RIEMANN-ROCH THEOREM ON COMPACT RIEMANN SURFACES

FEI SUN

Abstract. In this note, we will prove Riemann-Roch theorem for compact Riemann surfaces. We will first take a look at algebraic curves and Riemann-Roch theorem and briefly introduce the relationship between algebraic curves and compact Riemann surfaces. Then we will introduce some algebraic tools to study Riemann surfaces and eventually prove Riemann-Roch theorem and Abel-Jacobi theorem, and their application in classifying holomorphic line bundles.

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fei.sun@icloud.com.

Introduction

The study of algebraic curves dated to long time ago. In classical *algebraic geometry*, people are interested in global theories, i.e, to study curves as a whole. Audience must still remember in high school, we use algebraic equations like $x^2+y^2=1$ to depict a geometric object, and study their global properties, like its perimeter, the area it encloses, etc. Then people gradually begin to study the function on these curves, for example, holomorphic or meromorphic functions on $\mathbb{C}$ or on Riemann surfaces. And discovered that some local properties are easier than global ones, e.g, holomorphic and meromorphic function can be written as power or Laurent series at an arbitrary point on Riemann surface (given that we admit polynomials are *easier* than arbitrary functions). Then a natural question arises: under what condition a local property can be extended to global case? Tools and methods are developed to solve this question: we have already seen manifolds, which glues together local pieces that are easier to study. We will see later in this note that sheaves and cohomologies are other tools often used to study this topic.

And it is amazing that the above paths is not unique to geometry, people have found amazing resemblance in number theory. The most basic non-trivial arithmetic I can think of (although it's not at all trivial!) is pythagorus problem: find x, y, z such that $x^2+y^2=z^2$. The issue that makes this question hard is people require integer or rational solutions. But we know that $\mathbb{Z}$ is a ring and $\mathbb{Q}$ is a *field*, and in ring we can't do *division* and $\mathbb{Q}$ is a very good field (we can see that $\mathbb{Q}$ is not closed under square root). Thus, we may embed $\mathbb{Q}$ into $\mathbb{R}$ or to p-adique field which are *complete* and have better properties. To make analogous, we may compare numbers with functions as we have seen in above paragraph. Let us compare $\mathbb{Z}$ with the ring of polynomials $k[T]$ (k a field) so that both of them are principal ideal domain and unique factorization domain (we can factorize elements in them and such factorization is unique). In geometry case, consider the field of formal series $\mathbb{C}(T)$, we can add an *infinity point* and when we approach to it, polynomial becomes Laurent series which corresponds to the embedding of $\mathbb{C}(T)$ into $\mathbb{C}((\frac{1}{T}))$. Similarly, for any point $c \in \mathbb{C}$, we can consider embedding $\mathbb{C}(T)$ into $\mathbb{C}((T-c))$ of all formal Laurent series at c. Analogously in $\mathbb{Q}$, if we *complete* it by adding all limit points under usual Euclidean metric to get $\mathbb{R}$; under p-adique metric we would get $\mathbb{Q}_{\mathfrak{p}}$ for any $\mathfrak{p}$ a prime number.

We can thus see the idea of going from local to global plays a big role in the development of modern mathematics. And two seemingly afar fields, analytic geometry and number theory, are related. We hope through this note, you may get a feeling of this idea and have a taste of modern algebraic geometry!

1. Some Complex Analysis

1.1. **Holomorphic and Meromorphic Functions.** We are majorly interested in analysis and functions with value in $\mathbb{C}$. Let us recall some useful results from complex analysis.

Let Ω be a path connected domain in $\mathbb{C}$ (we will just call it a domain for short), then we have the following classical results:

Theorem 1 (Open mapping). *If f is a non-constant analytic map on Ω, then f sends an open set to an open set.*

Theorem 2. *If f is a non-constant analytic map on Ω, then f doesn't have maximal point in Ω.*

Corollary 3. *Holomorphic function on a compact Riemann surface is constant.*

Theorem 4 (Liouville). *If f is bounded analytic function on $\mathbb{C}$, then f is a constant map.*

Suppose f and g are meromorphic functions on Ω having the same principal parts of their Laurent series, then $f - g$ is holomorphic on Ω. That is to say, meromorphic functions are determined by the principal part of its Laurent series, up to a holomorphic function. Moreover, we have the following result

Theorem 5 (Mittag-Leffler). *Let $\{z_n\}$ be a series in $\mathbb{C}$ and $\lim_{n\to\infty} z_n = \infty$. Let*

$$L_n(z) = \frac{a_{n_1}}{z - z_n} + \cdots + \frac{a_{n_{m_n}}}{(z - z_n)^{m_n}}$$

Then there exists a meromorphic function $f(z)$ having $\{z_n\}$ as its poles and $L_n(z)$ the principal parts of its Laurent series at $\{z_n\}$.

Since holomorphic functions on $\mathbb{C}$ are constants, on Riemann sphere $\bar{\mathbb{C}}$, meromorphic functions are determined by principal parts of Laurent series up to a constant.

Theorem 6. *Meromorphic functions on Riemann sphere are rational functions.*

Theorem 7. *On compact Riemann surface, a meromorphic function has same number of zeros and poles (counting multiplicity).*

Proof. Let's consider more generally, that $f : X \to Y$ being a non-constant holomorphic map between Riemann surfaces, with X compact. Suppose $f(x) = y$, and in properly chosen neighborhood of x, y, we may consider $f(z) = z^k$, and we define the index to be $\nu(x) = k$. Since X is compact, the pre-image $f^{-1}(y)$ is finite and we define the degree on y as $d(y) = \sum_{f(x)=y} \nu(x)$. We will prove that d doesn't depend on the choice of local charts.

For any $y \in Y$ and $f^{-1}(y) = \{x_1, x_2, \cdots, x_m\}$. Let $y \in U$ and $x_i \in V_i$ be local charts and $f(V_i) \subset U$. By replace U by $\bigcap_i f(V_i)$ and V_i by $V_i - f^{-1}(\bigcap_i f(V_i))$, we may assume $U = f(V_i)$ for each i (f is an open map). Furthermore, replace U by $U - f(X - \bigcup_i V_i)$ and intersecting V_i with the pre-image of the new U, we may assume $f^{-1}(U) = \bigcup_i V_i$ (f is also locally closed). Thus, for each $y \in U$, it has index $\nu(x_i)$ in V_i. Thus $d(y)$ is locally constant. □

1.2. **Multi-Value Functions.** For any $z \in \mathbb{C}$, we have

$$f(z) = \sqrt{z} = \sqrt{re^{i\theta}} = (\sqrt{r}e^{i\frac{\theta}{2}}, \sqrt{r}e^{i\frac{\theta+2\pi}{2}}) = (f_1, f_2).$$

From this simple example, we see that complex functions may have multi-values. Actually, both f_1 and f_2 are single value holomorphic function on $\mathbb{C} - \mathbb{R}^+$. We see that

$$\lim_{\theta \to 0^+} f_1(z) = \sqrt{r}, \qquad \lim_{\theta \to 0^+} f_2(z) = -\sqrt{r}$$

and

$$\lim_{\theta \to 0^-} f_1(z) = -\sqrt{r}, \qquad \lim_{\theta \to 0^-} f_2(z) = \sqrt{r}.$$

So $f_1(z)$ and $f_2(z)$ have different limits on both sides of $\mathbb{R}^+$. In fact, in order to get a single valued function of $f(z)$, we need to take two copies of $\mathbb{C}$, call them $\mathbb{C}_1$ and $\mathbb{C}_2$ and cut both of them along $\mathbb{R}^+$. After cutting $\mathbb{C}$ open along $\mathbb{R}^+$, we get two boundaries $\mathbb{R}^+_+$ and $\mathbb{R}^+_-$. We then glue $\mathbb{C}_1 - \mathbb{R}^+_+$ with $\mathbb{C}_2 - \mathbb{R}^+_-$ and $\mathbb{C}_1 - \mathbb{R}^+_-$ with $\mathbb{C}_2 - \mathbb{R}^-_-$ to get a single surface on which $f(z)$ is univalent and holomorphic.

Theorem 8. *Suppose Ω is a simply connected domain on which $f(z)$ is everywhere non-zero homomorphic function, then $\ln(f(z))$ has univalent holomorphic solution on Ω. Equivalently, there exists holomorphic function $g(z)$ such that $e^{g(z)} = f(z)$, moreover, $g(z) + i2k\pi = \ln(f(x))$, $k \in \mathbb{Z}$ are all the univalent homomorphic solutions.*

1.3. **Residues.** Let $D(z_0, R)$ be a disk of radius R with center removed. If f is analytic in $D(z_0, R)$ and z_0 is a pole of f, then the residue of f on z_0 is defined as

$$\mathrm{Res}(f, z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} f(z)dz$$

where $0 < \rho < R$. When ∞ is a pole of f, i.e, f is analytic in $R < |z| < \infty$, the the residue of f at ∞ is defined as:

$$\mathrm{Res}(f, \infty) = -\frac{1}{2\pi i}\int_{|z|=\rho} f(z)dz$$

where $R < \rho < +\infty$.

In fact, if $z_0 \in \mathbb{C}$ is a pole of function f, then $\mathrm{Res}(f, z_0)$ is the coefficient a_{-1} of term $\frac{1}{z-z_0}$ in the Laurent series of f at z_0. And when $z_0 = \infty$, the Laurent series is $\sum_{n=-\infty}^{\infty} a_n z^n$, and $\mathrm{Res}(f, \infty) = -a_{-1}$.

Residues can be calculated in the following way. Suppose $z_0 \neq \infty$ is a pole of order m of f, then

$$\mathrm{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z)\right).$$

In particular when $m = 1$, we have

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

Theorem 9 (Cauchy Integral). *Suppose γ is a simple closed curve and $z_1, z_2, \cdots, z_n$ lie inside of domain D encircled by γ. Suppose $f(z)$ is analytic in D except on $z_1, \cdots, z_n$ and continuous in $\bar{D}$ except $z_1, \cdots, z_n$, then*

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, z_k).$$

Theorem 10. *On Riemann sphere ($\bar{\mathbb{C}}$), we have:*

$$\sum_{p} \mathrm{Res}(f, p) = 0.$$

This theorem holds true on any compact Riemann surface, and p might be ∞. In fact, on compact Riemann surface, let $w = f dz$ and take any simple loop γ. According to Cauchy integral theorem, we have

$$\int_\gamma w = \sum_{p \in \gamma} \mathrm{Res}(w, p).$$

Then the sum of residue on the entire compact Riemann surface can be thought of Cauchy integral along γ in opposite direction, hence equals 0.

The reason that we are in particular interested in the coefficient a_{-1} of the Laurent series can be seen from the following example:

$$\frac{1}{2\pi i} \int_{|z-a|=r} \frac{dz}{(z-a)^n} = \begin{cases} 1; & n = 1 \\ 0; & \text{otherwise} \end{cases}$$

When $n \neq 1$, the function $\frac{dz}{(z-a)^n}$ is the derivative of a univalent holomorphic function for $0 < |z - a| < +\infty$. However, when $n = 1$, $\frac{dz}{z-a}$ is the derivative of $\log(z - a)$ which is a multi-valued function. $\log(z - a)$ has real part $\log|z| - a$ and imaginary part $i\mathrm{Arg}(z - a)$ which is a multi-value function, hence the value of

$$\frac{1}{2\pi i} \int_{|z-a|=r} \frac{dz}{(z-a)^n}$$

is the increment of of $\mathrm{Im}(\log(z - a))$ when goes around a for one lap, i.e, $\frac{2\pi i}{2\pi i} = 1$.

More generally, we have:

Theorem 11. *Let* γ *and* D *be as above,* f *is analytic in* $\bar{D}$ *and doesn't have zeros on* γ*. Then*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = N$$

where N *is the number of zeros in* D *(counting multiplicity).*

2. Algebraic Curves and Riemann-Roch Theorem

2.1. **Intro.** Riemann-Roch theorem considers the question of "how many functions" are there on a compact Riemann surface. More precisely, how many functions with *given poles.* We first give some necessary definitions and background.

Definition 12. *A divisor* D *on a surface is a formal linear combination of points, i.e,*

$$D = \sum_{\mathfrak{p}} n_{\mathfrak{p}} \mathfrak{p}.$$

And its degree is defined as $\deg(D) = \sum_{\mathfrak{p}} n_{\mathfrak{p}}$. *If* f *is a meromorphic function, then we define* (f) *to be the divisor formed by formal linear combination of zeros and poles of* f *counting multiplicity.*

Recall that we have proved that on a compact Riemann surface, a meromorphic function has same number of zeros and poles, thus $\deg((f)) = 0$. We call a divisor $D = \sum n_{\mathfrak{p}} \mathfrak{p}$ positive (non-negative) if $n_{\mathfrak{p}} > 0$ ($n_{\mathfrak{p}} \geq 0$) for any $\mathfrak{p}$.

Definition 13. *We call* k *a canonical divisor if its the linear combination of zeros and poles of a meromorphic* 1*-form.*

We call two divisors D and D' are linearly equivalent if $D - D' = (f)$ for some meromorphic function f. Equivalent divisors have the same degree.

Given above definition, we further define

$$l(D) = \dim\{\text{meromorphic functions } f \mid (f) + D \geq 0\}.$$

Some immediate results/observations are

(1) $l(0) = 1$ since holomorphic functions are constant on compact Riemann surface.
(2) $l(D) = 0$ if $\deg(D) < 0$. Since in this case, $l(D)$ consists of functions with more zeros than poles which is impossible.

After introducing all the notations, we give the Riemann-Roch Theorem. The proof of this theorem is the main focus of this note, and we will give a not so rigorous proof at the end of this section, and a detailed proof at later stage of this note.

Theorem 14 (Riemann-Roch)**.** *Let* D *be any divisor and* k *a canonical divisor on a compact Riemann surface* X. *Suppose* g *is the genus of* X, *then*

$$l(D) = \deg(D) + 1 - g + l(k - D).$$

Some immediate consequences of Riemann-Roch Theorem:

(1) Let $D = 0$. Then Riemann-Roch says $1 = 0 + 1 - g + l(k)$. thus we get $l(k) = g$.
(2) Let $D = k$. Then Riemann-Roch says $g = \deg(k) + 1 - g + 1$, thus $\deg(k) = 2g - 2$.

2.2. **Genus 0 Curves - Riemmann Sphere.** When $g = 0$, Riemann-Roch theorem has the following form:

$$l(D) = deg(D) + 1 + l(k - D).$$

An obvious candidate for meromorphic 1-form on Riemann sphere is dz, which has no zeros on $\mathbb{C}$. However, via transform $z \to y = \frac{1}{z}$, we see that $dz = -\frac{1}{y^2}dy$ has an order 2 pole at ∞, hence the canonical divisor $k = dz = -2 \times \infty$ has order -2. From our previous discussion, $l(k) = 0$. Hence Riemann-Roch theorem can further be written as

$$l(D) = deg(D) + 1 + l(-2 \cdot \infty - D),$$

thus

$$l(D) = \begin{cases} 0; & deg(D) < 0 \\ deg(D) + 1; & deg(D) \geq 0 \end{cases}$$

We then have the following simple summary of the relationships between $deg(D)$ and $l(D)$:

$deg(D)$	-3	-2	-1	0	1	2	3
$l(D)$	0	0	0	1	2	3	4
$l(k - D)$	2	1	0	0	0	0	0
$l(D) - l(k - D)$	-2	-1	0	1	2	3	4

In particular, let us take an arbitrary point p so that $l(p) = 2$. This means there exists functions f on X such that f has a single pole at p and no other poles everywhere. Further, we know that f also has a single zero on X. Hence f gives a function

$$f : X \to \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1.$$

f is clearly injective since $f - c$ has only one zero for any $c \in \mathbb{C}$. Hence f is bijective and hence can be identified as Riemann sphere.

One other properties is that *genus* 0 is almost equivalent to unique factorization domain. Suppose y is $\mathbb{CP}^1 - \{p_1, \cdots, p_k\}$, and R be the ring of rational functions which are regular on y, i.e, those functions whose poles can only be in the set $\{p_1, \cdots, p_k\}$. For example, if we take $y = \mathbb{C}$ then $R = \mathbb{C}[t]$. Now, according to Riemann-Roch, for any point $p \in y$, there exists a function $g_p \in R$ such that g_p is 0 at p and has no other zeros and poles on y. It turns out for any $f \in R$, we can write f as:

$$f = u \prod_p g_p^{n_p}, \qquad n_p = \text{order of zeros of } g_p \text{ at } p, \quad u \in \mathbb{C}^\times$$

and this factorization is unique.

For curves with genus≥ 1, the assertion is not true. In general, we would have the following:

genus	0	1	2
$\deg(k)$	< 0	$= 0$	> 0
Gaussian Curvature	> 0	$= 0$	< 0
Automorphisms	$PGL_2(\mathbb{C})$	$\mathbb{C}/L$	few automorphisms
UFD	Yes	No	No

2.3. **Genus 1 Curves - Elliptic Curves.** In this case, Riemann-Roch theorem has the form

$$l(D) = \deg(D) + l(k - D).$$

In particular, $\deg(k) = 2g - 2 = 0$ and $l(k) = 1$. And if $\deg(D) > 0$, then $l(D) = \deg(D)$. Take any point $\mathfrak{p}$, we have $l(\mathfrak{p}) = \deg(\mathfrak{p}) = 1$ hence all functions such that $(f) + \mathfrak{p} \geq 0$ are constants. This means, on $g = 1$ curve, we can not find functions having poles of order 1 at $\mathfrak{p}$ and no other poles.

Let us pick $\mathfrak{p} \in X$, and consider $l(n\mathfrak{p})$ with $n = 0, 1, 2, 3, \dots$

n	$l(n\mathfrak{p})$	functions
0	1	constants
1	1	same as above
2	2	x, pole of order=2
3	3	y, pole of order=3
4	4	x^2
5	5	xy
6	6	$y^2,\ x^3$

There are 7 functions in 6 dimensional space, so there must be a linear relationships between these functions, let's assume

$$ay^2 + by + cxy = dx^3 + ex^2 + fx + g, \qquad a, b, c, d, e, f, g \in \mathbb{C}.$$

We can manipulate the above equation such as add or multiply a constant to x and y, eventually, the equation can take the form

$$y^2 = 4x^3 - g_2 x - g_3, \quad g_2, g_3 \in \mathbb{C} \tag{1}$$

which is an affine curve in $\mathbb{C}^2$. If we compactify it to $\mathbb{CP}^2$, we get

$$zy^2 = 4x^3 - g_2 xz^2 - g_3 z^3.$$

This is called an elliptic curve. The affine curve can also be written as

$$y^2 = (x - a)(x - b)(x - c)(x - d), \qquad a, b, c, d \in \mathbb{C} \cup \{\infty\}.$$

From complex analysis point of view, the above function is multi-valued for y. In order to recover a univalent function, we need to cut open two copies of $\mathbb{CP}^1$ along a, b and c, d, then glue the two copies together. We can now see that the resulting surface is a Riemann surface of genus 1, i.e, a torus.

Moreover, from equation (1), if we take derivative on both ends, we get

$$2y\,dy = ((x-b)(x-c)(x-d) + (x-a)(x-c)(x-d) + (x-a)(x-b)(x-d) \\ +(x-a)(x-b)(x-c))dx$$

hence $\frac{dx}{y}$ is non-zero at $y = 0$ and a holomorphic 1-form. Suppose we perform integral on torus

$$w = \int_{p_0}^{p} \frac{dx}{y}.$$

The result is not unique, and the ambiguity is caused by tunnel and handle loops of the torus, i.e,

$$w_1 = \int_{handle} \frac{dx}{y}, \qquad w_2 = \int_{tunnel} \frac{dx}{y}$$

have ambiguities on

$$L =< w_1, w_2 >:= n \cdot w_1 + m \cdot w_2, \qquad n, m \in \mathbb{Z}.$$

Hence w_1 and w_2 are well-defined on $\mathbb{C}/L$ and gives a map from elliptic curve to torus.

2.4. **Genus 2 Curves.** In this case, Riemann-Roch theorem has the form

$$l(D) = deg(D) - 1 + l(k - D).$$

As before, we can calculate $deg(k) = 2g - 2 = 2$ and $l(k) = g = 2$. Let us also count $deg(D)$ and $l(D)$:

$deg(D)$	$l(D)$
< 0	0
$= 0$	0 or 1 (when $D \equiv 0$)
> 2	$deg(D) - 1$
$= 2$	1 or 2 (when $D \equiv k$)
$= 1$	$0, 1$ (cannot be 2 otherwise we may find a map to $\mathbb{CP}^1$ which is generically one-to-one hence not genus= 2!)

Since $l(k) = 2$, it induces a map

$$X \to \mathbb{CP}^1, \qquad (a, b) \mapsto [a : b]$$

where $< a, b >= l(k)$, hence we still get a double cover of $\mathbb{CP}^1$. To be more concrete, let us consider a point p such that $2p = k$. We have

n	$l(n\mathfrak{p})$	functions
0	1	constants
1	1	same as above
2	2	x, pole of order=2
3	2	same as above
4	3	x^2
5	4	y, pole of order=5
6	5	x^3
7	6	xy
8	7	x^4
9	8	x^2y
10	9	$y^2,\ x^5$

Similar to $g = 1$ case, we now have enough functions to form a linear relationship, and after some simplification, we may write our curve as

$$y^2 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_6), \qquad \alpha_i \in \mathbb{C},\ \forall 1 \le i \le 6. \tag{2}$$

To obtain a univalent function, we may cut open two copies of $\mathbb{CP}^1$ along $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4)$ and (α_5, α_6) then glue the two copies together. This way we find that out *genus* 2 curve has indeed two genus in the sense of complex geometry. Notice that in this form, one of the α_i might be ∞. In this case, we consider the affine curve

$$y^2 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_5).$$

If we embed it to $\mathbb{CP}^2$, we get $y^2z^3 = (x-\alpha_1 z)(x-\alpha_2 z)\cdots(x-\alpha_5 z)$. Hence if we let $y = 1$, we can see that the curve $z^3 = (x-\alpha_1 z)(x-\alpha_2 z)\cdots(x-\alpha_5 z)$ is singular at $[0:1:0]$.

Moreover, from the equation (2), we can also deduce holomorphic 1-forms of X. First, $\frac{dx}{y}$ is holomorphic, which can be seen by taking differentials on both side of (2). Then $z = \frac{1}{x}$ and $w = \frac{y}{x^3}$, equation (2) becomes

$$w^3 = (1-\alpha_1 z)(1-\alpha_2 z)\cdots(1-\alpha_6 z).$$

Hence $x^n\frac{dx}{y} = -z^{1-n}\frac{dz}{w}$, and it's immediate that $x\frac{dx}{y}$ is holomorphic. And, $x^n\frac{dx}{y}$ will not be holomorphic for $n > 1$.

In general, there are multiple ways of representing a genus 2 curve:

(1) Double cover of $\mathbb{CP}^1$, an 8 shape Riemann surface.
(2) $\mathbb{H}/\Gamma$ with $\Gamma \subset \mathsf{PSL}(2,\mathbb{R})$. Automorphisms are called modular forms.
(3) Plane curve in $\mathbb{CP}^2$ with one double point.
(4) There exists an embedding $X \to \mathbb{C}/(L \simeq \mathbb{Z}^4)$ via

$$z \longmapsto \left(\int_{z_0}^{z}\frac{dx}{y},\ \int_{z_0}^{z} x\frac{dx}{y}\right).$$

The $\mathbb{C}/L$ is called the Jacobian of X, we will talk more about it in later sections (after the proof of Riemann-Roch theorem).

2.5. **Comparison with Complex Geometry.** From our discussion in previous sections, we could deduce a comparison of algebraic theory with analysis.

	Algebra	Analysis
Objects	Projective algebraic curves over $\mathbb{C}$	Compact Riemann surfaces
Genus	dimension of space of holomorphic 1-forms	number of handles
Functions	rational functions	meromorphic functions
Genus 0	projective lines over $\mathbb{C}$	Riemann sphere
Genus 1	elliptic curves	$\mathbb{C}/L$ with $L = nw_1 + mw_2,\ n, m \in \mathbb{Z}$
Higher genus	$xy^3 + yz^3 + zx^3 = 0 \in \mathbb{CP}^2$	$H = \frac{\{\tau \mid \text{Im}(\tau)>0\}}{\text{discrete groups in } PSL_2(\mathbb{R})}$

We hence obtain a nice correspondence between algebraic curves and complex geometry. In general, the algebra to analysis direction can be realized by imbedding curves in *big enough* projective spaces. The way to do it is to first imbed algebraic curves to something called *Jacobi*, which is a compact Riemann surface with corresponding genus. Then by Kodaira imbedding theorem, we could imbed a compact complex manifold M into a big enough projective space, and the dimension of the projective space is given by the *degree* of certain line bundle on M.

2.6. **Sloppy Proof of Riemann-Roch Theorem.** We will divide the proof into three parts, like how exactly was it proved.

Riemann:

$$l(D) = deg(D) + 1 - i(0) + i(D)$$

where $i(D)$ is called the index of speciality or the obstruction of Mittag-Laffler problem. We will talk more on this later.

Genus: There are different definitions of genus, which we will assume they are equivalent:

(1) Topological: number of handles.
(2) Geometric (p_g): dimension of space of holomorphic 1-forms. This is $l(k)$.
(3) Arithmetic (p_a): defined as $i(0)$.

Roch:

$$i(D) = l(k - D).$$

In particular, if $D \equiv 0$ then $i(0) = l(k)$ which means $p_g = p_a$.

2.6.1. *The Space* $i(D)$. Let us first study $i(D)$. We are interested in the problem that given some points $\{p_i\}$ on X and some integers $\{n_{p_i}\}$, can we find meromorphic functions having $\{p_i\}$ as poles with multiplicity at most $\{n_{p_i}\}$? We define $i(D)$ be the space of obstruction of solving the problem.

For example, let $D = \sum n_p p$ be a divisor, we would care about those $(z-p)^{-n}$ such that $n > n_p$. E.g, for $2p$, we would be interested in series of the form:

$$c_{-3}(z-p)^{-3} + \cdots + c_{-m}(z-p)^{-m}.$$

Let R_p denotes the space of all Laurent series at p, we define

$$R = \left\{ \prod_p f_p \in \prod_p R_p \mid \text{almost all } f_p \text{ are holomorphic} \right\}.$$

Let

$$R(D) := \left\{ \prod_p f_p \mid f_p \text{ has poles of order} \leq n_p \text{ at } p \right\}$$

and

$$K(C) := \{\text{meromorphic fucntions on } C\}.$$

Then the space of obstructions $i(D)$ can be written as

$$i(D) = \frac{R}{R(D) + K(C)}.$$

2.6.2. *Riemann Theorem.* Notice that $l(D) = deg(D) + 1 - i(0) + i(D)$ is always true for $D \equiv 0$. Also compare the following two equations

$$l(D) = deg(D) + 1 - i(0) + i(D) \tag{3}$$

and

$$l(D+p) = deg(D+p) + 1 - i(0) + i(D+p) \tag{4}$$

Since $l(D+p) \leq 1 + l(D)$ with equality holds if and only if there exists meromorphic function f such that $(f) + D + p \geq 0$ but $(f) + D \not\geq 0$. Moreover, $i(D+p) \leq i(D)$ with equality holds when exactly the same condition is satisfied, hence the theorem is true of D if and only if it's true for $D + p$. Thus it's enough to show that $dim(i(0))$ is finite.

Let us take a non-singular, irreducible curve $C \subset \mathbb{CP}^2$ defined by homogenous function $f(x, y, z)$ of degree d. For simplicity, we assume that we have moved all the poles to ∞. Let's consider functions on this curve:

1. The space of all possible poles at ∞ of order $\leq N$ has dimension dN.
2. The space of polynomials in x, y with deg $\leq N$ has dimension

$$\frac{(N+1)(N+2)}{2}.$$

(3) The space of polynomials on C given by polynomials of deg$\leq N$ has dimension

$$\frac{(N+1)(N+2)}{2} - \frac{(N-d+1)(N-d+2)}{2} = dN + \frac{3}{2}d - \frac{d^2}{2}.$$

(4) The space of poles of functions on C of polynomials of deg$\leq N$ has dimension

$$dN + \frac{3}{2}d - \frac{d^2}{2} - 1.$$

(5) Compare with dN poles at ∞, we get (via genus-degree formula)

$$i(0) = \frac{3}{2}d - \frac{d^2}{2} - 1 = \frac{(d-1)(d-2)}{2} = p_a < \infty.$$

2.6.3. *Roch Theorem.* Consider the following bilinear pairing

$$\{\mathrm{meromorphic\ 1-forms}\ w \mid (w) + D \geq 0\} \times \frac{R}{R(D) + K(C)} \to \mathbb{C}$$

sending $(w, f) \mapsto \sum_{\mathfrak{p}} \mathrm{Res}(fw, \mathfrak{p})$. This pairing induces a map

$$p : l(k-D) \to i(D)^*.$$

We know that the sum of residues $\sum \mathrm{Res}(fw, \mathfrak{p}) = 0$ if f is meromorphic function, but there is no such function in $i(D)$, hence the map p is injective since $\sum \mathrm{Res}(fw, \mathfrak{p}) = \sum \mathrm{Res}(fw', \mathfrak{p})$ if and only if $w = w'$. So we have $l(k-D) \leq i(D)$ and it leaves us to show that $l(k-D) \geq i(D)$.

Till now, we have

$$\begin{cases} l(D) = \deg(D) + 1 - g + i(D)\ (\mathrm{Riemann}) \\ l(k-D) \leq i(D),\ l(D) \leq i(k-D) \\ \deg(k) = 2g-2 \end{cases}$$

Hence

$$\begin{aligned} l(D) = \deg(D) + 1 - g + i(D) &\geq \deg(D) + 1 - g + l(k-D) \\ &= \deg(D) + 1 - g + \deg(k-D) + 1 - g + i(k-D) \\ &\geq \deg(D) + \deg(k-D) + 2 - 2g + l(D) \\ &= \deg(k) + 2 - 2g + l(D) = l(D). \end{aligned}$$

which forces $l(k-D) = i(D)$.

3. Sheaves

3.1. Presheaves and Sheaves.

Definition 15. *A presheaf $\mathcal{F}$ on a topological space X is the following data:*

- *For each open set $U \subset X$, we have an Abelian group $\mathcal{F}(U)$ which is called the group of section of $\mathcal{F}$ over U.*
- *For each inclusion of open sets $V \hookrightarrow U$ we have a restriction map $\mathrm{res}_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$.*

The data satisfies the following two conditions:

(1) *The restriction map limited to itself is identity: $\mathrm{res}_{U,U} = \mathrm{id}_U$ for $\forall U$ open in X.*
(2) *If $U \hookrightarrow V \hookrightarrow W$ are inclusions of open sets, then we have (cocyle condition):*

$$\mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V}.$$

We can see that a presheaf is a collection of data of local functions. For any point $p \in X$, there might exist multiple neighbourhoods containing it, we define the **germs** at p the set

$$\{(f, U) \mid p \in U \subset X, f \in \mathcal{F}(U)\}$$

modulo the relationship that $(f, U) \sim (g, V)$ if there exists open set $W \subset U \cap V$ such that $f|_W = g|_W$. We call the set of germs the **stalk** at p, denoted by $\mathcal{F}_p$.

Definition 16. *A presheaf $\mathcal{F}$ on a topological space X is a sheaf if it satisfies two more axioms:*

Identity: *If $\{U_i\}_{i\in I}$ is an open cover of U and $f_1, f_2 \in \mathcal{F}(U)$. If $\mathrm{res}_{U,U_i} f = \mathrm{res}_{U,U_i} g$ for any $i \in I$, then $f = g$.*

Gluability: *If $\{U_i\}_{i\in I}$ is an open cover of U and $f_i \in U_i$ for all i. If $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_j\cap U_i} f_j$ for all $i, j \in I$, then there exists some $f \in \mathcal{F}(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all $i \in I$.*

Espace etalé or sheaf of sections We can construct a sheaf from a presheaf by the following method (a fancier way of saying it is that we constructed an espace etalé): Suppose $\mathcal{F}$ is a presheaf (sheaf) on a topological space X. As a set, let F be the union of all stalks of $\mathcal{F}$, i.e, $F = \bigcup_p \mathcal{F}_p$. Clearly there is a map $\pi : F \to X$. Topology of F is as follows: for each open U, it defines a set $\{(x, s_x)|s \in \mathcal{F}(U)\}$, such that for any for each $y \in F$ there exist an open neighborhood $V \subset F$ and open neighborhood $U \subset X$ so that $\pi|_V : V \to U$ is a homeomorphism. Moreover, for any $x \in X$ and any $s_1, s_2 \in \pi^{-1}(x)$, there exist neighborhoods U_1, U_2 of s_1, s_2 and V of x, such that $U_1 \to V$ and $U_2 \to V$ are homeomorphisms and for each $y \in V$, the multiplication of elements in $\pi^{-1}(y) \cap U_1$ and $\pi^{-1}(y) \cap U_2$ lies in the neighborhood of $s_1 \cdot s_2$.

The topological space F is called *the space of sections*, or *sheaf of sections*. It worth noticing that lots of people use this as the definition of sheaf (especially analysis people). It has advantages and disadvantages, advantage being it's naturally linked to sheafification (will talk later), and disadvantage being it's not general enough (we see that our definition of sheaves doesn't require $\mathcal{F}$ being a topological space).

We will talk more on morphisms between sheaves later, but here's an immediate example: suppose $\pi : X \to Y$ is a continuous map between topological space, and $\mathcal{F}$ is a sheaf on X. Then we define the pushforward sheaf $\pi_*(\mathcal{F})(V) := \mathcal{F}(\pi^{-1}(V))$ for an open set $V \in Y$.

Example 17 (Skyscraper Sheaf)**.** *Suppose* X *is a topological space,* $\mathfrak{p} \in X$ *a point and* $i_\mathfrak{p} : \mathfrak{p} \hookrightarrow X$ *the inclusion map. Suppose* A *an abelian group, then* $i_{\mathfrak{p},*}(A)$ *defined as*

$$i_{\mathfrak{p},*}(U) = \begin{cases} A, & \mathfrak{p} \in U \\ \{0\}, & \text{otherwise} \end{cases}$$

is a sheaf. Here, $\{0\}$ *denotes the one element group. We call this sheaf a skyscraper sheaf, and denote it by* $\mathsf{Sk}(\mathfrak{p})$.

Example 18 (Constant Sheaf)**.** *Let* $\mathcal{F}(U)$ *be the maps to* A *which is locally constant, i.e, for any* $\mathfrak{p} \in U$ *there is an open neighborhood of* $\mathfrak{p}$ *such that the function is constant. This structure forms a sheaf. We call this sheaf the constant sheaf associated to* A*, and denote it as* $\underline{A}$.

Ringed-spaces, $\mathcal{O}_X$-modules Suppose $\mathcal{O}_X$ is a sheaf of rings on topological space X, the structure $(X, \mathcal{O}_X)$ is called a ringed space. The sheaf $\mathcal{O}_X$ is called the structural sheaf, and the restriction of $\mathcal{O}_X$ on an open $U \subset X$ is denoted by $\mathcal{O}_X|_U = \mathcal{O}_U$. Since $\mathcal{O}_X$ is a sheaf of rings, we can have it act on sheaf of Abelian groups $\mathcal{F}$ such that the following diagram commute ($\mathcal{O}_X(U)$ is a ring and $\mathcal{F}(U)$ is an Abelian group for every open set $U \subset X$):

$$\begin{array}{ccc} \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\text{action}} & \mathcal{F}(V) \\ \downarrow {\scriptstyle res_{V,U} \times res_{V,U}} & & \downarrow {\scriptstyle res_{V,U}} \\ \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow[\text{action}]{} & \mathcal{F}(U) \end{array}$$

More concretely, suppose $(X, \mathcal{O}_X)$ is a differential manifold with $\mathcal{O}_X$ the sheaf of differentiable functions, and $\pi : V \to X$ a vector bundle on X. Then the sheaf of differentiable section $\sigma : X \to V$ is an $\mathcal{O}_X$-module. For any section s of π over U, and a function $f \in \mathcal{O}_X(U)$, we can get a new section fs on U.

3.2. **Morphisms of Sheaves.**

Definition 19. *A morphism of presheaf $\phi : \mathcal{F} \to \mathcal{G}$ is the data of maps $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ on open sets $U \subset X$ such that the diagram*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle res_{U,V}} & & \downarrow{\scriptstyle res_{U,V}} \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

Morphisms of sheaves are defined exactly the same: it's the data of maps satisfying the same commutative diagram between sheaves. The morphism of presheaves/sheaves induce morphisms of stalks $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$, since it's easy to see that the sheave/presheaf morphism sends sections to sections and preserves equivalence relationship that defines stalks.

Abelian Category Since our goal is to do cohomology of sheaves, we would need the notion of kernels and cokernels. We will not go into the technical details of Abelian categories, it suffices to know that such a thing admits basic arithmetic, kernels, cokernels, images and quotients. Firstly, it is easy to define sum of presheaves such that $(\mathcal{F} + \mathcal{G})(U) = \mathcal{F}(U) + \mathcal{G}(U)$. Secondly, if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, we define $(\mathsf{Ker}\phi)(U) := \mathsf{Ker}(\phi(U))$. Then the kernel presheaf is uniquely determined as we may chase the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathsf{Ker}\phi(V) & \longrightarrow & \mathcal{F}(V) & \longrightarrow & \mathcal{G}(V) \\ & & \downarrow{\scriptstyle \exists!} & & \downarrow{\scriptstyle res_{V,U}} & & \downarrow{\scriptstyle res_{V,U}} \\ 0 & \longrightarrow & \mathsf{Ker}\phi(U) & \longrightarrow & \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \end{array}$$

Presheaf Cokernel can be defined similarly by chasing the symmetric diagram as the above one: $\mathcal{F}(U) \xrightarrow{\psi} \mathcal{G}(U) \to \mathsf{Coker}\psi(U) \to 0$. Similarly, **images and quotients of presheaves** can be defined open sets by open sets.

However, we would like to study things locally and can be extended to globally. Thanks to gluability axiom, sheaf should be the right object to focus on. We then need to extend kernels, cokernels, image and quotients to sheaves, and since there are two more axioms to verify, this makes it harder to do so than in presheaf case. Actually, Kernel is the easiest, and if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the above definition gives a sheaf $\mathsf{Ker}\phi$. Cause the kernel presheaf is a sub-sheaf of $\mathcal{F}$, hence identify and gluability passes from $\mathcal{F}$ to $\mathsf{ker}\phi$.

Example 20. *Let $X = \mathbb{C}$ given the ordinary topology. Let $\mathcal{O}_X$ be the structural sheaf of holomorphic functions and $\mathcal{F}$ be the presheaf of functions admitting complex logarithm. The following sequence is exact:*

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_X \xrightarrow{\exp^{2\pi i f}} \mathcal{F} \to 0$$

where $\mathbb{Z}$ is the constant sheaf. Another thing worth noticing is that $\mathcal{F}$ is not a sheaf, since there are holomorphic functions that only has local logarithm but not globally.

3.3. **Properties at Level of Stalks and Sheafification.** The most important property this subsection will show is that the properties of sheaves can be checked on the level of stalks. This is not true for presheaves and reflects the local nature of sheaves.

Lemma 21. *The section of sheaves of sets is determined by it's germs, i.e, the map*

$$\mathcal{F}(U) \to \prod_{p \in U} \mathcal{F}_p$$

is injective.

Proof. Use identity axiom of sheaves, if s, s' are two sections in $\mathcal{F}(U)$ who images under the above map is the same, then for any $p \in U$ there exists an open neighborhood U_i such that s, s' restrict on U_i coincide. These $\{U_i\}$ cover the entire U and hence $s = s'$. □

Notice that we only used identity axiom and not gluability. Also the use of identity axiom is the key, so the above assertion is not true in general for presheaves.

Definition 22. *We say that an element $\prod_p s_p$ of $\prod_p \mathcal{F}_p$ consists of compatible germs if for all $p \in U$ there is a representative*

$$(U_p \text{ open in } U,\ \tilde{s}_p \in U_p)$$

for s_p such that the germs of $\tilde{s}_p$ at all $q \in U_p$ is s_q. Equivalently, there exists an open covering $\{U_i\}$ and sections $f_i \in U_i$ such that for any $p \in U_i$ we have s_p is the germ of f_i at p.

Morphisms are determined by stalks If $\phi_1, \phi_2 : \mathcal{F} \to \mathcal{G}$ are morphisms from a presheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$, then we have the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \prod_p \mathcal{F}_p & \longrightarrow & \prod_p \mathcal{G}_p \end{array}$$

So if ϕ_1 and ϕ_2 agree on each stalk, then $\phi_1 = \phi_2$. We can further show that two morphisms of sheaves are isomorphism if and only if they are isomorphic on each stalk.

Sheafification Suppose $\mathcal{F}$ is a presheaf. We define $\mathcal{F}^{sh}$ by defining $\mathcal{F}^{sh}(U)$ as the set of **compatible germs** of the presheaf $\mathcal{F}$ over U. Explicitly:

$$\mathcal{F}^{sh}(U) = \{(f_\mathfrak{p} \in \mathcal{F}_\mathfrak{p})_{\mathfrak{p}\in U}, \ \exists V \subset U \text{ containing } \mathfrak{p} \text{ and } s \in \mathcal{F}(V), \\ \text{such that } s_\mathfrak{q} = f_\mathfrak{q}, \forall \mathfrak{q} \in V\}$$

The morphism $\mathcal{F} \to \mathcal{F}^{sh}$ will send $s \in \mathcal{F}(U)$ to its compatible germ in $\mathcal{F}^{sh}(U)$. We can show that **espace etalé** is actually the sheafification of $\mathcal{F}$.

Definition 23. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. We say that ϕ is a monomorphism or injective if one of the following equivalent conditions holds*

(1) $\phi_\mathfrak{p} : \mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ *is injective of stalks for all* $\mathfrak{p} \in X$.
(2) $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ *is injective on all open set* $U \subset X$.

and in this case we call $\mathcal{F}$ a subsheaf of $\mathcal{G}$. And we say that ϕ is an epimorphism or surjective if $\phi_\mathfrak{p} : \mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ is surjective of stalks for all $\mathfrak{p} \in X$, and in this case we call $\mathcal{G}$ a quotient sheaf of $\mathcal{F}$.

Notice that we actually need to show that (1) and (2) are equivalent, but since we know that **morphisms are determined by stalks**, it's easy to check. Also, for surjectivity, we cannot check it on open sets. A (counter)-example is as follows: Consider X being the complex plane and $\mathcal{O}_X$ its structural sheaf of holomorphic functions. Let $\mathcal{O}_X^*$ be the sheaf of invertible (nowhere zero) holomorphic functions, then the sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1$$

is exact. For the map $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$, there exists open set V such that $\exp(V)$ is not surjective.

4. Some Homological Algebra and Cohomology of Sheaves

4.1. Čech Cohomology.

Definition 24. *Suppose $\mathcal{F}$ is a sheaf of Abelian group on a topological space X. Let $\mathcal{U}$ be an open cover of X. We define a p-cochain with respect to this cover as linear combinations of sections of the form*

$$h_{0,1,\dots,p} \in \Gamma(U_{0,1,\dots,p}, \mathcal{F})$$

where

$$U_{0,1,\dots,p} := U_0 \cap U_1 \cap \cdots \cap U_p \neq \emptyset, \qquad U_i \in \mathcal{U}.$$

We denote the (additive) Abelian group of p-cochains as $\check{C}^p(\mathcal{U}, \mathcal{F})$. Define the coboundary map $\delta : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^{p+1}(\mathcal{U}, F)$ as

$$\delta(h)_{0,1,\dots,p+1} = (-1)^i \sum_{i=0}^{p+1} h_{0,1,\dots,\hat{i},\dots,p+1} \in \Gamma(U_{0,\dots,p+1}, \mathcal{F}), \qquad \forall h_{0,\dots,p} \in \Gamma(U_{0,\dots,p}, \mathcal{F}).$$

As a standard procedure, it's easy to check that $\delta^2 \equiv 0$. Then, we define

$$\check{Z}^p(\mathcal{U}, \mathcal{F}) = \ker\{\check{C}^p(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^{p+1}(\mathcal{U}, \mathcal{F})\}$$

as the group of closed p-cochains and

$$\check{B}^p(\mathcal{U}, \mathcal{F}) = \mathrm{Im}\{\check{C}^{p-1}(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \check{C}^p(\mathcal{U}, \mathcal{F})\}$$

as the group of exact p-cochains. So

$$\check{H}^p(\mathcal{U}, \mathcal{F}) := \check{Z}^p(\mathcal{U}, \mathcal{F})/\check{B}^p(\mathcal{U}, \mathcal{F})$$

as the p-Čech cohomology group. For simplicity, we will denote $\mathcal{U}_I$ a non-empty intersection of finitely many opens indexed by I, and Čech complex can be written as

$$0 \to \prod_{|I|=1} \mathcal{F}(\mathcal{U}_I) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \prod_{|I|=i} \mathcal{F}(\mathcal{U}_I) \xrightarrow{\delta} \prod_{|I|=i+1} \mathcal{F}(\mathcal{U}_I) \xrightarrow{\delta} \cdots$$

and denote $\check{H}_{\mathcal{U}}(X, \mathcal{F})$ the homology associated to this chain complex.

Lemma 25. *For any cover $\mathcal{U}$, we have*

$$\check{H}^0_{\mathcal{U}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

Proof. According to definition, 0-cohomology group is the kernel of δ. By axioms of sheaves, a global section can be decomposed to $\{f_i \in U_i\}_{i\in I}$ such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$, and $\{f_i\}$ under δ is 0. □

Notice that the definition of Čech complex and cohomology can be defined in the same way for presheaves, however, the above lemma doesn't hold anymore for presheaf cohomology.

4.2. **Some Homological Algebra.** Suppose now we have two coverings $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ such that there exists a map $r : J \to I$ and for each $j \in J$ we have $V_j \subset U_{r(j)}$, then we call $\mathcal{V}$ a refinement of $\mathcal{U}$, or the cover $\mathcal{V}$ is finer than $\mathcal{U}$, denoted by $\mathcal{V} \leq \mathcal{U}$.

Suppose $\mathcal{U}$ is an open cover of X and for any $\mathfrak{p} \in X$ there exists an open neighborhood of X which intersects with only finite many elements in $\mathcal{U}$, then we call $\mathcal{U}$ a **locally finite** open cover. Moreover, if for any cover $\mathcal{U}$ there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is locally finite, then we call X a paracompact space. From now on, we will suppose that we are working on paracompact spaces.

In previous section, we have defined kernels, cokernels, images and quotients of sheaves, which can be checked on level of stalks. Now, let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are sheaves on some topological space X, we say the sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a **short exact sequence** if for any $\mathfrak{p} \in X$

(1) $\mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p}$ is injective.

(2) $\mathcal{G}_\mathfrak{p} \to \mathcal{H}_\mathfrak{p}$ is surjective.
(3) $\mathsf{Ker}(\mathcal{G}_\mathfrak{p} \to \mathcal{H}_\mathfrak{p}) = \mathsf{Im}(\mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p})$.

More generally, for a sequence of morphisms of sheaves

$$\cdots \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{r} \mathcal{H} \to \cdots$$

we say it's exact if $\mathsf{Ker}(\mathcal{G}_\mathfrak{p} \to \mathcal{H}_\mathfrak{p}) = \mathsf{Im}(\mathcal{F}_\mathfrak{p} \to \mathcal{G}_\mathfrak{p})$ for any $\mathfrak{p} \in X$.

In this section, we would like to prove the following theorem which is widely used in homological algebra:

Theorem 26. *Suppose* X *is a paracompact space, and*

$$0 \to \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{r} \mathcal{H} \to 0$$

is a short exact sequence. Then there exists group homomorphism

$$\delta^p : H^p(X, \mathcal{H}) \to H^{p+1}(X, \mathcal{F})$$

such that the long sequence

$$0 \to \check{H}^0(X, \mathcal{F}) \xrightarrow{i^*} \check{H}^0(X, \mathcal{G}) \xrightarrow{r^*} \check{H}^0(X, \mathcal{H}) \xrightarrow{\delta^0} \check{H}^1(X, \mathcal{F}) \xrightarrow{i^*}$$
$$\check{H}^1(X, \mathcal{G}) \xrightarrow{r^*} \check{H}^1(X, \mathcal{H}) \xrightarrow{\delta^1} \check{H}^2(X, \mathcal{F}) \xrightarrow{i^*} \cdots \xrightarrow{r^*} \check{H}^p(X, \mathcal{H}) \xrightarrow{\delta^p} \check{H}^{p+1}(X, \mathcal{F}) \xrightarrow{i^*} \cdots$$

is exact.

We start by proving the following lemma:

Lemma 27. *Let* X *be a paracompact space and* $0 \to \mathcal{F} \xrightarrow{i} \mathcal{G} \xrightarrow{r} \mathcal{H} \to 0$ *is a short exact sequence such that for any open set in a covering* $\mathcal{U}$ *of X, the sequence*

$$0 \to \mathcal{F}(U) \xrightarrow{i} \mathcal{G}(U) \xrightarrow{r} \mathcal{H}(U) \to 0$$

is exact, then there is a long exact sequence

$$0 \to \check{H}^0_{\mathcal{U}}(X, \mathcal{F}) \xrightarrow{i^*} \check{H}^0_{\mathcal{U}}(X, \mathcal{G}) \xrightarrow{r^*} \check{H}^0_{\mathcal{U}}(X, \mathcal{H}) \xrightarrow{\delta^0} \check{H}^1_{\mathcal{U}}(X, \mathcal{F}) \xrightarrow{i^*}$$
$$\check{H}^1_{\mathcal{U}}(X, \mathcal{G}) \xrightarrow{r^*} \check{H}^1_{\mathcal{U}}(X, \mathcal{H}) \xrightarrow{\delta^1} \check{H}^2_{\mathcal{U}}(X, \mathcal{F}) \xrightarrow{i^*} \cdots \xrightarrow{r^*} \check{H}^p_{\mathcal{U}}(X, \mathcal{H}) \xrightarrow{\delta^p} \check{H}^{p+1}_{\mathcal{U}}(X, \mathcal{F}) \xrightarrow{i^*} \cdots$$

Proof. We will use a technique called *chase the graph or diagramming chasing.* Consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \check{C}^{p-1}(\mathcal{U},\mathcal{F}) & \xrightarrow{\delta} & \check{C}^{p}(\mathcal{U},\mathcal{F}) & \xrightarrow{\delta} & \check{C}^{p+1}(\mathcal{U},\mathcal{F}) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\
\cdots & \longrightarrow & \check{C}^{p-1}(\mathcal{U},\mathcal{G}) & \xrightarrow{\delta} & \check{C}^{p}(\mathcal{U},\mathcal{G}) & \xrightarrow{\delta} & \check{C}^{p+1}(\mathcal{U},\mathcal{G}) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} & & \\
\cdots & \longrightarrow & \check{C}^{p-1}(\mathcal{U},\mathcal{H}) & \xrightarrow{\delta} & \check{C}^{p}(\mathcal{U},\mathcal{H}) & \xrightarrow{\delta} & \check{C}^{p+1}(\mathcal{U},\mathcal{H}) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

such that each vertical line is a short exact sequence. Let's choose an arbitrary $h \in \check{Z}^p(\mathcal{U},\mathcal{H})$, from the exactness of the p-column there exists $g \in \check{C}^p(\mathcal{U},\mathcal{G})$ such that $r^*(g) = h$. By commutativity of the diagram, we have

$$r^*\delta(g) = \delta r^*(g) = \delta(h) = 0.$$

By exactness of the $(p+1)$-column, we can find an $f \in \check{C}^{p+1}(\mathcal{U},\mathcal{F})$ such that $i^*(f) = \delta(g)$ and

$$i^*\delta(f) = \delta i^*(f) = \delta\delta(g) = 0.$$

By column exactness, i^* must be injective and hence $\delta(f) = 0$, meaning $f \in \check{Z}^{p+1}(\mathcal{U},\mathcal{F})$. Similar reasoning will shows that if $h \in \check{B}^p(\mathcal{U},\mathcal{H})$ then $f \in \check{B}^{p+1}(\mathcal{U},\mathcal{F})$. Thus we get a morphism

$$\delta^p : \check{H}^p(\mathcal{U},\mathcal{H}) \to \check{H}^{p+1}(\mathcal{U},\mathcal{F})$$

and furthermore a long sequence

$$
\begin{aligned}
&0 \to \check{H}^0_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{i^*} \check{H}^0_{\mathcal{U}}(X,\mathcal{G}) \xrightarrow{r^*} \check{H}^0_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\delta^0} \check{H}^1_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{i^*} \\
\check{H}^1_{\mathcal{U}}(X,\mathcal{G}) \xrightarrow{r^*} \check{H}^1_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\delta^1} \check{H}^2_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{i^*} \cdots \xrightarrow{r^*} \check{H}^p_{\mathcal{U}}(X,\mathcal{H}) \xrightarrow{\delta^p} \check{H}^{p+1}_{\mathcal{U}}(X,\mathcal{F}) \xrightarrow{i^*} \cdots
\end{aligned}
$$

We need to show that this sequence is exact. Consider the piece

$$\check{H}^p(\mathcal{U},\mathcal{H}) \xrightarrow{\delta^p} \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \xrightarrow{i^*} \check{H}^{p+1}(\mathcal{U},\mathcal{G})$$

From the above construction, we get $i^*(f) = \delta(g)$ and pass it to homology groups shows $i^*(f) = 0 \in \check{H}^{p+1}(\mathcal{U},\mathcal{G})$, so

$$\mathrm{Im}(\check{H}^p(\mathcal{U},\mathcal{H}) \xrightarrow{\delta^p} \check{H}^{p+1}(\mathcal{U},\mathcal{F})) \subset \mathrm{Ker}(\check{H}^{p+1}(\mathcal{U},\mathcal{F}) \xrightarrow{i^*} \check{H}^{p+1}(\mathcal{U},\mathcal{G})).$$

On the other hand, suppose $f \in \mathrm{Ker}(i^*)$, then we have

$$\delta^p(c) = \delta^p(r^*(g)) = r^*\delta^p(g) = r^*(i^*(f)) = 0$$

thus $c \in \check{Z}^p(\mathcal{U}, \mathcal{H})$ and hence in $\mathrm{Im}(\delta^p)$. □

We'd have to stop here first and prove an important theorem in Čech cohomology in the next subsection. We will continue our proof of Theorem 26 after that.

4.3. **Important Theorem.** It is obvious that the above definition depends on the choice of covering $\mathcal{U}$. We will then study how the cohomology behave under different cover. Suppose $\mathcal{V}$ is a refinement of covering $\mathcal{U}$. Every time we have such an refinement, we can define a group homomorphism $r^* : \check{C}^p(\mathcal{U}, \mathcal{F}) \to \check{C}^p(\mathcal{V}, \mathcal{F})$ as follows:

for any section $h_{0,\dots,p} \in \Gamma(U_{0,\dots,p}, \mathcal{F})$, the map $r : J \to I$ induces an inclusion $V_{0,\dots,p} \hookrightarrow U_{r(0),\dots,r(p)}$, and a cochain morphism:

$$r^*(h)_{0,\dots,p} = h_{r(0),\dots,r(p)}|_{V_{0,\dots,p}}.$$

We say that two covers $\mathcal{V} \leq \mathcal{U}$ induce a chain map $\check{C}^*_{\mathcal{U}}(X, \mathcal{F}) \to \check{C}^*_{\mathcal{V}}(X, \mathcal{F})$ if

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \prod_{|I|=1} \mathcal{F}(\mathcal{U}_I) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \prod_{|I|=i} \mathcal{F}(\mathcal{U}_I) & \xrightarrow{\delta} & \prod_{|I|=i+1} \mathcal{F}(\mathcal{U}_I) & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle r^*} & & & & \downarrow{\scriptstyle r^*} & & \downarrow{\scriptstyle r^*} & & \\
0 & \longrightarrow & \prod_{|J|=1} \mathcal{F}(\mathcal{V}_J) & \xrightarrow{\delta} & \cdots & \xrightarrow{\delta} & \prod_{|J|=i} \mathcal{F}(\mathcal{V}_J) & \xrightarrow{\delta} & \prod_{|J|=i+1} \mathcal{F}(\mathcal{V}_J) & \xrightarrow{\delta} & \cdots
\end{array}$$

commutes where the vertical maps are given by r^*. Readers are encouraged to getting familiar with commutative maps of chain complexes and verify the map r and δ actually make the diagram commutative.

Theorem 28. $\check{H}^p_{\mathcal{U}}(X, \mathcal{F})$ *is independent of the cover* $\mathcal{U}$*. That is, if we have a refinement* $\mathcal{V} \leq \mathcal{U}$*, then the induced map* $\check{H}^p_{\mathcal{U}}(X, \mathcal{F}) \to \check{H}^p_{\mathcal{V}}(X, \mathcal{F})$ *is an isomorphism. We define Čech cohomology group as* $\check{H}^p(X, \mathcal{F})$ *regardless of choice of cover.*

Proof. We will show the case that $|\mathcal{V}| = |\mathcal{U}| + 1$. Let us fix an open set U_0, and for an open cover $\mathcal{U} = \{I_i\}_{1 \leq i \leq n}$, the map $\check{H}^*_{\{U_i\}_{0 \leq i \leq n}}(X, \mathcal{F}) \to \check{H}^*_{\{U_i\}_{0 \leq i \leq n}}(X, \mathcal{F})$ is an isomorphism.

Consider the exact sequences of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \prod_{|I|=i-1,0\in I}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i,0\in I}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i+1,0\in I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \prod_{|I|=i-1}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i+1}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \prod_{|I|=i-1,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \prod_{|I|=i+1,0\notin I}\mathcal{F}(U_I) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Notice that the cohomology groups associated with the bottom two row chain complexes are the cohomologies in question, hence it suffices to show that the top row is exact and induces trivial cohomologies. However, we notice that the cohomology groups of the top row is actually $\check{H}^p_{U_0\cap\mathcal{U}_I}(U_0,\mathcal{F})$ except at step $p=0$, and since U_0 is a contractible open set of X its higher cohomology groups are trivial. □

By Theorem 28, we ignore the index of covering when writing Čech cohomology group unless we need to specify which cover we are using.

After showing Čech cohomology doesn't depend on specific cover, we shall continue our proof of Theorem 26. Remember that the lemma 27 requires that $0\to\mathcal{F}(U)\xrightarrow{i}\mathcal{G}(U)\xrightarrow{r}\mathcal{H}(U)\to 0$ is exact for any open U in the open cover $\mathcal{U}$. This is generally not true for sheaf, but for any sheaf or presheaf, the *left truncated* sequence

$$0\to\mathcal{F}(U)\xrightarrow{i}\mathcal{G}(U)\xrightarrow{r}\mathcal{H}(U)$$

is exact. Hence we could define $\tilde{\mathcal{H}}:=\operatorname{Im}(\mathcal{G}\xrightarrow{r}\mathcal{H})$ as a presheaf, and the short sequence

$$0\to\mathcal{F}(U)\xrightarrow{i}\mathcal{G}(U)\xrightarrow{r}\tilde{\mathcal{H}}(U)\to 0$$

is exact for any $U\in\mathcal{U}$. Lemma 27 shows the long sequence

$$\begin{aligned}
&0\to H^0(X,\mathcal{F})\xrightarrow{i^*}\check{H}^0(X,\mathcal{G})\xrightarrow{r^*}\check{H}^0(X,\tilde{\mathcal{H}})\\
&\xrightarrow{\delta^0}\check{H}^1(X,\mathcal{F})\xrightarrow{i^*}\check{H}^1(X,\mathcal{G})\xrightarrow{r^*}\check{H}^1(X,\ r\tilde{H})\\
&\xrightarrow{\delta^1}\check{H}^2(X,\mathcal{F})\xrightarrow{i^*}\cdots\xrightarrow{r^*}\check{H}^p(X,\tilde{\mathcal{H}})\xrightarrow{\delta^p}\check{H}^{p+1}(X,\mathcal{F})\xrightarrow{i^*}\cdots
\end{aligned}\qquad(5)$$

is exact. The following lemma will complete the proof of Theorem 26,

Lemma 29. *The cohomology groups* $\check{H}^*(X, \mathcal{H})$ *and* $\check{H}^*(X, \tilde{\mathcal{H}})$ *are isomorphic.*

Proof. Since $\tilde{\mathcal{H}}$ is a sub-presheaf of $\mathcal{H}$, the monomorphism $\iota : \tilde{\mathcal{H}} \to \mathcal{H}$ induces a cochain map sending cocycles (resp. coboundaries) to cocycles (resp. coboundaries). Thus it further induces a map $\iota^* : \check{H}^*(X, \tilde{\mathcal{H}}) \to \check{H}^*(X, \mathcal{H})$.

First, let's show that ι^* is surjective. Take $h \in \check{Z}^p(X, \mathcal{H})$ and $\mathcal{U}$ is an open cover of X. We may suppose $\mathcal{U}$ is locally finite since X is paracompact. By exactness of sheaves, for any $\mathfrak{p} \in X$, the sequence

$$0 \to \mathcal{F}_{\mathfrak{p}} \to \mathcal{G}_{\mathfrak{p}} \to \mathcal{H}_{\mathfrak{p}} \to 0$$

is exact. For any $h \in \mathcal{H}_{\mathfrak{p}}$, we take an a compatible germ s on a neighborhood O of $\mathfrak{p}$ such that $s(\mathfrak{p}) = h$, and $s \in \mathrm{Im}(\mathcal{G}(O) \to \mathcal{H}(O)) = \tilde{\mathcal{H}}(O)$. We can take this O to be sufficiently small, so that $h = h_{0,\dots,p}$ restrict on O is actually in $\Gamma(O, \tilde{\mathcal{H}})$. Since this argument works for every $\mathfrak{p} \in X$, we can then obtain an open cover $\mathcal{V}$ such that every $h_{0,\dots,p}$ restrict on $\mathcal{V}$ belongs to $\check{Z}^p(\mathcal{V}, \tilde{\mathcal{H}})$. Thus ι^* is surjective.

It leaves us to prove the injectivity. Suppose $\tilde{h} \in \check{H}^p(X, \mathcal{H})$ and $\iota^*(\tilde{h}) = 0$. Then there exists an $f_{0,\dots,p-1} \in \check{C}^{p-1}(\mathcal{U}, \mathcal{H})$ and $h_{0,\dots,p} \in \check{Z}^p(\mathcal{U}, \mathcal{H})$ such that $\delta(f) = h$. Similar as above, we can find a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $f \in \check{C}^{p-1}(\mathcal{V}, \mathcal{H})$ and $h = \delta(f) \in \check{C}^{p-1}(\mathcal{V}, \tilde{\mathcal{H}})$. Hence ι^* is also injective. $\square$

In fact, if we examine the above proof, we used the construction of compatible germs, so if we pass it to presheaves and shifification, we actually proved

Lemma 30. *If the two presheaves have isomorphic shifification then their cohomology groups are isomorphic.*

4.4. **More Homological Algebra and Sheaf Cohomology.**

Definition 31. *We say a sheaf* $\mathcal{F}$ *on* X *is acyclique if its higher cohomology groups are zero, i.e, if* $\check{H}^i(X, \mathcal{F}) = 0$ *for* $i > 0$. *If a sheaf can be written as the kernel of some exact acyclique sequence then the sequence is called an acyclique resolution of* $\mathcal{F}$.

Suppose

$$\mathcal{I}^* = \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

is a long exact sequence of acyclique sheaves, and $0 \to \mathcal{F} \to \mathcal{I}^*$ is exact, i.e, $\mathcal{I}^*$ is an acyclique resolution of $\mathcal{F}$, then for any $U \in \mathcal{U}$ of X, only the truncated short sequence

$$0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{I}^0) \to \Gamma(U, \mathcal{I}^1)$$

is exact.

Theorem 32 (De Rham)**.** *Let* X *be a paracompact space and* $\mathcal{F}$ *is an Abelian sheaf on* X. *If* $\mathcal{I}^*$ *is an acyclique resolution of* $\mathcal{F}$, *then we have group isomorphism*

$$\check{H}^i(X, \mathcal{F}) \simeq \mathrm{Ker}\{\Gamma(X, \mathcal{I}^i) \to \Gamma(X, \mathcal{I}^{i+1})\}/\mathrm{Im}\{\Gamma(X, \mathcal{I}^{i-1}) \to \Gamma(X, \mathcal{I}^i)\}.$$

Proof. For $i = 0$, since the resolution sequence is exact, $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}^0)$ is injective, so

$$\check{H}^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathrm{Ker}(\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1))$$

Now, suppose $i > 0$. Take $\mathcal{K}^i := \mathrm{Ker}(\mathcal{I}^i \to \mathcal{I}^{i+1})$, we get a short exact sequence:

$$0 \to \mathcal{K}^i \to \mathcal{I}^i \to \mathcal{K}^{i+1} \to 0.$$

Use Theorem 26, we obtain a long exact sequence

$$0 \to \check{H}^0(X, \mathcal{K}^i) \to \check{H}^0(X, \mathcal{I}^i) \to \check{H}^0(X, \mathcal{K}^{i+1}) \to \check{H}^1(X, \mathcal{K}^i) \to \check{H}^1(X, \mathcal{I}^i) \to$$
$$\check{H}^1(X, \mathcal{K}^{i+1}) \to \check{H}^2(X, \mathcal{K}^i) \to \cdots \to \check{H}^p(X, \mathcal{K}^{i+1}) \to \check{H}^{p+1}(X, \mathcal{K}^i) \to \cdots$$

Since $\mathcal{I}^i$ is acyclique, we get $\check{H}^p(X, \mathcal{K}^{i+1}) \simeq \check{H}^{p+1}(X, \mathcal{K}^i)$. Also, from short exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{K}^1 \to 0$ we obtain

$$\check{H}^p(X, \mathcal{F}) \simeq \check{H}^{p-1}(X, \mathcal{K}^1) \simeq \check{H}^{p-2}(X, \mathcal{K}^2) \simeq \cdots \simeq \check{H}^1(X, \mathcal{K}^{p-1}).$$

Next, from exact sequence

$$0 \to \check{H}^0(X, \mathcal{K}^{p-1}) \to \check{H}^0(X, \mathcal{I}^{p-1}) \to \check{H}^0(X, \mathcal{K}^p) \to \check{H}^1(X, \mathcal{K}^{p-1}) \to 0,$$

we have

$$\check{H}^1(X, \mathcal{K}^{p-1}) \simeq \check{H}^0(X, \mathcal{K}^p)/\mathrm{Im}\{\check{H}^0(X, \mathcal{I}^{p-1}) \to \check{H}^0(X, \mathcal{K}^p)\}$$

Moreover, since $H^0(X, \mathcal{K}^p) = \mathrm{Ker}(\Gamma(X, \mathcal{I}^p) \to \Gamma(X, \mathcal{I}^{p+1}))$, we obtain

$$H^p(X, \mathcal{F}) \simeq \mathrm{Ker}\{\Gamma(X, \mathcal{I}^p) \to \Gamma(X, \mathcal{I}^{p+1})\}/\mathrm{Im}\{\Gamma(X, \mathcal{I}^{p-1}) \to \Gamma(X, \mathcal{I}^p)\}.$$

□

De Rham theorem allows us to define cohomology theory without considering open covers, it shows that cohomology is a *gloabl* property. Our next question is: what sheaves are acyclique?

Definition 33. *We call a sheaf $\mathcal{F}$ flasque if for any open set inclusion $V \subset U$, the morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective.*

In particular, we may take $U = X$ and a flasque sheaf satisfies $\mathcal{F}(X) \to \mathcal{F}(V)$ is surjective, which implies that any local section can be extended to a global section for flasque sheaf $\mathcal{F}$.

Theorem 34. *All flasque sheaves are acyclique for Čech cohomology.*

Proof. Suppose $h = h_{0,\dots,p} \in \check{Z}^p(X, \mathcal{F})$ which is defined on $U_{0,1,\dots,p} = \bigcap_i U_i$. For any small open subset O, we denote $O(\mathcal{U}) := \{O \cap U_i, U_i \in \mathcal{U}\}$. Then $O(\mathcal{U})$ is an open cover of O.

Let $p \in U_a$ and let $O = U_a$. We define a p-cochain $f_{0,\dots,p-1}$ on $O(\mathcal{U})$ as

$$f_{0,\dots,p-1} := h_{a,0,\dots,p-1} \in U_a \cap U_0 \cap \cdots \cap U_{p-1}.$$

Because h is a cocyle, we have

$$0 = \delta(h)_{a,0,\dots,p} = h_{0,\dots,p} - \sum_{i=1}^{p} (-1)^i h_{a,0,\dots,\hat{i},\dots,p} = h_{0,\dots,p} - \delta(f).$$

Thus, locally we have $\delta(f) = h|_{O(\mathcal{U})}$. Suppose for another open subset V with $O \cap V \neq \emptyset$, there exists $g \in \check{C}^{p-1}(V(\mathcal{U}), \mathcal{F})$ such that $\delta(g) = h|_{V(\mathcal{U})}$. We want to show that for open cover $(O \cup V)(\mathcal{U})$, there exists an $s \in \check{C}^{p-1}((O \cup V)(\mathcal{U}), \mathcal{F})$ such that $\delta(s) = h|_{O\cup V(\mathcal{U})}$.

For $p = 1$, let a be an index such that $(O \cup V) \cap U_a \neq \emptyset$, and $f_a - g_a$ is defined on $(O \cup V) \cap U_a$. Since $\mathcal{F}$ is flasque, there exists $t \in \mathcal{F}(X)$ such that $f_a - g_a = t$ on $(O \cup V) \cap U_a$. Thus if we define

$$s_a(\mathfrak{p}) = \begin{cases} f_a(\mathfrak{p}); & \mathfrak{p} \in O \cap U_a \\ g_a(\mathfrak{p}) + t(\mathfrak{p}); & \mathfrak{p} \in V \cap U_a \end{cases}$$

then after going over all such a, we could find $s \in \check{C}^0((O \cup V)\mathcal{U}, \mathcal{F})$ such that $\delta(s) = h|_{(O\cup V)(\mathcal{U})}$.

For $p > 1$, we have $\delta(f) - \delta(g) = 0$ on $(O \cap V)(\mathcal{U})$. Thus there exists t on $\check{C}^{p-2}((O \cap V)(\mathcal{U}), \mathcal{F})$ such that $\delta(t) = f - g$ on $(O \cap V)(\mathcal{U})$. Since $\mathcal{F}$ is flasque, we can extend t to $\check{C}^{p-2}(V(\mathcal{U}), \mathcal{F})$ and let

$$s_{0,\dots,p-1}(\mathfrak{p}) = \begin{cases} f_{0,\dots,p-1}(\mathfrak{p}); & \mathfrak{p} \in O \cap U_{0,\dots,p-1} \\ g_{0,\dots,p-1}(\mathfrak{p}) + \delta(t)_{0,\dots,p-1}(\mathfrak{p}); & \mathfrak{p} \in V \cap U_{0,\dots,p-1} \end{cases}$$

Then we have $s = s_{0,\dots,p-1} \in \check{C}^{p-1}((O \cup V)(\mathcal{U}), \mathcal{F})$ such that $\delta(s) = h|_{(O\cup V)(\mathcal{U})}$.

To finish the proof, we apply Zorn's lemma so that s can be extended to $\mathcal{U}$, hence every cocycle is also co-exact and $\mathcal{F}$ is acyclique. □

We next show a result that only true for smooth manifold but not complex manifold. We suppose our smooth manifold admits compact support for smooth functions, which in turn gives us a partition of unity. For complex manifold however, holomorphic functions with compact support are identically 0 (consider any point outside of the support, on which holomorphic functions must have all derivatives zero hence itself is zero).

Theorem 35. *Any $\mathcal{O}_X$-module over a smooth manifold is acyclique.*

Proof. Since we always assume our space is paracompact, we can only consider locally finite covers. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be such a cover and $\{f_i \in U_i\}$ be a partition of unity on this cover. Take $h = h_{0,\dots,p} \in \check{Z}^p(X, \mathcal{F})$ be a p-cocyle. For any $U_{0,\dots,p-1} \neq \emptyset$, define

$$g_{0,\dots,p-1} := \sum_{a\in I} f_a h_{a,0,\dots,p-1}.$$

Since $\mathcal{F}$ is an $\mathcal{O}_X$-module, we have $g \in \check{C}^{p-1}(\mathcal{U}, \mathcal{F})$. Moreover,

$$0 = \delta(h)_{a,0,\dots,p} = h_{0,\dots,p} - \sum_{i=0}^{p} (-1)^i h_{a,\dots,\hat{i},\dots,p},$$

hence

$$\begin{aligned}\delta(g)_{0,\dots,p} &= \sum_{i=0}^{p} (-1)^i g_{0,\dots,\hat{i},\dots,p} = \sum_{i=0}^{p} (-1)^i \sum_{a \in I} f_a h_{a,0,\dots,\hat{i},\dots,p} \\ &= \sum_{a \in I} f_a \sum_{i=0}^{p} (-1)^i h_{a,0,\dots,\hat{i},\dots,p} = \sum_{a \in I} h_{0,\dots,p} = h_{0,\dots,p}.\end{aligned}$$

So h is also co-exact. □

This theorem shows that if we have a sheaf $\mathcal{F}$ which is an $\mathcal{O}_X$-module of a smooth manifold X then it's acyclique. And if we have a resolution of such $\mathcal{O}_X$-module, by De Rham Theorem 32, we can calculate cohomology groups from it. We will use this theorem to show some classical cohomology theories can be obtained from Čech cohomology.

4.5. **Some Classical Cohomology Theories.** Recall that for any point $\mathfrak{p}$ on a complex manifold X, we have

$$\wedge^r T_p^*(X) = \bigoplus_{p+q=r} \{ \left(\wedge^p T^*_{(1,0),\mathfrak{p}}(X)\right) \wedge \left(\wedge^q T^*_{(0,1),\mathfrak{p}}(X)\right) \}$$

Here, the sub-index $(0,1)$ and $(1,0)$ emphasizes the holomorphic and anti-holomorphic part of the exterior products. In terms of local coordinates, suppose $(z^1, \dots, z^n)$ is a local coordinate of $p \in X$, then $(dz^1, \dots, dz^n)$ and $(d\bar{z}^1, \dots, d\bar{z}^n)$ are local coordinates of $T^*_{(1,0),\mathfrak{p}}(X)$ and $T^*_{(0,1),\mathfrak{p}}(X)$. Then on $\mathfrak{p}$, a (p,q)-form can be written as

$$\omega = \sum_{1 \le i_1 \le \dots \le i_p \le n} \sum_{1 \le \bar{j}_1 \le \dots \le \bar{j}_q \le n} a_{i1\dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \dots \wedge d\bar{z}^{\bar{j}_q}$$

with $a_{i1\dots i_p \bar{j}_1 \dots \bar{j}_q} \in \mathbb{C}$.

We denote by $\mathcal{A}^r(X)$ the sheaf of r-differential form on X, $A^{(p,q)}(X)$ the sheaf of smooth (p,q)-forms on X, and $\Theta(X)$ the sheaf of holomorphic functions on X. Since a constant function can be thought of those functions which vanishes under differential, we have a long sequence

$$0 \to \mathbb{C} \hookrightarrow \mathcal{A}^0(X) \xrightarrow{d} \mathcal{A}^1(X) \xrightarrow{d} \mathcal{A}^2(X) \xrightarrow{d} \cdots$$

Moreover, holomorphic functions can be thought of those functions which vanishes under $\bar{\partial}$ operator, where

$$\bar{\partial}(f) = \sum_i \frac{\partial f}{\bar{\partial} z^i} d\bar{z}^i$$

we have another long sequence

$$0 \to \Theta(X) \hookrightarrow \mathcal{A}^{(0,0)}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)}(X) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,q)}(X) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,q+1)}(X) \xrightarrow{\bar{\partial}} \cdots$$

More generally, let $E \to X$ be a holomorphic vector bundle, and $\Theta(E)$ is the sheaf of holomorphic sections, similarly we have a long sequence

$$0 \to \Theta(E) \hookrightarrow \mathcal{A}^{(0,0)}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)}(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,q)}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,q+1)}(E) \xrightarrow{\bar{\partial}} \cdots$$

In particular, $E \otimes \wedge^p T^*_{(1,0)}(X)$ is a holomorphic vector bundle on X, if we denote $\Omega^p(X)$ the sheaf of sections of $E \otimes \wedge^p T^*_{(1,0)}(X)$, i.e, sheaf of E-valued $(p,0)$-forms, then we've got another long sequence

$$0 \to \Omega^p(E) \hookrightarrow \mathcal{A}^{(p,0)}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,1)}(E) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,q)}(E) \xrightarrow{\bar{\partial}} \mathcal{A}^{(p,q+1)}(E) \xrightarrow{\bar{\partial}} \cdots$$

Notice that in the above construction, we only consider smooth forms (in particular, holomorphic functions would be 0 under $\bar{\partial}$, so if we are interested in anything holomorphic here, we would get almost everything 0). Hence the $\mathcal{A}^r(X), \mathcal{A}^{(p,q)}(X)$ and $\mathcal{A}^{(p,q)}(M)$ are modules of sheaf of smooth functions over X. By Theorem 35, if we can show that the above sequences are exact, we would have obtained acyclique resolutions of $\Theta(X), \Theta(E)$ and $\Omega^p(E)$. And furthermore by De Rham Theorem 32, we can calculate cohomologies of $\Theta(X), \Theta(E)$ and $\Omega^p(E)$ by Čech cohomology.

Thus, but Theorem 32, we have the following theorems connecting Čech cohomology and some classical cohomologies.

Theorem 36. *Suppose X is a differentiable manifold, then*

$$\check{H}^r(X,\mathbb{C}) \simeq \frac{\mathrm{Ker}(\mathcal{A}^r(X) \xrightarrow{d} \mathcal{A}^{r+1}(X))}{\mathrm{Im}(\mathcal{A}^{r-1}(X) \xrightarrow{d} \mathcal{A}^r(X))} = H^r_{dr}(X,\mathbb{C}).$$

Theorem 37. *Let X be a complex manifold, $E \to X$ is a holomorphic vector bundle, then we have the following isomorphisms between Čech cohomology and Dolbeault cohomology:*

$$\check{H}^r(X,\Theta(X)) \simeq \frac{\mathrm{Ker}(\mathcal{A}^{(0,q)}(X) \xrightarrow{d} \mathcal{A}^{(0,q+1)}(X))}{\mathrm{Im}(\mathcal{A}^{(0,q-1)}(X) \xrightarrow{d} \mathcal{A}^{(0,q)}(X))} = H^{(0,q)}(X),$$

$$\check{H}^r(X,\Theta(E)) \simeq \frac{\mathrm{Ker}(\mathcal{A}^{(0,q)}(E) \xrightarrow{d} \mathcal{A}^{(0,q+1)}(E))}{\mathrm{Im}(\mathcal{A}^{(0,q-1)}(E) \xrightarrow{d} \mathcal{A}^{(0,q)}(E))} = H^{(0,q)}(E),$$

and

$$\check{H}^r(X,\Omega^p(E)) \simeq \frac{\mathrm{Ker}(\mathcal{A}^{(p,q)}(E) \xrightarrow{d} \mathcal{A}^{(p,q+1)}(E))}{\mathrm{Im}(\mathcal{A}^{(p,q-1)}(E) \xrightarrow{d} \mathcal{A}^{(p,q)}(E))} = H^{(p,q)}(E).$$

From Hodge theory, we have the following duality result

Corollary 38 (Kodaira-Serre Duality). *Suppose* X *is an* n*-dimensional compact complex manifold,* $E \to X$ *is a holomorphic vector bundle, then*

$$H^{(p,q)}(X, E) = H^{(n-p,n-q)}(X, E^*), \qquad \forall 0 \le p, q \le n.$$

In particular,

$$\dim \left(H^{(p,q)}(X, E)\right) = \dim \left(H^{(n-p,n-q)}(X, E^*)\right), \qquad \forall 0 \le p, q \le n.$$

5. Riemann-Roch Theorem

5.1. **Divisors and Line Bundles.** We will constrain our conversation within the scope of compact Riemannian surface, a.k.a, one dimensional complex closed manifold. Suppose R is such a surface, $\mathbb{C}$ is a constant sheaf on R. From our discussion in the previous section, we know

$$\check{H}^r(R, \mathbb{C}) \simeq \frac{\mathrm{Ker}(\mathcal{A}^r(R) \xrightarrow{d} \mathcal{A}^{r+1}(R))}{\mathrm{Im}(\mathcal{A}^{r-1}(R) \xrightarrow{d} \mathcal{A}^r(R))} = H^r_{dr}(R, \mathbb{C}).$$

The real dimension of R is 2, and for any $r > 2$ we have $\check{H}^r(R, \mathbb{C}) = 0$. When $r = 0$

$$\Gamma(R, \mathbb{C}) = H^0(R, \mathbb{C}) = \{f \mid df = 0, \ f \ \mathrm{smooth}\} = \mathbb{C},$$

hence $\dim(H^0(R, \mathbb{C})) = 1$. Then according to Poincaré duality theorem, $\dim H^0(R, \mathbb{C}) = \dim H^2(R, \mathbb{C}) = 1$. On the other hand, we have Hodge decomposition:

$$H^1(R, \mathbb{C}) \simeq \bigoplus_{p+q=1} H^{(p,q)}(R),$$

and $H^{(p,q)}(R) = \overline{H^{(q,p)}(R)}$. Thus according to Kodaira-Serre duality theorem (Theorem 38), we have

$$H^1(R, \mathbb{C}) \simeq H^{(1,0)}(R) \oplus H^{(0,1)}(R) = H^{(1,0)}(R) \oplus \overline{H^{(1,0)}(R)} = 2g,$$

where g is the genus of the surface.

Definition 39. *Let* M *and* E *be complex manifolds, and* $\pi : M \to E$ *is holomorphic map. If there exists an open cover* $\mathcal{U}$ *of* E *such that for any* $U_\alpha \in \mathcal{U}$*, there is a diffeomorphism*

$$g_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$$

and for $U_\alpha \cap U_\beta$*, there are diffeomorphisms*

$$g_\beta \circ g_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$

Then we call $\pi : M \to E$ *is called an* r*-dimensional holomorphic vector bundle.*

For any such a vector bundle, the map $g_\beta \circ g_\alpha^{-1}$ has the form

$$g_\beta \circ g_\alpha^{-1}(e, v) = (e, g_{UV}(v)), \qquad e \in E,\ v \in \mathbb{C}^r$$

where $g_{UV} \in GL(r, \mathbb{C})$. Theses $\{g_{UV}\}$ are called transition functions of the vector bundle and satisfy:

$$g_{UU} = Id, \qquad g_{WU} \circ g_{VW} \circ g_{UV} = Id, \qquad U, V, W \in \mathcal{U}. \tag{6}$$

In particular, if $r = 1$, we call E a holomorphic line bundle. It is easy to see that a line bundle is equivalent to a collection of $c \in GL(1, \mathbb{C}) \simeq \mathbb{C}^\times$ satisfying the cocycle condition (6).

Now, given a compact Riemannian surface of genus g, and $p_i \in R$, $n_i \in \mathbb{Z}$ for $i = 1, \ldots, s$,

$$D = n_1 p_1 + \cdots + n_s p_s$$

is a divisor on R. We define $deg(D) = n_1 + \cdots + n_s$. For any divisor D, let $\mathcal{U}$ be an open cover of R such that on every $U_\alpha \in \mathcal{U}$ we can find a meromorphic function f_α having some $p_i's$ as all its zeros (multiplicity $n_i > 0$) and poles (multiplicity $n_i < 0$) if $p_i \in U_\alpha$. When $U_\alpha \cap U_\beta \neq \emptyset$, let f_α and f_β have same multiplicities on all zeros/poles on $U_\alpha \cap U_\beta$, and define

$$h_\beta^\alpha = \frac{f_\alpha}{f_\beta}$$

then h_β^α is an everywhere non-zero holomorphic function on $U_\alpha \cap U_\beta$. The $\{h_\beta^\alpha\}$ satisfy the cocycle condition (6) and define a line bundle, denoted by [D]. We call $s = \{f_\alpha\}$ a section of [D]. When f_α has only zeros on U_α, it is a holomorphic function, and s is called a **holomorphic section**, otherwise it's called a **meromorphic section**. On the other hand, suppose $\mathcal{L}$ is a holomorphic line bundle, $\mathcal{U}$ is an open cover, $\{h_\beta^\alpha\}$ is the transition function of the line bundle with respect to this cover. If there exists a meromorphic section $s' = \{f'_\alpha\}$, then f'_α is a meromorphic function on U_α and

$$f'_\alpha = h_\beta^\alpha f'_\beta.$$

Thus f'_α and f'_β have same multiplicities on zeros and poles on $U_\alpha \cap U_\beta$, and we could glue the $\{f'_\alpha\}$ to get a divisor on R.

5.2. **Riemann-Roch Theorem.** According to our discussion on divisors and line bundles, we can assign a line bundle [D] to a divisor D. Suppose D' is another divisor, [D] and [D′] are defined by local meromorphic functions $\{f_\alpha\}$ and $\{h_\alpha\}$ respectively. We say a map between two vector bundles over M

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle \pi_2} & \\ & M & \end{array}$$

is isomorphic if for any open set $U_\alpha \in M$ there exists a diffeomorphism $g_\alpha : \pi_1^{-1}(U_\alpha) \to \pi_2^{-1}(U_\alpha)$. Hence if $F : [D] \to [D']$ is an isomorphism of holomorphic line bundles, then

$$g_\alpha \frac{f_\alpha}{f_\beta} = \frac{h_\alpha}{h_\beta} g_\beta$$

thus on $U_\alpha \cap U_\beta$, we get

$$g_\alpha \frac{f_\alpha}{h_\alpha} = g_\beta \frac{f_\beta}{h_\beta}.$$

Using the above argument, we define a function on R as $f|_{U_\alpha} = g_\alpha \frac{f_\alpha}{h_\alpha}$, then f is meromorphic on R. And if we write $div(f)$ as the linear sum of the zeros and poles of f, we have $div(f) = D - D'$. En effet, $[D]$ and $[D']$ are isomorphic if and only if there exists a meromorphic function f such that $div(f) = D' - D$. Since any meromorphic function on a compact Riemannian surface has same number of zeros and poles, we get $deg(D') - Deg(D) = deg(div(f)) = 0$.

Theorem 40. *If $\mathcal{L}$ is a holomorphic line bundle over a compact Riemannian surface R, then there exists a divisor D such that $\mathcal{L} = [D]$.*

Proof. Take a point $\mathfrak{p} \in R$ and $n > 0$, and s is a section of the line bundle $[n\mathfrak{p}]$. We consider the line bundle $\mathcal{L} \otimes [n\mathfrak{p}]$, and consider the short exact sequence

$$0 \to \Theta(\mathcal{L}) \xrightarrow{\times s} \Theta(\mathcal{L} \otimes [n\mathfrak{p}]) \to SK(n\mathfrak{p}) \to 0$$

where $SK(n\mathfrak{p})$ is the *skyscraper sheaf.*

For $SK(n\mathfrak{p})$, if we take an open cover $\mathcal{U}$, such that $\mathfrak{p}$ is not in the intersection of any two open sets in $\mathcal{U}$, then by Čech cohomology, we have $\check{H}^1(\mathcal{U}, SK(n\mathfrak{p})) = 0$ hence

$$H^1(R, SK(n\mathfrak{p})) = 0.$$

By Theorem 26, we obtain the following long exact sequence from the above short exact sequence

$$0 \to H^0(R, \Theta(\mathcal{L})) \to H^0(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) \to H^0(R, SK(n\mathfrak{p}))$$
$$\to H^1(R, \Theta(\mathcal{L})) \to H^1(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) \to 0.$$

Since the alternating sum of dimensions is zero for a long exact sequence, we have

$$dimH^0(R, \Theta(\mathcal{L})) - dimH^0(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) + dimH^0(R, SK(n\mathfrak{p}))$$
$$-dimH^1(R, \Theta(\mathcal{L})) + dimH^1(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) = 0.$$

Since $dimH^0(R, SK(n\mathfrak{p})) = n$, we get

$$dimH^0(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) - dimH^1(R, \Theta(\mathcal{L}))$$
$$= dimH^0(R, \Theta(\mathcal{L})) - dimH^0(R, \Theta(\mathcal{L})) + n.$$

So, for big enough $n > 0$, we can guarantee $dimH^0(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}])) > 0$ and hence the line bundle $\mathcal{L} \otimes [n\mathfrak{p}]$ admits non-zero meromorphic section. Take such a section

$\tilde{S} \in \Gamma(R, \Theta(\mathcal{L} \otimes [n\mathfrak{p}]))$, it turns out $\tilde{s}/s$ is a non-zero meromorphic section of $\mathcal{L}$, then there exists [D] defined by zeros and poles of $\tilde{s}/s$. $\square$

For any holomorphic line bundle $\mathcal{L}$, we could choose a divisor D such that $\mathcal{L} = [D]$ and its degree is independent of the choice of D. Thus we can define $\deg(L) := \deg(D)$.

Theorem 41 (Riemann-Roch)**.** *Suppose* R *is a compact Riemannian surface with genus* g *and* $\mathcal{L}$ *is a holomorphic line bundle over* R*. Then*

$$\dim H^0(R, \Theta(\mathcal{L})) - \dim H^1(R, \Theta(\mathcal{L})) = \deg(L) - g + 1.$$

Proof. Suppose $\mathcal{L} = [D]$ with $D > 0$ and $s \in \Gamma(R, [D])$. Consider the short exact sequence

$$0 \to \Theta(R \times \mathbb{C}) \xrightarrow{\times s} \Theta([D]) \to SK(D) \to 0$$

where $SK(D)$ is defined as

$$SK(D) = \bigoplus_{i=1}^{q} SK(n_i \mathfrak{p}_i), \qquad D = n_1\mathfrak{p}_1 + \cdots + n_q\mathfrak{p}_q.$$

By Theorem 26, we obtain a long exact sequence

$$\begin{aligned} 0 \to H^0(R, \Theta(R \times \mathbb{C})) \to H^0(R, \Theta([D])) \to H^0(R, SK(D)) \\ \to H^1(R, \Theta(R \times \mathbb{C})) \to H^1(R, \Theta([D])) \to 0 \end{aligned}$$

from which we get

$$\begin{aligned} \dim H^0(R, \Theta(R \times \mathbb{C})) - \dim H^0(R, \Theta([D])) + \dim H^0(R, SK(D)) \\ - \dim H^1(R, \Theta(R \times \mathbb{C})) + \dim H^1(R, \Theta([D])) = 0. \end{aligned}$$

Since $\dim H^0(R, SK(D)) = n_1 + \cdots + n_q = \deg(D)$, we further have

$$\begin{aligned} &\dim H^0(R, \Theta(\mathcal{L})) - \dim H^1(R, \Theta(\mathcal{L})) \\ &= \dim H^0(R, \Theta(R \times \mathbb{C})) - \dim H^1(R, \Theta(R \times \mathbb{C})) + \dim H^0(R, SK(D)) \\ &= \dim H^0(R, \Theta(R \times \mathbb{C})) - \dim H^1(R, \Theta(R \times \mathbb{C})) + \deg(D). \end{aligned}$$

According to the acyclique resolution of holomorphic sections

$$0 \to \Theta(R \times \mathbb{C}) \hookrightarrow \mathcal{A}^{(0,0)}(R) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)}(R) \to 0$$

we have

$$\dim H^1(R, \Theta(R \times \mathbb{C})) = \dim H^0(R, \mathcal{A}^{(0,1)}(R)) = \frac{1}{2} \dim H^1(R, \mathbb{C}) = g.$$

Moreover, $\dim H^0(R, \Theta(R \times \mathbb{C})) = 1$, thus we obtain

$$\dim H^0(R, \Theta(\mathcal{L})) - \dim H^1(R, \Theta(\mathcal{L})) = \deg(D) - g + 1,$$

i.e, Riemann-Roch theorem holds for positive divisors.

To show Riemann-Roch for arbitrary divisor, suppose $\mathcal{L} = [D]$ with $D = D_1 - D2$ with D_1 and D_2 two positive divisors. Let s be the canonical section of $[D_2]$, the short exact sequence

$$0 \to \Theta([D_1 - D_2]) \xrightarrow{\times s} \Theta([D_1]) \to SK(D_2) \to 0$$

induces a long exact sequence, and the alternating sum of it gives

$$\begin{aligned} &\dim H^0(R, \Theta([D_1 - D_2])) - \dim H^1(R, \Theta([D_1 - D_2])) \\ &= \dim H^0(R, \Theta([D_1])) - \dim H^1(R, \Theta([D_1])) - \dim H^0(SK(D_2)) \\ &= deg(D_1) - g + 1 - deg(D_2) \\ &= deg(D) - g + 1. \end{aligned}$$

□

We may further define the following two linear spaces

$$l(D) := \{f \mid f \text{ meromorphic}, \ div(f) \geq -D\}$$

and

$$i(D) := \{u \mid u \text{ meromorphic differential form}, \ div(u) \geq D\}$$

It can be show that

$$l(D) \simeq H^{(0,0)}(R, [D]), \qquad i(D) \simeq H^{(0,1)}(R, [D])$$

as linear spaces, hence

Corollary 42. *Suppose* R *is a compact Riemannian surface with genus* g *and* D *is a divisor on* R. *Then*

$$\dim(l(D)) - \dim(i(D)) = deg(L) - g + 1.$$

6. Abel Theorem and Classification of Holomorphic Line Bundles

6.1. **Abel Theorem.** We still consider compact Riemann surface X, with genus= g. From algebraic topology, the fundamental group of X can be represented as

$$< a_1, b_1, \cdots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} >$$

Suppose dw is a holomorphic 1-form, let's define the **periodic vector of dw** as

$$V(dw) = \left(\int_{a_1} dw, \cdots, \int_{a_g} dw, \int_{b_1} dw, \cdots, \int_{b_g} dw \right).$$

For any closed path L in X, there exists $n_i, m_i \in \mathbb{Z}$ such that

$$\int_L dw = \sum_{i=1}^{g} \left(n_i \int_{a_i} dw + m_i \int_{b_i} dw \right),$$

since we have the homotopy relationship

$$L \sim \sum_{i=1}^{g} \left(n_i \int_{a_i} + m_i \int_{b_i} \right).$$

Furthermore, we define

$$L(dw) := \left\{ \sum_{i=1}^{g} \left(n_i \int_{a_i} dw + m_i \int_{b_i} dw \right) \mid n_i, m_i \in \mathbb{Z}, 1 \leq i \leq g \right\}.$$

Then for any fixed point p_0, the integration from p_0 to $p \in X$

$$\int_{p_0}^{p} dw (\mathrm{mod} L(dw))$$

doesn't depend on the choice of paths thus gives a well defined map

$$w(-): \qquad X \xrightarrow{\int_{p_0}^{-} dw} \mathbb{C}/L(dw)$$

Since our surface has genus= g, there exists $dw_1, \cdots, dw_g$ which form a basis of $H^{(1,0)}(X)$. Let us define the **periodic matrix** with respect to $a_1, \cdots, a_g, b_1, \cdots, b_g$

$$\Omega = \begin{pmatrix} V(dw_1) \\ V(dw_2) \\ \vdots \\ V(dw_g) \end{pmatrix}_{g \times 2g} \tag{7}$$

From algebraic topology, we may obtain X from an n-gon by gluing its boundaries in a proper way. Let's denote the n-gon as X^* and its boundary

$$\partial X^* = \{a_1^+, b_1^+, a_1^-, b_1^-, \cdots, a_g^+, b_g^+, a_g^-, b_g^-\}.$$

Then for any p on ∂X^*, integration on $\mathbb{C}/L(dw)$ satisfy:

$$w(p)|_{a_i^+} + w(p)|_{b_i^+} - w(p)|_{a_i^-} = 0, \qquad p \in a_i^+$$

and

$$w(\mathfrak{p})|_{b_i^+} - w(\mathfrak{p})|_{b_i^-} = -w(\mathfrak{p})|_{a_i^-} = w(\mathfrak{p})_{a_i^+}, \qquad \mathfrak{p} \in b_i^+.$$

Use these relationships, if dv is another holomorphic form on R we have:

$$\begin{aligned}\int_{\partial X^*} w dv &= \sum_{i=1}^{g} \left(\int_{a_i} w dv + \int_{b_i} w dv + \int_{a_i^{-1}} w dv + \int_{b_i^{-1}} w dv \right) \\ &= \sum_{i=1}^{g} \left(\int_{a_i} \left(w|_{a_i} - w|_{a_i^{-1}} \right) dv + \int_{b_i} \left(w|_{b_i} - w|_{b_i^{-1}} \right) dv \right) \\ &= \sum_{i=1}^{g} \left(-\int_{a_i} dv \cdot \int_{b_i} dw + \int_{b_i} dv \cdot \int_{a_i} dw \right)\end{aligned}$$

Moreover, since $d(wdv) = 0$, by Stokes formula, we get

$$\int_{\partial X^*} w dv = \int_{X^*} d(wdv) = 0,$$

hence

$$\sum_{i=1}^{g} \left(-\int_{a_i} dv \cdot \int_{b_i} dw + \int_{b_i} dv \cdot \int_{a_i} dw \right) = 0. \tag{8}$$

Similarly, for $dw \neq 0$, we have

$$\frac{i}{2} \int_{\partial X^*} w \overline{dw} = \frac{i}{2} \int_{X^*} dw \wedge \overline{dw} > 0$$

and hence

$$\frac{i}{2} \sum_{i=1}^{g} \left(-\int_{a_i} dw \cdot \int_{b_i} \overline{dw} + \int_{b_i} dw \cdot \int_{a_i} \overline{dw} \right) = 0. \tag{9}$$

If we substitute dw by dw, dv in periodic matrix (7) and denote

$$J = \begin{bmatrix} 0 & I_g \\ I_g & 0 \end{bmatrix}$$

We may write (8) and (9) as

$$\Omega J \Omega^T = 0, \qquad \frac{i}{2} \Omega J \overline{\Omega} > 0$$

where the second term means the matrix is positive definite Hermitian. These relations are called **Riemann Bilinear Relations**, which gives a necessary condition for a complex $g \times 2g$ matrix to be a periodic matrix of certain compact Riemann surface. If we further defnote

$$A = \begin{bmatrix} \int_{a_i^+} dw_1 & \cdots & \int_{a_g^+} dw_1 \\ \vdots & & \vdots \\ \int_{a_i^+} dw_g & \cdots & \int_{a_g^+} dw_g \end{bmatrix} \qquad B = \begin{bmatrix} \int_{b_i^+} dw_1 & \cdots & \int_{b_g^+} dw_1 \\ \vdots & & \vdots \\ \int_{b_i^+} dw_g & \cdots & \int_{b_g^+} dw_g \end{bmatrix}$$

Then $\Omega = [A\ B]$, and Riemann bilinear relations can be written as

$$AB^T - BA^T = 0, \qquad \frac{i}{2}\left(A\overline{B}^T - B\overline{A}^T\right) > 0.$$

Moreover, if we choose $d2_1, \cdots, dw_g$ so that $A = I_g$, we would have

$$\int_{a_i} dw_j = \delta_{ij}, \qquad \int_{b_i} dw_j = \int_{b_j} dw_i \tag{10}$$

for $1 \le i, j \le g$.

The following lemma gives another necessary condition that Ω satisfies.

Lemma 43. *If we denote*

$$\Omega = [V_1\ V_2\ \cdots\ V_{2g}]$$

Then $V_1, \cdots, V_{2g}$ *are real-independent.*

Proof. Suppose otherwise, then there exists $r_1, \cdots, r_{2g} \in \mathbb{R}$, such that for any $1 \le i \le g$, we have

$$\sum_{j=1}^{g}\left(r_j \int_{a_j} dw_i + r_{g+j}\int_{b_j} dw_i\right) = 0,$$

and

$$\sum_{j=1}^{g}\left(r_j \int_{a_j} \overline{dw_i} + r_{g+j}\int_{b_j} \overline{dw_i}\right) = 0.$$

Thus,

$$\mathrm{rank}\left(V(dw_1), \cdots, V(dw_g), V(\overline{dw_1}), \cdots, V(\overline{dw_g})\right)^T < 2g.$$

So, there exist $c_1, \cdots, c_{2g} \in \mathbb{C}$, such that

$$(c1, \cdots, c_{2g})\left(V(dw_1), \cdots, V(dw_g), V(\overline{dw_1}), \cdots, V(\overline{dw_g})\right)^T = 0.$$

So if we let

$$V = \sum_{i=1}^{g}\left(c_i dw_i + c_{g+i}\overline{dw_i}\right),$$

we have $\int_{a_j} V = 0$, $\int_{b_j} V = 0$ and $dV = 0$. If we fix arbitrary p_0 and the path integral $f(p) := \int_{p_0}^{p} V$ is well defined and $df = V$, i.e, V is exact.

On the other hand, according to Hodge decomposition

$$H^1(X, \mathbb{C}) = H^{(1,0)}(X) \oplus H^{(0,1)}(X)$$

hence $\{dw_1, \cdots, dw_g, \overline{dw_1}, \cdots, \overline{dw_g}\}$ is a basis of $H^1(X, \mathbb{C})$. So $\sum_{i=1}^{g}\left(c_i dw_i + c_{g+i}\overline{dw_i}\right)$ is not exact, and we have a contradiction.

□

Lemma 44. *Let* dw *be a holomorphic* 1*-form and* du *a meromorphic* 1*-form on* X*, and suppose for any* $1 \le i \le g$ *the poles of* du *do not lie on* $\{a_i, b_i\}$*. Then*

$$\int_{\partial X^*} w du = \sum_{i=1}^{g} \left(-\int_{a_i} du \cdot \int_{b_i} dw + \int_{b_i} du \cdot \int_{a_i} dw \right)$$
$$= 2\pi i \sum_{p \in X} \mathrm{Res}(w du, p).$$

Theorem 45. *Let*

$$D = p_1 + \cdots p_n - q_1 - \cdots - q_n$$

is a divisor on X*. Then* D *is defined by zeros and poles of a meromorphic function if and only if: there exists a closed curve* $L \subset R$ *such that for any holomorphic* 1*-form* dw*, we always have*

$$\sum_{i=1}^{n} \int_{q_i}^{p_i} dw = \int_{L} dw.$$

Here, $\int_{q_i}^{p_i} dw$ *is well defined integral which does not depend on the choice of paths.*

Proof. Without loss of generality, we suppose all p_i, q_i are not on $\{a_i, b_i\}$, and the set $\{p_1, \cdots, p_n\} \cap \{q_1, \cdots, q_n\} = \emptyset$.

We suppose there exists meromorphic function f such that $\mathrm{div}(f) = D$. Let $du = d(\ln(f))$, then du is meromorphic 1-form on X, and $\int_L du \in \mathbb{Z}$ for any closed curve L (its residues). We denote

$$n_i := \frac{1}{2\pi i} \int_{a_i} du \in \mathbb{Z}, \qquad m_i := \frac{1}{2\pi i} \int_{b_i} du \in \mathbb{Z}$$

Now, let $L = \sum_{i=1}^{g} (m_i a_i - n_i b_i)$ which is a closed curve in X. If dw is a holomorphic 1-form, then

$$\frac{1}{2\pi i} \int_{\partial X^*} w du = \sum_{i=1}^{g} \left(-\frac{1}{2\pi i} \int_{a_i} du \cdot \int_{b_i} dw + \frac{1}{2\pi i} \int_{b_i} du \cdot \int_{a_i} dw \right)$$
$$= \sum_{i=1}^{g} \left(-n_i \int_{b_i} dw + m_i \int_{a_i} dw \right) = \int_L dw.$$

And by residue calculation

$$\frac{1}{2\pi i} \int_{\partial X^*} w du = \sum_{j=1}^{n} \left(\mathrm{Res}(w d(\ln(f)), p_j) - \mathrm{Res}(w d(\ln(f)), q_j) \right)$$
$$= \sum_{j=1}^{g} (w(p_j) - w(q_j)) = \sum_{j=1}^{n} \int_{q_i}^{p_i} dw.$$

Thus the equality in Abel theorem holds.

To prove the other direction, we define divisors $D_j := -p_j - q_j$. According to Riemann-Roch,

$$\dim H^0(X, \theta([D_j])) - \dim H^1(X, \theta([D_j])) = -2 - g + 1.$$

Since $\deg(D_j) < 0$, we have

$$\dim H^0(X, \theta([D_j])) = 0, \qquad \dim H^1(X, \theta([D_j])) = i(D_j) = g + 1.$$

On the other hand, the space of all holomorphic 1-form is g dimensional, and $i(D) = g + 1$ implies that there exists meromorphic 1-form having exactly two poles p_j and q_j (counting multiplicity). We denote such a meromorphic form by $E(p_j, q_j)$, and may suppose that $\mathrm{Res}(E, p_j) = 1$ and $\mathrm{Res}(E, q_j) = -1$.

Now, let $du = \sum_{j=1}^{n} E(p_j, q_j)$ which is a meromorphic 1-form on X. According to (10), we may add $\int_{a_i} dw_j = \delta_{ij}$ to du so that $\int_{a_i} du = 0$ and $\mathrm{Res}(du, p_i)$ and $\mathrm{Res}(du, q_i)$ for $1 \le i, j \le n$ don't change.

For an arbitrary holomorphic 1-form dw, we also have

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial X^*} w du &= \sum_{j=1}^{n} (\mathrm{Res}(w du, p_j) - \mathrm{Res}(w du, q_j)) \\ &= \sum_{j=1}^{g} (w(p_j) - w(q_j)) = \sum_{j=1}^{n} \int_{q_i}^{p_i} dw. \end{aligned} \tag{11}$$

Take $dw = dw_s$ for $1 \le s \le n$, according to (10), the left hand side of (11) is

$$\frac{1}{2\pi i} \sum_{i=1}^{g} \left(-\int_{b_i} dw_s \int_{a_i} du + \int_{a_i} dw_s \int_{b_i} du \right) = \frac{1}{2\pi i} \int_{b_s} du.$$

Also, by the equality of the Abel theorem, if we let L be a closed curve homotopic to $\sum_{i=1}^{g} (m_i a_i - n_i b_i)$, then

$$\int_L dw_s = \sum_{i=1}^{g} \left(m_i \int_{a_i} dw_s - n_i \int_{b_i} dw_s \right).$$

Hence for $s = 1, \cdots, g$, from the equality in Abel theorem, we obtain

$$\frac{1}{2\pi i} \int_{b_s} du = \sum_{i=1}^{g} \left(m_i \int_{a_i} dw_s - n_i \int_{b_i} dw_s \right).$$

Thus, if we define

$$du_1 = \frac{1}{2\pi i} du + \sum_{i=1}^{g} n_i dw_i,$$

then for $s = 1, \cdots, g$,

$$\int_{a_i} du_1 = n_s.$$

Again, use (10), we deduce

$$\int_{b_s} du_1 = \frac{1}{2\pi i}\int_{b_s} du + \sum_{i=1}^{g} n_i \int_{b_s} dw_i$$

$$= \sum_{i=1}^{g}\left(m_i \int_{a_i} dw_s - n_i \int_{b_i} dw_s\right) + \sum_{s=1}^{g} n_i \int_{b_i} dw_s = m_s.$$

We thus constructed a meromorphic 1-form, and its integral along any closed curve is an integer. Hence on X the integral $\int_{p_0}^{p} du_1$ doesn't depend on the choice of path, and

$$f(p) = \exp\left(2\pi i \int_{p_0}^{p} du_1\right)$$

is a meromorphic function with $\{p_1, \cdots, p_n\}$ as zeros and $\{q_1, \cdots, q_n\}$ as poles. □

6.2. **Classification of Holomorphic Line Bundles on Compact Riemann Surfaces.** We adopt the notations from above, in particular,

$$\Omega = [V_1 \ V_2 \ \cdots \ V_{2g}].$$

We denote

$$L := \left\{n_1 V_1^T + \cdots + n_g V_g^T + n_{g+1} V_{g+1}^T + \cdots + n_{2g} V_{2g}^T \in \mathbb{C}^g \mid n_i \in \mathbb{Z}\right\}$$

and call it the lattice generated by $\{V_1^T, \cdots, V_g^T, V_{g+1}^T, \cdots, V_{2g}^T\}$. We view $\mathbb{C}^g$ as an Abelian group with respect to vector addition, then $L \subset \mathbb{C}^g$ is a subgroup. We define

$$J(X) := \mathbb{C}^g / L.$$

We have shown that vectors $\{V_1^T, \cdots, V_g^T, V_{g+1}^T, \cdots, V_{2g}^T\}$ are real independent, we may view $J(X)$ as a product of $2g$ unit circles, and hence a dimension g compact complex manifold.

The map

$$X \xrightarrow{\left(\int_{p_0}^{p} dw_1, \cdots, \int_{p_0}^{p} dw_g\right)} J(X)$$

doesn't depend on the choice of paths and is well defined. We define $D_0(X)$ the group of all order zero divisors, the above map induces a map $F : D_0(X) \to J(X)$ in the following way

For any $D = p_1 + \cdots + p_n - q_n - \cdots - q_n \in D_0(X)$,

$$F(D) = \left(\sum_{i=1}^{n}\int_{p_0}^{p_i} dw_1, \cdots, \sum_{i=1}^{n}\int_{p_0}^{p_i} dw_g\right) - \left(\sum_{i=1}^{n}\int_{p_0}^{q_i} dw_1, \cdots, \sum_{i=1}^{n}\int_{p_0}^{q_i} dw_g\right)$$
$$= \left(\sum_{i=1}^{n}\int_{q_i}^{p_i} dw_1, \cdots, \sum_{i=1}^{n}\int_{q_i}^{p_i} dw_g\right) \in J(X).$$

Abel theorem can be then expressed in the form:

Theorem 46. *A divisor* $D \in D_0(X)$ *is defined by a meromorphic function if and only if* $F(D) = 0$*. That is to say, the group homomorphism* $F : D_0(X) \to J(X)$ *satisfies*

$$\mathrm{Ker}(F) = \{D \in D_0(X) \mid D \text{ is defined by a meromorphic function}\}.$$

Let us denote $Z(X)$ the subgroup of $D_0(X)$ which consists of divisors that are defined by meromorphic functions. Hence Abel theorem 46 induces a group homomorphism

$$F : D_0(X)/Z(X) \to J(X).$$

Theorem 47. *The group homomorphism*

$$F : D_0(X)/Z(X) \to J(X)$$

is an isomorphism.

We have shown that all holomorphic line bundles are defined by divisors, and two holomorphic bundles are homeomorphism if and only if the divisors define them are equivalent, i.e, the difference of the two divisors is given by zeros and poles of some meromorphic function. Thus, the group $D_0(X)/Z(X)$ is isomorphic to all deg 0 holomorphic line bundles (group structure defined by tensor products of line bundles).

Moreover, for $d \in \mathbb{Z}$, let $J_d(X)$ be the set of all deg d holomorphic line bundles up to homeomorphisms. Then for arbitrary deg d holomorphic line bundles l_d, the map

$$J(X) \xrightarrow{\cdot l_d} J_d(X)$$

is an isomorphism. This gives us the classification result:

Theorem 48 (Classification of Holomorphic Line Bundles on Compact Riemann Surface)**.** *The Abelian group of all holomorphic line bundles on* X *via tensor product is isomorphic to* $J(X) \times \mathbb{Z}$*.*

Moreover, two holomorphic line bundles l_1, l_2 are diffeomorphism if and only if $\deg(l_1) = \deg(l_2)$. So actually $J(X)$ describes the set of non-analytical homeomorphism holomorphic line bundles, or furthermore, all the complex structures on holomorphic line bundles of X gives $J(X)$ which is a dimension g complex manifold. $J(R)$ is called the **moduli space** of holomorphic line bundles of X.