I|=2} \mathcal{F}(U_I) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{p-1}} \prod_{|U_I|=p} \mathcal{F}(U_I) \longrightarrow \cdots$$

$\boxed{\text{⚠ Doesn't work for presheaves}}$

- Our space X, we assume for any open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X, $\forall p \in X$, $\exists$ an open neighborhood of p which intersects with finitely many elements of $\mathcal{U}$.

- Two coverings $\mathcal{U} = \{U_i\}_{i \in I}$, $\mathcal{V} = \{V_j\}_{j \in J}$. If $\exists$ a map $r : J \to I$ s.t. $\forall j \in J$, $V_j \subseteq U_{r(j)}$, we call the covering $\mathcal{V}$ is finer than $\mathcal{U}$, or call $\mathcal{V}$ a refinement of $\mathcal{U}$, denoted as $\mathcal{V} \leq \mathcal{U}$.

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

sequence of sheaves. We call it a short exact sequence if

- $\mathcal{F}$ a subsheaf of $\mathcal{G}$.
- $\mathcal{H}$ a quotient sheaf of $\mathcal{G}$.
- $\ker(\mathcal{F} \to \mathcal{G}) = \operatorname{Im}(\mathcal{G} \to \mathcal{H})$.

Theorem: $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ a short exact sequence, then $\exists\ \delta^p: \check{H}^p(X, \mathcal{H}) \longrightarrow \check{H}^{p+1}(X, \mathcal{F})$ s.t. the long sequence:

$$
\begin{aligned}
0 \to \check{H}^0(X, \mathcal{F}) &\longrightarrow \check{H}^0(X, \mathcal{G}) \longrightarrow \check{H}^0(X, \mathcal{H}) \\
\xrightarrow{\delta^0} \check{H}^1(X, \mathcal{F}) &\longrightarrow \check{H}^1(X, \mathcal{G}) \longrightarrow \check{H}^1(X, \mathcal{H}) \\
\xrightarrow{\delta^1} &\cdots\cdots \\
&\cdots \qquad\qquad \longrightarrow \check{H}^p(X, \mathcal{H}) \\
\xrightarrow{\delta^p} \check{H}^{p+1}(X, \mathcal{F}) &\longrightarrow \cdots\cdots
\end{aligned}
$$

is exact.

$\boxed{\text{! We droped the subscript } \mathcal{U} \text{ in } \check{H}^p_{\mathcal{U}}(X, \mathcal{F})}$

Lemma: If for $\forall\, U \subset \mathcal{U}$, the sequence

$$0 \to \mathcal{F}(U) \xrightarrow{i} \mathcal{G}(U) \xrightarrow{r} \mathcal{H}(U) \to 0$$

is exact, then

$$0 \to \check{H}^0_{\mathcal{U}}(X,\mathcal{F}) \to \check{H}^0_{\mathcal{U}}(X,\mathcal{G}) \to \check{H}^0_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^1_{\mathcal{U}}(X,\mathcal{F})$$

$$\to \cdots \longrightarrow \check{H}^p_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\delta} \check{H}^{p+1}_{\mathcal{U}}(X,\mathcal{F}) \to \cdots$$

is exact.

Proof:

$i^*(\delta(f)) = \delta(i^*(f)) = \delta\circ\delta(g) = 0$

i^* inj $\Rightarrow \delta(f) = 0$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \check{C}^{p-1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "i^*"] & \check{C}^{p}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "i^*"] & \check{C}^{p+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "i^*"] & \cdots \\
\cdots \arrow[r] & \check{C}^{p-1}(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "r^*"] & \check{C}^{p}(\mathcal{U},\mathcal{G}) \arrow[r, "\delta"] \arrow[d, "r^*"] & \check{C}^{p+1}(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "r^*"] & \cdots \\
\cdots \arrow[r] & \check{C}^{p-1}(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d] & \check{C}^{p}(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d] & \check{C}^{p+1}(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$\exists f$

$\exists g$

$\delta(g)$

$h \in Z^p(\mathcal{U},\mathcal{H})$

0

0

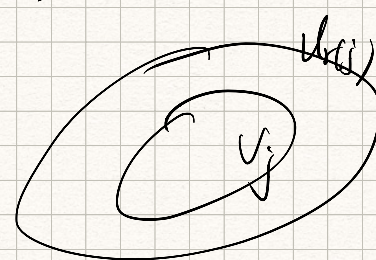
exact

Should cohomology depend on the choice of covering?

- Suppose $\mathcal{V} \le \mathcal{U}$. $\Rightarrow$ $\exists\ r : J \to I$, $V_j \subset U_{r(j)}$

$\rightsquigarrow$ $r^* : \check{C}^\bullet_{\mathcal{U}}(X, \mathcal{F}) \longrightarrow \check{C}^\bullet_{\mathcal{V}}(X, \mathcal{F})$ s.t.

$$r^*(h)_{0,\dots,p} = h_{r(0),\dots,r(p)}\Big|_{V_{0,\dots,p}}$$



Thm: $\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) \longrightarrow \check{H}^p_{\mathcal{V}}(X, \mathcal{F})$ is an isomorphism.

proof:

exact : $\check{H}_{U_0 \cap U_I}$ and U_0 contractible

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \prod\limits_{|I|=i-1,\, 0\in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i,\, 0\in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i+1,\, 0\in I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod\limits_{|I|=i-1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i+1} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \prod\limits_{|I|=i-1,\, 0\notin I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i,\, 0\notin I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \prod\limits_{|I|=i+1,\, 0\notin I} \mathcal{F}(U_I) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$|\mathcal{V}| = |\mathcal{U}| + 1$

Define $\tilde{\mathcal{H}} := \mathrm{Im}(\mathcal{G} \to \mathcal{H})$

Then sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \tilde{\mathcal{H}}(U) \to 0$

is exact sequence of presheaves.

<u>Lemma</u>: $\check{H}^*(X, \mathcal{H}) \cong \check{H}^*(X, \tilde{\mathcal{H}})$.

More generally, if two presheaves have isomorphic sheafification, then their cohomology groups are isomorphic.

· Complete the proof of original theorem by choosing an arbitrary covering.

Sheaf Cohomology

⚠ Another technical section

Will see how to calculate $H^i(X, \mathcal{F})$ given a sheaf $\mathcal{F}$ on X.

- We call a sheaf acyclique if $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

- Let $\mathcal{J}^\bullet = \mathcal{J}^0 \to \mathcal{J}^1 \to \mathcal{J}^2 \to \cdots$ be a long exact sequence of acyclique sheaves. If $0 \to \mathcal{F} \to \mathcal{J}^\bullet$ is exact, we call $\mathcal{J}^\bullet$ an acyclique resolution of $\mathcal{F}$.

Theorem (De Rham):

$$\check{H}^i(X,\mathcal{F}) \cong \frac{\ker\left(\Gamma(X,\mathcal{J}^i) \to \Gamma(X,\mathcal{J}^{i+1})\right)}{\operatorname{Im}\left(\Gamma(X,\mathcal{J}^{i-1}) \to \Gamma(X,\mathcal{J}^i)\right)}$$

proof: Consider $\mathcal{K}^i := \ker(\mathcal{J}^i \xrightarrow{\delta^i} \mathcal{J}^{i+1})$

- short exact sequence:

$$0 \to \mathcal{K}^i \to \mathcal{J}^i \to \mathcal{K}^{i+1} \to 0$$

and associated long exact sequence

$$\Rightarrow \check{H}^p(X,\mathcal{K}^{i+1}) \cong \check{H}^{p+1}(X,\mathcal{K}^i)$$

- $0 \to \mathcal{F} \to \mathcal{J}^0 \to \mathcal{K}^1 \to 0$ short exact

$$\Rightarrow \check{H}^p(X,\mathcal{F}) \cong \check{H}^{p-1}(X,\mathcal{K}^1) \cong \check{H}^{p-2}(X,\mathcal{K}^2) \cong \dots \cong \check{H}^1(X,\mathcal{K}^{p-1})$$

- $0 \to \check{H}^0(X,\mathcal{K}^{p-1}) \to \check{H}^0(X,\mathcal{J}^{p-1}) \to \check{H}^0(X,\mathcal{K}^p) \to \check{H}^1(X,\mathcal{K}^{p-1}) \to 0$

$$\Rightarrow \check{H}^1(X,\mathcal{K}^{p-1}) \cong \check{H}^0(X,\mathcal{K}^p) \Big/ \operatorname{Im}\left(\check{H}^0(X,\mathcal{J}^{p-1}) \to \check{H}^0(X,\mathcal{K}^p)\right)$$

$$\cong \frac{\ker\left(\Gamma(X,\mathcal{J}^p) \to \Gamma(X,\mathcal{J}^{p+1})\right)}{\operatorname{Im}\left(\Gamma(X,\mathcal{J}^{p-1}) \to \Gamma(X,\mathcal{J}^p)\right)}$$

De Rham $\Rightarrow$ cohomologie without bother considering open covers.

Cohomology is a global property.

? What kind of sheaves are acyclique?

Def: $\mathcal{F}$ is called flasque if $\forall\ V \overset{\circ}{\hookrightarrow} U$ open inclusion, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.

- Take $U = X$, $\mathcal{F}(X) \to \mathcal{F}(V)$ surjective for $\mathcal{F}$ flasque $\Rightarrow$ local section can be lifted to global!

<u>Thm</u> : All flasque sheaves are acyclique for Čech cohomology.

<u>proof</u>: Let $h = h_{0,\dots,p} \in \check{Z}^p(X, \mathcal{F})$,

define $f_{0,\dots,p-1} = h_{a,0,\dots,p-1} \in U_a \cap U_0 \cap \cdots \cap U_{p-1}$.

Then :

$$0 = \delta(h)_{a,0,\dots,p} = h_{0,\dots,p} - \sum_{i=1}^{p} (-1)^i h_{a,0,\dots,\hat{i},\dots,p} = h_{0,\dots,p} - \delta(f)$$

$\Rightarrow$ locally we have $\delta(f) = h|_{\mathcal{U}}$.

Suppose $\delta(g) = h|_V$.

need to show for open cover of $V \cup U$,

there exists $s \in \check{C}^{p-1}(V \cup U, \mathcal{F})$ s.t $\delta(s) = h|_{V \cup U}$.

Thm: Any $\mathcal{O}_X$-module over a smooth manifold is acyclique.

proof: $\boxed{\text{Partition of unity is the key!}}$

$\{f_i\} \in \{\mathcal{U}_i\}_{i\in I}$ s.t $\sum_i f_i = 1$.

· Consider $g_{0,\dots,p-1} := \sum_{a\in I} f_a h_{a,0,\dots,p-1}$. $\in \check{C}^{p-1}(\mathcal{U}, \mathcal{F})$.

$$0 = \delta(h)_{a,0,\dots,p} = h_{0,\dots,p} - \sum_{i=0}^{p} (-1)^i h_{a,0,\dots,\hat{i},\dots,p}$$

for $h_{0,\dots,p} \in \check{Z}^p(\mathcal{U}, \mathcal{F})$

Hence:
$$\begin{aligned}
\delta(g)_{0,\dots,p} &= \sum_{i=0}^{p} (-1)^i g_{0,\dots,\hat{i},\dots,p} \\
&= \sum_{i=0}^{p} (-1)^i \sum_{a\in I} f_a g_{a,0,\dots,\hat{i},\dots,p} \\
&= \sum_{a\in I} f_a \sum_{i=0}^{p} (-1)^i g_{a,0,\dots,\hat{i},\dots,p} \\
&= h_{0,\dots,p}
\end{aligned}$$

∎

Divisors and Line Bundles

- $\pi : M \longrightarrow E$ holomorphic map between complex manifolds. If $\{U_\alpha\} =: \mathcal{U}$ open covering of E and $\forall \alpha$, $\exists$ diffeomorphism $g_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$ and for $U_\alpha \cap U_\beta$,

$$g_\beta \circ g_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \longrightarrow (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

is diffeomorphism, we call M a holomorphic vector bundle over E.

- For an such vector bundle,

$$g_\beta \circ g_\alpha^{-1} : (e, v) \longmapsto (e, g_{uv}(v)), \; g_{uv} \in GL(r, \mathbb{C})$$

- $\{g_{uv}\}$ are called transition functions:

$$g_{uu} = id_u, \quad g_{wu} \circ g_{vw} \circ g_{uv} = id.$$

If $r=1$, $E \leftarrow$ holomorphic line bundle.

In this case, $g_{UV} \in GL(1, \mathbb{C}) \cong \mathbb{C}^{\times}$.

- $D = n_1 P_1 + \cdots + n_s P_s$ a divisor.

$\mathcal{U}$ an open covering s.t. $\forall\, U_\alpha \in \mathcal{U}$, $\exists$ meromorphic function f_α having some P_i's as its zeros/poles.

On $U_\alpha \cap U_\beta$, f_α & f_β have same zeros/poles.

Then $h^\alpha_\beta = \dfrac{f_\alpha}{f_\beta}$ everywhere non-zero holomorphic on $U_\alpha \cap U_\beta$ $\rightsquigarrow$ $h^\alpha_\beta \in \mathbb{C}^\times$ gives a line bundle, called $[D]$.

- $s = \{f_\alpha\}$ a section of $[D]$.

- If L a line bundle, $\{h^\alpha_\beta\}$ transition functions.

If $\{f_\alpha\}$ meromorphic functions on $\{U_\alpha\}$, and

$$f_\alpha = h^\alpha_\beta f_\beta$$

$\Rightarrow \{f_\alpha\}$ gives a divisor.

· suppose D & D' two divisors, and $[D]$, $[D']$ defined by locally meromorphic functions $\{f_\alpha\}$ and $\{f'_\alpha\}$.

$$\begin{tikzcd} {[D]} \arrow[rr, "G=\{g_\alpha\}"] \arrow[dr, "\pi_1"'] & & {[D']} \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

$U_\alpha \overset{\circ}{\hookrightarrow} X$

G is an isomorphism if $\exists\ g_\alpha : \pi_1^{-1}(U_\alpha) \xrightarrow{\sim} \pi_2^{-1}(U_\alpha)$

s.t $\quad g_\alpha \dfrac{f_\alpha}{f_\beta} = \dfrac{f'_\alpha}{f'_\beta} g_\beta$

$\Rightarrow \quad g_\alpha \dfrac{f_\alpha}{f'_\alpha} = g_\beta \dfrac{f_\beta}{f'_\beta} \rightsquigarrow \quad \{F_\alpha\}$ meromorphic

$\rightsquigarrow \quad div(F) = D - D'$.

D, D' are isomorphic iff $\exists$ meromorphic function F

s.t $\quad D - D' = div(F)$.

Thm: $\mathcal{L}$ a holomorphic line bundle, then $\exists$ divisor D s.t. $\mathcal{L} = [D]$.

proof: Take p a point, s a section of $[np]$.

$$0 \to \mathcal{O}(\mathcal{L}) \xrightarrow{\times s} \mathcal{O}(\mathcal{L} \otimes [np]) \to sk(np) \to 0$$

$sk(np)_p = \{ a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} \mid a_i \in \mathbb{C} \}$

$$\Rightarrow 0 \to H^0(X, \mathcal{O}(\mathcal{L})) \to H^0(X, \mathcal{L} \otimes [np]) \to H^0(X, sk(np))$$
$$\to H^1(R, \mathcal{O}(\mathcal{L})) \to H^1(X, \mathcal{L} \otimes [np]) \to 0$$

$$\rightsquigarrow \dim H^0(X, \mathcal{O}(\mathcal{L})) - \dim H^0(X, \mathcal{O}(\mathcal{L} \otimes [np]))$$
$$+ \underbrace{\dim H^0(X, sk(np))}_{= n} - \dim H^1(X, \mathcal{O}(\mathcal{L}))$$
$$+ \dim H^1(X, \mathcal{L} \otimes [np]) = 0$$

$\rightarrow \dim H^0(X, \mathcal{O}(\mathcal{L} \otimes [np])) > 0$ for $n \gg 0$.

$\exists$ non-trivial section $\tilde{s}$

Take $[D]$ defined by $\tilde{s}/s$.

Theorem (Riemann - Roch):

$$\dim H^0(X, \mathcal{O}(L)) - \dim H^1(X, \mathcal{O}(L)) = \deg(L) - g + 1$$

proof: Consider short exact sequence:

$$0 \to \underset{\text{holomorphic functions on } X}{\mathcal{O}(X \times \mathbb{C})} \xrightarrow{\times s} \mathcal{O}(\underset{\underset{L > 0}{\parallel}}{[D]}) \to sk(D) \to 0$$

Still by dimension argument of associate long exact sequence:

$$\dim H^0(X, \mathcal{O}(L)) - \dim H^1(X, \mathcal{O}(L))$$

$$= \underbrace{\dim H^0(X, \mathcal{O}(X \times \mathbb{C}))}_{=1} - \underbrace{\dim H^1(X, \mathcal{O}(X \times \mathbb{C}))}_{= \dim H^{(0,1)}(X) = \frac{1}{2} \dim H^1(X, \mathbb{C}) = g} + \underbrace{\dim H^0(X, sk(D))}_{= \deg(D)}$$

Hodge decomposition:

$$H^1(X, \mathbb{C}) \simeq \bigoplus_{p+q=1} H^{(p,q)}(X) = H^{(1,0)}(X) \oplus \overline{H^{(1,0)}}(X) = 2g.$$

$$0 \to \mathcal{O}(X \times \mathbb{C}) \hookrightarrow A^{(0,0)}(X) \xrightarrow{\bar{\partial}} A^{(0,1)}(X) \to 0$$

For arbitrary divisor D, let $D = D_1 - D_2$ with $D_1 > 0$ and $D_2 > 0$.

$$0 \to \mathcal{O}([D_1 - D_2]) \xrightarrow{\times s} \mathcal{O}([D_1]) \to sk(D_2) \to 0$$

$$\Rightarrow \dim H^0(X, \mathcal{O}([D_1 - D_2])) - \dim H^1(X, \mathcal{O}([D_1 - D_2]))$$

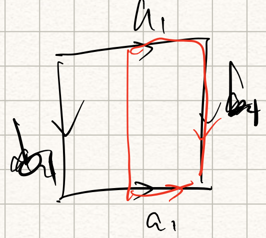
$$= \dim H^0(X, \mathcal{O}([D_1])) - \dim H^1(X, \mathcal{O}([D_1])) - \dim H^0(sk(D_2))$$

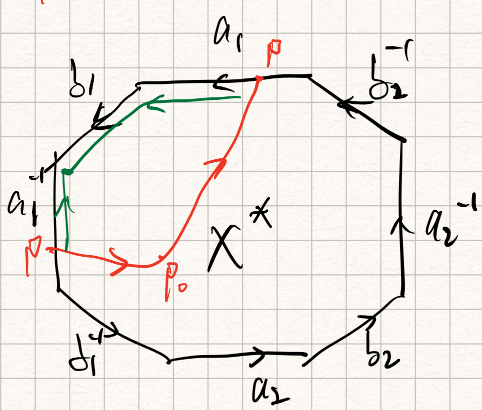
$$= \deg(D_1) - g + 1 - \deg(D_2)$$

$$= \deg(D) - g + 1.$$

$$\cdot \begin{cases} l(D) \cong H^{(0,0)}(X, [D]) \\ i(D) \cong H^{(0,1)}(X, [D]) \end{cases}$$

Abel theorem





For any closed curve L on X, $d\omega$ a holomorphic 1-form. From algebraic topology, L is homotopic to $\sum_{i=1}^{g}(n_i a_i + m_i b_i)$, $n_i, m_i \in \mathbb{Z}$.

$$\int_L d\omega = \sum_{i=1}^{g}\left(n_i \int_{a_i} d\omega + m_i \int_{b_i} d\omega\right), \quad a_i, b_i \in \mathbb{Z}.$$

since $L \sim \sum_{i=1}^{g}(n_i a_i + m_i b_i)$

Def: $L(d\omega) = \left\{ \sum_{i=1}^{g}\left(n_i \int_{a_i} d\omega + m_i \int_{b_i} d\omega\right), n_i, m_i \in \mathbb{Z} \right\}$

$$w(-) : X \xrightarrow{\int_{p_0}^{-} dw} \mathbb{C}/L(dw).$$

. For any point p on ∂X^*, the integration on $\mathbb{C}/L(dw)$ satisfy:

$$\begin{cases} w(p)\big|_{b_i} + w(p)\big|_{a_i} - w(p)\big|_{a_i^{-1}} = 0, & p \in a_i \\ w(p)\big|_{b_i} + w(p)\big|_{a_i^{-1}} - w(p)\big|_{b_i^{-1}} = 0 & \\ \Rightarrow w(p)\big|_{b_i} - w(p)\big|_{b_i^{-1}} = -w(p)\big|_{a_i^{-1}} = w(p)\big|_{a_i}. & \end{cases}$$

$$\Rightarrow \int_{\partial X^*} w dv = \sum_{i=1}^{g} \left(\int_{a_i} w dv + \int_{b_i} w dv + \int_{a_i^{-1}} w dv + \int_{b_i^{-1}} w dv \right)$$

$$= \sum_{i=1}^{g} \left(\int_{a_i} (w|_{a_i} - w|_{a_i^{-1}}) dv + \int_{b_i} (w|_{b_i} - w|_{b_i^{-1}}) dv \right)$$

$$= \sum_{i=1}^{g} \left(- \int_{b_i} dw \cdot \int_{a_i} dv + \int_{a_i} dw \cdot \int_{b_i} dv \right)$$

Let $V(dw) := \left(\int_{a_1} dw, \ \dots, \ \int_{a_g} dw, \ \int_{b_1} dw, \ \dots, \ \int_{b_g} dw \right)$

and $\Omega := \begin{pmatrix} v(dw_1) \\ \vdots \\ v(dw_g) \end{pmatrix}$ with $\{dw_1, \dots, dw_g\}$ basis of holomorphic forms.

<u>Lemma</u>: If we write $\Omega = [V_1 \ V_2 \ \dots\dots \ V_{2g}]$

then $V_1, \dots, V_{2g}$ are real-independent.

<u>proof</u>: Suppose $\exists \ c_1, \dots, c_{2g}$ s.t

$$(c_1, \dots, c_{2g}) \left(v(dw_1), \dots, v(dw_g), \overline{v(dw_1)}, \dots, \overline{v(dw_g)} \right)^T = 0$$

So if we let $V = \sum_{i=1}^{g} (c_i dw_i + c_{g+i} d\overline{w}_i)$

then $\int_{a_j} V = 0$, $\int_{b_j} V = 0$ and $dV = 0$.

$f(P) := \int_{P_0}^{P} V$ is well-defined and $d(f) = V$.

But $\{dw_1, \dots dw_g, d\overline{w}_1, \dots, d\overline{w}_g\}$ basis of $H^1(X, \mathbb{C})$. $\boxed{\text{contradiction}}$

Theorem (Abel): $D = p_1 + \cdots + p_n - q_1 - \cdots - q_n$ is defined by zeros/poles of a meromorphic function iff $\exists$ closed curve L s.t $\forall$ holomorphic form $d\omega$, we always have:

$$\sum_{i=1}^{n} \int_{q_i}^{p_i} d\omega = \int_L d\omega. \quad (*)$$

proof: $D_j = -p_j - q_j$

$$RR \Rightarrow \dim H^0(X, \mathcal{O}([D_j])) - \dim H^1(X, \mathcal{O}([D_j])) = -2 - g + 1$$

$\dim H^0(X, \mathcal{O}([D_j])) = 0$ since $\deg(D_j) < 0$.

$\Rightarrow \dim H^1(X, \mathcal{O}([D_j])) = i(D_j) = g+1$

space of all holomorphic forms ~ $\dim = g$

$i(D_j) = g+1$

$\Rightarrow \exists$ meromorphic form having exactly 2 poles: p_j & q_j.

Recall $\Omega = [v_1 \ v_2 \ \cdots \ v_{2g}]$ real independent.

Denote $L := \langle v_1, \cdots, v_{2g} \rangle$ a lattice, and

$$J(X) := \mathbb{C}^g / L .$$

- The map $X \xrightarrow{\left(\int_{p_0}^{p} dw_1, \cdots, \int_{p_0}^{p} dw_g \right)} J(X)$ is well-defined.

- Let $D_0(X)$ the group of all divisors with $\deg = 0$.

take $D = p_1 + \cdots + p_n - q_1 - \cdots - q_n \in D_0(X)$

$$\begin{aligned} F(D) := & \left(\sum_{i=1}^{n} \int_{p_0}^{p_i} dw_1, \cdots, \sum_{i=1}^{n} \int_{p_0}^{p_i} dw_g \right) \\ & - \left(\sum_{i=1}^{n} \int_{p_0}^{q_i} dw_1, \cdots, \sum_{i=1}^{n} \int_{p_0}^{q_i} dw_g \right) \\ = & \left(\sum_{i=1}^{n} \int_{q_i}^{p_i} dw_1, \cdots, \sum_{i=1}^{n} \int_{q_i}^{p_i} dw_g \right) \in J(X) \end{aligned}$$

Abel: D is defined by meromorphic function iff $F(D) = 0$.

If let $Z(X) := \ker(\mathcal{F})$, then $\mathcal{F}$ induces

$$D_0(X)/Z(X) \longrightarrow J(X).$$

<u>Thm</u>: The above map is bijective.

- All divisors of $\deg = 0$ up to $\underbrace{\text{isomorphisms}}_{\text{or equivalence}}$ gives rise to $J(X)$.

- Morover, all holomorphic line bundles are given by divisors, so $\deg = 0$ holomorphic line bundles up to isomorphism $\cong J(X)$.

<u>Thm</u>: $\{\text{Holomorphic line bundles on } X\} \cong J(X) \times \mathbb{Z}$.