Algebraic Geometry

1. The study of Geometric spaces can be done by studying (nice) functions on them.
   - Riemann-Roch Theorem
   - Sheaves
   - cohomologies

2. Local-to-Global.
   - Manifolds
   -

Numbers	Geometry
$\mathbb{R} \supset \mathbb{Q} \subset \mathbb{Q}_2$	$\mathbb{C}((\frac{1}{T})) \supset \mathbb{C}(T) \subset \mathbb{C}((T-2))$
$\mathbb{Q} \subset \mathbb{Q}_3$	$\mathbb{C}(T) \subset \mathbb{C}((T-3))$
$\mathbb{Q} \subset \mathbb{Q}_5$	$\mathbb{C}(T) \subset \mathbb{C}((T-5))$

plan :

- Algebraic curves over $\mathbb{C}$
- Examples genus $= 0, 1, 2, \ldots$
- Study them using Riemann-Roch Theorem
- (sloppy) proof of RR

- Presheaves and sheaves
- Cohomology of sheaves
- Proof of RR
- Abel theorem.

Algebraic Curves

- Riemann - Roch Theorem :

$\boxed{\text{How many functions with given poles ?}}$

- $l(D) = \deg(D) + 1 - g + l(K - D)$

D : divisor $\rightsquigarrow \deg(D)$

K : canonical divisor

g : genus

- f a rational function, divisor
  (f) = zeros/poles of f counting multiplicities.

On compact Riemann surface, $\deg(f) = 0$.

- $l(D) = \dim($ space of functions f, $(f)+D \geq 0)$
  we say a divisor $D = \sum n_p \cdot p \geq 0$ if $n_p \geq 0$.

- Immediate observations about $l(D)$:

  - $l(0) = 1$ (holomorphic functions are constant!)

  - $l(D) = 0$ if $\deg(D) < 0$.

$\uparrow$

$l(D) = \dim($ space of functions having more zeros than poles $)$

- Immediate consequences from RR:

- $D \equiv 0 \Rightarrow 1 = 0 + 1 - g + l(K)$

$$\Rightarrow \boxed{l(K) = g}$$

- $D \equiv K \Rightarrow g = \deg(K) + 1 - g + 1$

$$\Rightarrow \boxed{\deg(K) = 2g - 2}$$

Some Examples

$$\boxed{\text{Genus} = 0}$$

– RR : $l(D) = \deg(D) + 1 - \cancel{g} + l(K - D)$

$$\boxed{l(K) = g = 0}$$

– Meromorphic 1-form : dz

$$z \longrightarrow y = \frac{1}{z} \rightsquigarrow dz = -\frac{1}{y^2}\, dy$$

ord $= 2$ pole at ∞.

$$\Rightarrow \quad l(D) = \deg(D) + 1 + l(-2\cdot\infty - D)$$

$$l(D) = \begin{cases} 0 & , \quad \deg(D) < 0 \\ \deg(D) + 1 & , \quad \deg(D) \geq 0 . \end{cases}$$

$\deg(D)$	-3	-2	-1	0	1	2	3
$l(D)$							
$l(K-D)$							
$l(D)-l(K-D)$							

- Take a point p s.t $l(p) = 2$

  (why such a point exists?)

$\rightsquigarrow$ function f s.t f has a single pole at p and no other poles.

$\Rightarrow$ f also has a single zero.

$$f: X \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$$

- f injective since $f - c$ has only one zero for $\forall c \in \mathbb{C}$.

$\Rightarrow f$ bijective and identifies X with Riemann sphere.

genus $= 0 \implies$ Unique Factorization Domain.

$\mathbb{R} = \mathbb{C}[t]$

$\implies \forall f \in R, \quad f = u \prod_p g_p^{n_p}, \quad n_p = \mathrm{ord}_p(f), \ u \in \mathbb{C}^\times$.

unique factorization.

genus	0	1	> 1
deg(K)	< 0	0	> 0
Curvature	> 0	= 0	< 0
Automorphisms	PGL_2	$\mathbb{C}/L$	few automorphisms
UFD	Yes	No	No

$\boxed{\text{Kodaira dimension ?}}$

$$\boxed{\text{Genus} = 1}$$

- $\ell(D) = \deg(D) \cancel{+1-g} + \ell(K-D)$

  $\deg(K) = 0 \quad , \quad \ell(K) = 1$

- $\deg(D) > 0 \implies \ell(D) = \deg(D)$.

- P point, $\ell(P) = \deg(P) = 1$

  $\implies$ all functions s.t. $(f) + P \geq 0$ are constants!

$\rightsquigarrow$ no functions with poles of order $= 1$ at P and no other poles.

Take $p \in X$, consider $l(np)$ with $n = 0, 1, 2, \ldots$

n	$l(np)$	functions
0	1	constants
1	1	nothing new
2	2	x; pole of order $= 2$
3	3	y; pole of order $= 3$
4	4	x^2
5	5	xy
6	6	y^2, x^3

7 functions in 6 dimensional space.

$\rightsquigarrow$ linear relationships

$$ay^2 + by + cxy = dx^3 + ex^2 + fx + g$$

$$a, b, c, d, e, f, g \in \mathbb{C}$$

$y \longrightarrow y + \text{const}$, eliminate by

$y \longrightarrow y + \text{const} \cdot x$, eliminate cxy

$x \longrightarrow x + \text{const}$, eliminate x^2

$y \longrightarrow y \cdot \text{const} \rightsquigarrow 1 \cdot y^2$

$x \longrightarrow x \cdot \text{const} \longrightarrow 4 \cdot x^3$

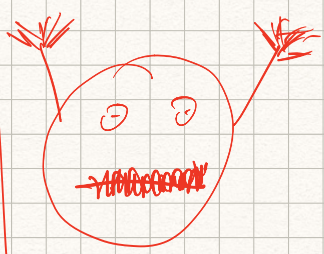
$\rightsquigarrow$ $y^2 = 4x^3 - g_2 x - g_3$, $g_2, g_3 \in \mathbb{C}$

or $zy^2 = 4x^3 - g_2 x z^2 - g_3 z^3$ curve in $\mathbb{CP}^2$.

Elliptic curve C:

$$y^2 = (x-a)(x-b)(x-c)(x-d)$$

one of the a, b, c, d might be ∞.



C: glue 2 copies of $\mathbb{P}^1$.

holomorphic 1-forms on C : $\boxed{\dfrac{dx}{y}}$

$\boxed{\text{singularity at } y=0 \text{ ?}}$

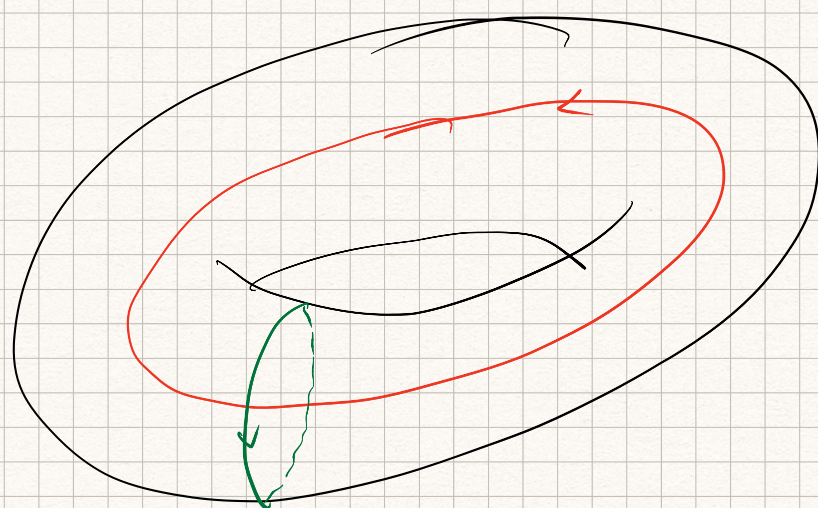
$y^2 = (x-a)(x-b)(x-c)$, $a \neq b$, $b \neq c$, $c \neq a$.

$2y\,dy = \Big((x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)\Big)\,dx$

$\Rightarrow \dfrac{dx}{y} = \dfrac{2y}{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}$

$\boxed{\text{non-zero at } y=0 \text{ !}}$

$\int_{p_0}^{p} \frac{dx}{y}$ ambiguity



$w_1 = \int_{red} \frac{dx}{y}$, $w_2 = \int_{green} \frac{dx}{y}$

Ambiguities $= \{ n_1 w_1 + n_2 w_2 \} =: L = \langle w_1, w_2 \rangle$

Genus $= 1 \iff \mathbb{C}/L$.

Genus = 2

$$l(D) = \deg(D) + \underbrace{\boxed{1-g}}_{=-1} + l(K-D)$$

$\deg(K) = 2g-2 = 2$, $\quad l(K) = g = 2$

$\deg(D)$	$l(D)$
< 0	0
$= 0$	0 or $\underbrace{1 \text{ (when } D \equiv 0)}_{\text{constants}}$
> 2	$\deg(D) - 1$
$= 2$	1 or 2 (when $D \equiv K$)
$= 1$	0 or 1 Can't be 2, otherwise can define a map to $\mathbb{CP}^1$ which is generically 1-1 $\Rightarrow$ genus $= 0$!

Since $l(k)=2$, it induces a map:

$$X \longrightarrow \mathbb{CP}^1$$

Again a double cover!

How to realize it?

Take point p s.t. $2p = K$.

n	$l(np)$	functions
0	1	constants
1	1	nothing new
2	2	x, pole of order $=2$
3	2	nothing new
4	3	x^2
5	4	y, pole of order $=5$
6	5	x^3
7	6	xy
8	7	x^4
9	8	x^2y
10	9	y^2, x^5

$$\rightsquigarrow \quad y^2 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_6), \quad \alpha_i \in \mathbb{C}$$

$$y^2z^3 = (x-\alpha_1 z)(x-\alpha_2 z)\cdots(x-\alpha_6 z) \in \mathbb{CP}^2$$

NOT a non-singular curve. $([0:1:0])$

1-forms on $y^2 = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_6)$

$$\boxed{\frac{dx}{y}\,, \quad \frac{x}{y}\,dx}$$

- $2y\,dy = \Big[(x-\alpha_2)\cdots(x-\alpha_6) + (x-\alpha_1)(x-\alpha_3)\cdots(x-\alpha_6)$

$\quad + \cdots\cdots$

$\quad + (x-\alpha_1)\cdots\cdots(x-\alpha_5) \Big]\, dx$

$$\Rightarrow \quad \frac{dx}{y} = \frac{1}{[-]}\,\frac{2y\,dy}{y} = \frac{2\,dy}{[-]}$$

regular at $x = \alpha_1, \ldots, \alpha_6$.

- at $z = \frac{1}{x}$, set $w = \frac{y}{x^3}$,

$$w^2 = (1-\alpha_1 z)\cdots\cdots(1-\alpha_6 z)$$

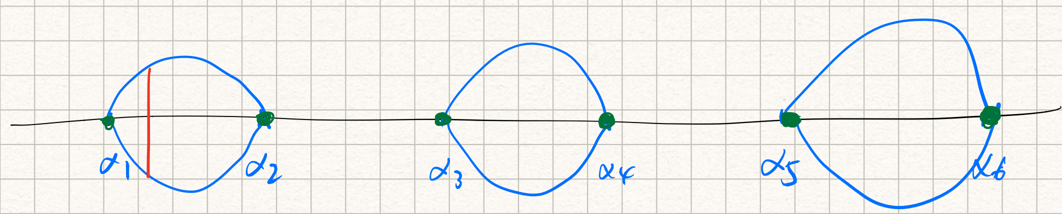
$$\Rightarrow \quad x^n \frac{dx}{y} = - z^{1-n}\,\frac{dz}{w}$$

$$\boxed{\text{holomorphic} \iff n \le 1}$$

1-form : $\dfrac{a+bx}{y}\,dx$

- 2 zeros at $(x, \pm y)$

- double zero at $(\alpha_i, 0)$



Those α_i's are pts s.t $2P \equiv K$.

Ways to represent a genus $=2$ curve:

1. Double cover of $\mathbb{CP}^1$.

2. $\mathbb{H}/\Gamma$ with $\Gamma < PSL_2(\mathbb{R})$.

3. Plane curve in $\mathbb{CP}^2$ with one double point.

4. Imbedding $X \hookrightarrow \mathbb{C}^2/(L \cong \mathbb{Z}^4)$

$$z \longmapsto \left(\int_{p_0}^{p} \frac{dx}{y} , \int_{p_0}^{p} \frac{x}{y} dx \right)$$

	Algebra	Analysis
Objects	projective curves / $\mathbb{C}$	Compact Riemann Surface
Genus	dimension of space of holomorphic 1-forms	# of handles
Functions	rational functions	meromorphic functions
Genus = 0	projective lines / $\mathbb{C}$	Riemann sphere
Genus = 1	Elliptic curves	$\mathbb{C}/L$ $L = n_1 w_1 + n_2 w_2 \ , \ n_1, n_2 \in \mathbb{Z}$
Higher genus	$xy^3 + yz^3 + zx^3 = 0 \in \mathbb{CP}^2$ genus = 3	$H = \dfrac{\{\tau \mid \mathrm{Im}\,\tau > 0\}}{\text{disc grp of } PSL_2(\mathbb{R})}$

Proof of RR:

1. Riemann: $l(D) = \deg(D) + 1 - i(0) + i(D)$

$\boxed{i(D) = ?}$

2. Genus:
   - Topology: # of handles
   - Geometry (P_g): dim of space of holomorphic 1-forms. $(= l(K))$
   - Arithmetic (P_a): $i(0)$ or $\frac{(d-1)(d-2)}{2}$

3. Roch: $i(D) = l(K-D)$

in particular, $D \equiv 0 \Rightarrow i(0) = l(K)$

$$\Updownarrow$$

$P_g = P_a$.

$\boxed{i(D)}$

- given $\{p_1, \dots, p_k\} \subset C$ and $\{n_{p_1}, \dots, n_{p_k}\}$,

  can we find meromorphic functions with poles at $\{p_i\}$ and order at most n_{p_i} ?

- $i(D)$ is the space of obstruction to solve this problem (Mittag-Leffler).

Define:

Laurent Series

$$R = \{ \prod_p f_p \in \prod_p R_p \mid \text{almost all } f_p \text{ are holomorphic} \}$$

$$R(D) = \{ \prod_p f_p \in \prod_p R_p \mid f_p \text{ has poles of ord} \le n_p \text{ at } p \}.$$

$K(C)$ = space of meromorphic functions on C.

$$\rightsquigarrow \boxed{i(D) = \frac{R}{R(D) + K(C)}}$$

e.g.: $D = 2p$, we care about (at p):

$$c_{-3}(z-p)^{-3} + c_{-4}(z-p)^{-4} + \cdots + c_{-n}(z-p)^{-n}$$

$\rightsquigarrow$ $i(D)$: space of functions that are locally of the above form.

$\boxed{\text{Riemann}}$

- theorem is true for $D=0$.

- $l(D) = \deg(D) + 1 - i(0) + i(D)$

$$l(D+p) = \deg(D+p) + 1 - i(0) + i(D+p)$$

. increase by 0 or 1

decrease by 0 or 1

- 1 if $\exists f$ s.t.
  $(f)+D+p \geq 0$ but $(f)+D \not\geq 0$
  f has pole of ord $= n_p + 1$ at p

In fact, $l(D+p) \leq l(D) + 1$

Consider R as defined before.

$\Rightarrow$ suffice to show $\dim(i(0)) < +\infty$.

- Consider C is defined by homogeneous function $f(x,y,z)$ in $\mathbb{CP}^2$, $\underbrace{\text{all poles at } \infty}$. ($\deg f = d$)

- space of all possible poles at ∞ of ord $\leq N$.

- space of $\underbrace{\text{polynomials in } x, y \text{ with } \deg \leq N}_{\text{poles at } \infty \text{ of ord} \leq N \text{ and holomorphic elsewhere}}$.

- space of polynomials on C with $\deg \leq N$.

- space of poles of functions on C of polynomials with $\deg \leq N$.

- compare with dN poles at ∞ (Obstruction to the above)

$$i(D) = \frac{3}{2}d - \frac{d^2}{2} - 1 = \frac{(d-1)(d-2)}{2} = g_a < \infty.$$

$\boxed{\text{Roch}}$ bilinear pairing :

$$\left\{ \text{meromorphic 1-forms } \omega \,\middle|\, (\omega) \geq D \right\} \times \frac{R}{R(D) + k(C)} \longrightarrow \mathbb{C}$$

$$(\omega, f) \longmapsto \sum_p \mathrm{Res}(f\omega, p)$$

- if f is meromorphic, then $\sum_p \mathrm{Res}(f\omega, p) = 0$

  But $\nexists$ such functions in $i(D)$, so the above pairing is injective :

  $\sum \mathrm{Res}(f\omega, p) = \sum \mathrm{Res}(f\omega', p)$ iff $\omega = \omega'$.

- Pairing induces

$$\underbrace{\left\{ 1\text{-forms } \omega \,\middle|\, (\omega) \geq D \right\}}_{\ell(K-D)} \xrightarrow{P} \underbrace{\mathrm{Dual}\left(\frac{R}{R(D) + k(C)} \right)}_{i(D)}$$

$\boxed{P \text{ is injective !}}$

$$\begin{cases} l(D) = \deg(D) + 1 - g + i(D) \\ l(K-D) \le i(D), \quad l(D) \le i(K-D) \\ \deg(K) = 2g-2 \end{cases}$$

$$\begin{aligned} \cdot\ l(D) &= \deg(D) + 1 - g + i(D) \\ &\geq \deg(D) + 1 - g + l(K-D) \\ &= \deg(D) + 1 - g + \deg(K-D) + 1 - g + i(K-D) \\ &\geq \underbrace{\deg(K) + 2 - 2g}_{=\,0} + l(D) \end{aligned}$$

<u>Remark</u>: $i(D) \cong \{\, 1\text{-forms } \omega \mid (\omega) \geq D \,\}$