Existence of the Solution to Discrete Surface Ricci Flow

David Gu

Yau Mathematics Science Center
Tsinghua University
Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

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Discrete Surface Curvature Flow Theorem

Vertex Scaling

Definition (Vertex Scaling)

Two triangulated PL surface $(S, V, \mathcal{T}, d)$ and $(S, V, \mathcal{T}, d')$ are said to differ by a vertex scaling, if $\exists\lambda : V(\mathcal{T}) \to \mathbb{R}_{>0}$, such that $d' = \lambda * d$ on $E(\mathcal{T})$, where

$$\lambda * d(u, v) = \lambda(u)\lambda(v)d(u, v).$$

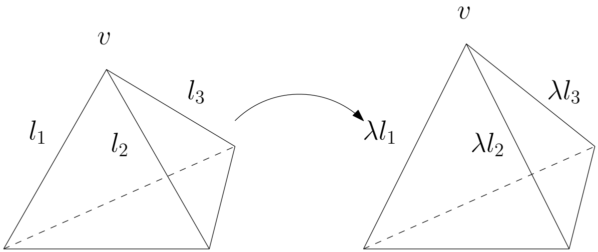


Figure: vertex scaling.

Discrete Conformal Equivalence

Definition (Gu-Luo-Sun-Wu)

Two PL metrics d, d' on a closed marked surface (S, V) are *discrete conformal*, if they are related by a sequence of two types of moves: vertex scaling and edge flip preserving Delaunay property.

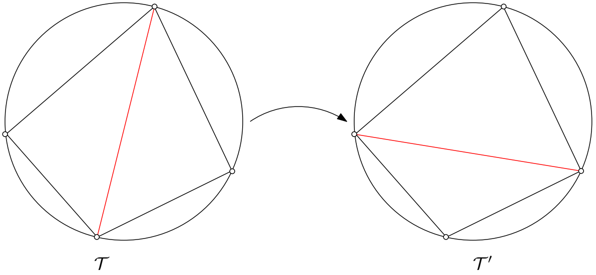


Figure: Edge flip, both triangulations are Delaunay.

Discrete Conformal Equivalence

Given a PL metric d on (S, V), produce a Delaunay triangulation $\mathcal{T}$ of (S, V),

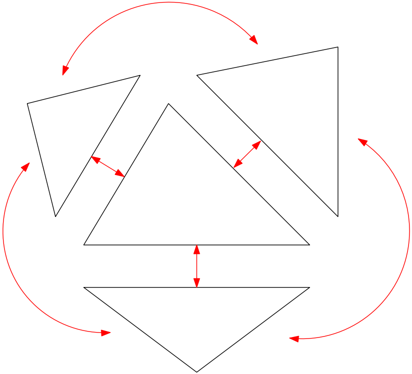
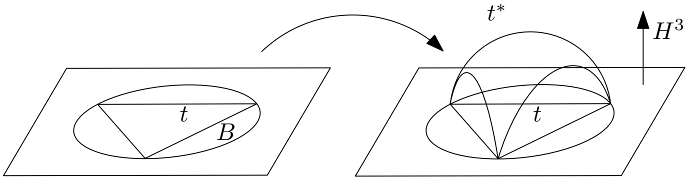


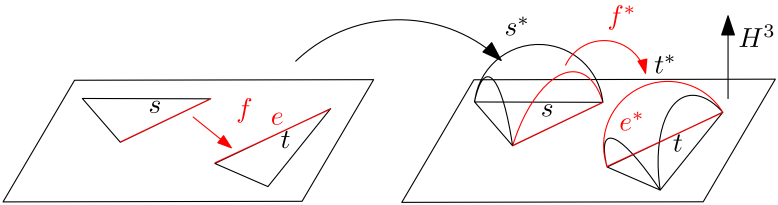
Figure: (S, V) with PL metric d, the triangulation is Delaunay.

Discrete Conformal Equivalence

Each face $t \in \mathcal{T}$ is associated an ideal hyperbolic triangle:

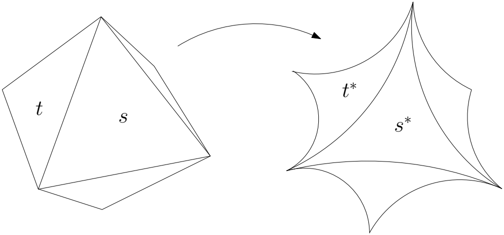


If $t, s \in \mathcal{T}$ glued by isometry f along e, then t^* and s^* are glued by the same f^* alonge e^*,



Discrete Conformal Equivalence

This induces a hyperbolic metric d^* on $S - V$.



Motivated by the important work of Bobenko-Pinkall-Springborn, equivalent to the previous defintion using vertex scaling and Delaunay condition.

Definition (Gu-Luo-Sun-Wu, JDG 2018)

Two PL metrics d_1 and d_2 on (S, V) are *discrete conformal* iff d_1^* and d_2^* are isometric by an isometry homotopic to identity on $S - V$.

Existence of the metric

Theorem (Gu-Luo-Sun-Wu)

Given a PL metric d on a closed marked surface (S, V)*, and curvature* $K^* : V \to (-\infty, 2\pi)$*, such that K satisfies the Gauss-Bonnet condition* $\sum K(v) = 2\pi\chi(S)$*, there there is a* d^* *discrete conformal to d, and* d^* *realizes the curvature* K^**.*

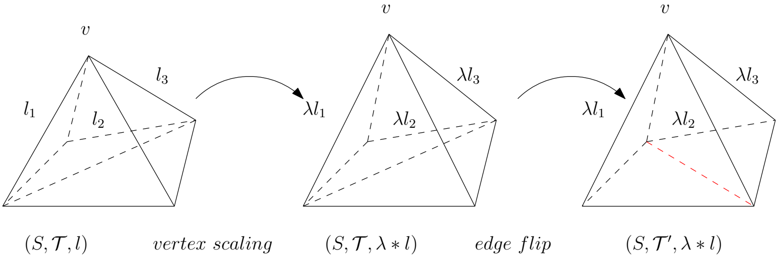


Figure: Discrete surface Yamabe flow.

Discrete Conformal Equivalence

Convex Optimization

Using Newton's method to minimize the following energy

$$\min_{\lambda} \int^{(\lambda_1,\lambda_2,\ldots,\lambda_n)} \sum_{v} (K^*(v) - K(v)) d\log\lambda(v),$$

such that $\Pi_v \lambda(v) = 1$. During the optimization, keep the triangulation always to be Delaunay.

Proof of the Discrete Surface Curvature Flow Theorem

Marked Surface

Definition (Marked Surface)

Let S be a closed topological surface, $V = \{v_1, v_2, \cdots, v_n\} \subset S$ is the set of distinct points, satsifying negative Euler number condition $\chi(S - V) < 0$. We call (S, V) a marked surface.

We consder the polyhedral metric **d** on the marked surface (S, V), with cone singularities at vertices.

Discrete Conformal Equivalence

Definition (Discrete Conformal Equivalence)

Two polyhedral metrics $\mathbf{d}$ and $\mathbf{d}'$ on a marked surface (S, V) are discrete conformal equivalent, if there is a series polyhedral metrics on (S, V),

$$\mathbf{d} = \mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m = \mathbf{d}'$$

and a series of triangulations $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m$, such that

1. every triangulation $\mathcal{T}_k$ is Delaunay on the metric $\mathbf{d}_k$;
2. if $\mathcal{T}_i = \mathcal{T}_{i+1}$, then there is a conformal factor $\mathbf{u} : V \to \mathbb{R}$, such that $\mathbf{d}_{i+1} = \mathbf{u} * \mathbf{d}_i$, namely the two polyhedral metrics differ by a vertex scaling operation;
3. if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometric transformation $h : (S, V, \mathbf{d}_i) \to (S, V, \mathbf{d}_{i+1})$, this transformation is homotopic to the identity map of (S, V), preserving the vertices.

Main Theorem

Existence and Uniqueness of the Solution to the Discrete Surface Ricci Flow:

Theorem (Gu-Luo-Sun)

Suppose $(S, V, \mathbf{d})$ *is a closed polyhedral surface, the for any* $K^* : V \to (-\infty, 2\pi)$, *satisfying the Gauss-Bonnet condition* $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, *there exists a polyhedral metric* $\mathbf{d}^*$

1. $\mathbf{d}^*$ *is discrete conformal equivalent to the metric* $\mathbf{d}$*;*
2. $\mathbf{d}^*$ *induces the discrete Gaussian curvature* K^**.*

All such kind of polyhedral metrics differ by a global scaling. Furthermore, $\mathbf{d}^*$ *can be obtained by discrete surface Ricci flow.*

Uniformization

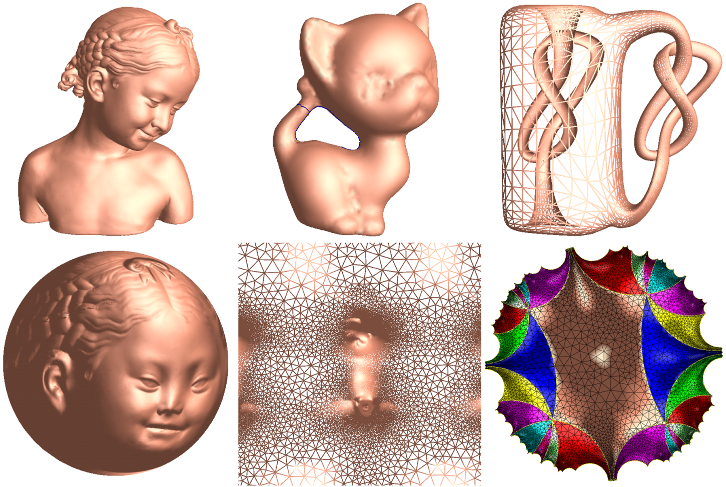


Figure: Closed surface uniformization.

Discrete Uniformization Theorem

Corollary (Gu-Luo-Sun)

Suppose $(S, V, \mathbf{d})$ *is a closed polyhedral surface, then there exists a polyhedral metric* $\mathbf{d}^*$, $\mathbf{d}^*$ *and the metric* $\mathbf{d}$ *are discrete conformal equivalent,* $\mathbf{d}^*$ *induces constant discrete Gaussian curvature* $2\pi\chi(S)/|V|$. *Such kind of polyhedral metrics differ by a global scaling.*

Teichmüller Space of Polyhedral Metrics

Definition (Equivalent Polyhedral Metrics)

Two polyhedral metrics $\mathbf{d}$ and $\mathbf{d}'$ on a marked surface (S, V) are equivalent, if there is an isometric tranformation $h : (S, V, \mathbf{d}) \to (S, V, \mathbf{d}')$, and h is homotopic to the identity map of (S, V), namely h preserves V.

Definition (Teichmüller Space of Polyhedral Metrics)

All the equivalence classes of polyhedral metrics on a marked surface (S, V) form the Teichmüller Space of polyhedral metrics.

$$T_{pl}(S, V) = \{\mathbf{d} | \text{polyhedral metrics on } (S, V)\} / \{\text{isometries} \sim \text{identity}\,(S, V)\}$$

Atlas of the Teichmüller Space of PL Metrics

Theorem (Troyanov)

Suppose (S, V) is a closed marked surface, the Teichmüller space of polyhedral metrics $T_{pl}(S, V)$ is homeomorphic to the Euclidean space $\mathbb{R}^{-3\chi(S-V)}$.

Definition (Local Chart of the Teichmüller Space of PL Metrics)

Suppose $\mathcal{T}$ is a triangulation of (S, V), its edge length function defines a polyhedral metric,

$$\Phi_{\mathcal{T}} : \mathbb{R}_{\triangle}^{E(\mathcal{T})} \to T_{pl}(S, V) \tag{1}$$

this gives a local chart of the Teichmüller space. Where the domain

$$\mathbb{R}_{\triangle}^{E(\mathcal{T})} = \left\{ x \in \mathbb{R}_{>0}^{E(\mathcal{T})} \middle| \text{for any } e_i, e_j, e_k \text{form a triangle }, x(e_i) + x(e_j) > x(e_k) \right\} \tag{2}$$

is a convex set, and is injective. We use $\mathcal{P}_{\mathcal{T}}$ to represent the image of $\Phi_{\mathcal{T}}$. Then $(\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})$ is a local chart of $T_{pl}(S, V)$.

Atlas of the Teichmüller Space of PL Metrics

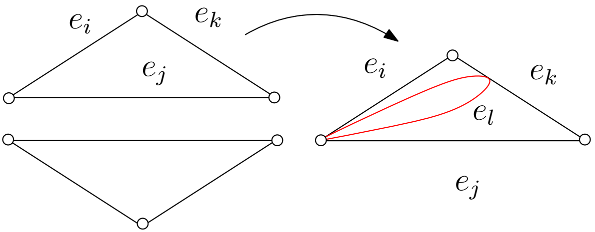


Figure: topological, not geometric triangulation.

If we edge swap e_k to e_l ot obtain the new triangulation $\mathcal{T}'$. Then under the metric $\mathbf{d}$, the topological triangle $\{e_j, e_l, e_j\}$ doesn't satisfy the triangle inequality. This shows the topological triangulation $\mathcal{T}'$ is not geometric.

$$\mathcal{P}(\mathcal{T}) \neq T_{pl}(S, V)$$

One chart can't cover the whole Teichmüller space $T_{pl}(S, V)$.

Definition (Atlas of Teichmüller Space of PL Metrics)

Suppose (S, V) is a closed marked surface, the atlas of $T_{pl}(S, V)$ consists of local coordinate charts $(\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1})$, where $\mathcal{T}$ exhausts all possible triangulation.

$$\mathcal{A}(T_{pl}(S, V)) = \bigcup_{\mathcal{T}} (\mathcal{P}_{\mathcal{T}}, \Phi_{\mathcal{T}}^{-1}). \tag{3}$$

Lemma (Real Analytic Manifold)

Suppose (S, V) is a closed marked surface, then the Teichmüller space of polyhedral metrics $T_{pl}(S, V)$ is a real analytic manifold.

Teichmüller Space of Decorated Hyperbolic Metrics

Definition (Equivalent decorated hyperbolic metrics)

Two decorated hyperbolic metrics $(\mathbf{h}, \mathbf{w})$ and $(\mathbf{h}', \mathbf{w}')$ on a closed marked surface (S, V) are equivalent, if there is an isometric transformation

$$h : (S, V, \mathbf{h}, \mathbf{w}) \rightarrow (S, V, \mathbf{d}', \mathbf{w}'),$$

which is homotopic to the identity map of (S, V), and preserves the horospheres.

Definition (Teichmüller Space of Decorated Hyperbolic Metrics)

Given a closed marked surface (S, V), $\chi(S - V) < 0$, then all the decorated hyperbolic metric on it form the Teichmüller space:

$$T_D(S, V) = \frac{\{(\mathbf{h}, \mathbf{w}) | (S, V) \text{decorated hyperbolic metrics}\}}{\{\text{isometries} \sim \text{identity of } (S, V) \text{preserving horospheres}\}} \quad (4)$$

Teichmüller Space of Decorated Hyperbolic Metrics

Definition (Local Chart of the Teichmüller Space)

Suppose $\mathcal{T}$ is a triangulation of (S, V), the hyperbolic edge length function determines a decorated hyperbolic metric,

$$\Psi_{\mathcal{T}} : \mathbb{R}^{E(\mathcal{T})} \to T_D(S, V) \tag{5}$$

which gives a local coordinate of the Teichmüller space. Let $\mathcal{Q}_{\mathcal{T}}$ be the image of $\Psi_{\mathcal{T}}$, then $(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1})$ form a local chart of $T_D(S, V)$.

Definition (Atlas of the Teichmüller Space)

Every triangulation of the marked closed surface (S, V) corresponds to a local chart $(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1})$. By exhausting all the possible triangulations, the union of all the local charts forms the atlas:

$$\mathcal{A}(T_D(S, V)) = \bigcup_{\mathcal{T}} \left(\mathcal{Q}_{\mathcal{T}}, \Psi_{\mathcal{T}}^{-1}\right).$$

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Equivalent Complete Hyperbolic Metrics)

Two complete hyperbolic metrics $\mathbf{h}$ and $\mathbf{h}'$ with finite area on a marked surface $(S - V)$ are equivalent, if there is an isometric transformation

$$h : (S - V, \mathbf{h}) \to (S - V, \mathbf{h}'),$$

furthermore h is homotopic to the identity automorphism of $S - V$.

Definition (Teichmüller Space of Complete Hyperbolic Metrics)

All the complete hyperbolic metrics with finite area on a marked surface $S - V$, $\chi(S - V) < 0$, form the Teichmüller space,

$$T_H(S - V) = \frac{\{\mathbf{h} | \text{complete hyperbolic metrics with finite area on } (S - V)\}}{\{\text{isometries} \sim \text{identity of } (S - V)\}} \tag{6}$$

Teichmüller Space of Complete Hyperbolic Metrics

Lemma (Local Coordinates)

Suppose $\mathbf{h}$ *is a complete hyperbolic metric on* $S - V$ *with finite area, the shear coordinate function is* $s : E(\mathcal{T}) \to \mathbb{R}$*, then for any* $v \in V$*, we have the relation*

$$\sum_{e \sim v} s(e) = 0. \tag{7}$$

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Local Chart of the Teichmüller Space)

Let $\mathcal{T}$ be a triangulation of (S, V), its shear coordinates uniquely determines a complete hyperbolic metric with finite area,

$$\Theta_{\mathcal{T}} : \Omega_{\mathcal{T}} \to T_H(S - V) \tag{8}$$

this gives local coordinates of the Teichmüller space, where

$$\Omega_{\mathcal{T}} = \left\{ x \in \mathbb{R}^{E(\mathcal{T})} \middle| \sum_{e \sim v} x(e) = 0,\ \forall v \in V(\mathcal{T}) \right\}.$$

Then $(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1})$ form a local chart of $T_H(S - V)$.

Teichmüller Space of Complete Hyperbolic Metrics

Definition (Atlas of the Teichmüller Space)

Let $\mathcal{T}$ be an arbitrary triangulation of (S,V), then $\mathcal{T}$ corresponds to a local chart $(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1})$. By exhausting all possible triangualtions of (S,V), all the local charts form an atlas of the Teichmüller space $T_H(S-V)$,

$$\mathcal{A}(T_H(S-V)) = \bigcup_{\mathcal{T}} \left(\Omega_{\mathcal{T}}, \Theta_{\mathcal{T}}^{-1}\right).$$

Teichmüller Space of Complete Hyperbolic Metrics

Lemma

Given a closed marked surface (S, V), $\chi(S - V) < 0$

$$T_D(S, V) = T_H(S - V) \times \mathbb{R}_{>0}^{|V|}. \tag{9}$$

Proof.

Any decorated hyperbolic metric on $(S, V, \mathcal{T})$ can be represented as $(\mathbf{h}, \mathbf{w})$, where $\mathbf{h}$ is a complete hyperbolic metric on $S - V$ with finite area, $\mathbf{h} \in T_H(S - V)$; $\mathbf{w}$ is the lengths of intersections between the horospheres and the surface. □

Diffeomorphisms Among Teichmüller Spaces

The Teichmüller space of all PL metrics has a cell decomposition, each cell

$$D_{pl}(\mathcal{T}) = \{[\mathbf{d}] \in T_{pl}(S, V) | \mathcal{T} \text{ is Delaunay under } \mathbf{d}\}$$

We show $D_{pl}(\mathcal{T})$ is simply connected. We change the edge length $x(e)$ to Rivin coordinates $y(e)$, $y(e) = \alpha + \alpha'$. Then the edge lengths of $(S, V, \mathcal{T}, \mathbf{d})$ are determined by the Rivin's coordinates unique to a scaling,

$$D_{pl}(\mathcal{T}) = \{y(e) \in (0, \pi) | e \in E(\mathcal{T})\} \times \mathbb{R}_{>0}$$

is a convex set. D_{pl} is simply connected.

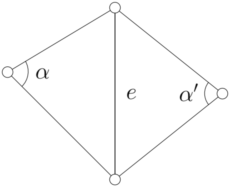


Figure: Rivin coordinates.

Diffeomorphisms Among Teichmüller Spaces

Cell Decomposition of $T_{pl}(S, V)$

The Teichmüller of the PL metrics has the cell decomposition:

$$T_{pl}(S, V) = \bigcup_{\mathcal{T}} D_{pl}(\mathcal{T}).$$

Cell Decomposition of $T_D(S, V)$

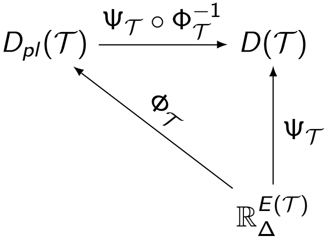
The Teichmüller space of the decorated hyperbolic metrics has the cell decomposition:

$$T_D(S, V) = \bigcup_{\mathcal{T}} D(\mathcal{T}).$$

where the cell

$$D(\mathcal{T}) = \{(\mathbf{d}, \mathbf{w}) \in T_D(S, V) | \mathcal{T} \text{ is Delaunay under } (\mathbf{d}, \mathbf{w})\}.$$

Diffeomorphisms Among Teichmüller Spaces



We use Penner's λ-length to establish the diffeomorphism between two cells,

$$A_{\mathcal{T}} = \Psi_{\mathcal{T}} \circ \Phi_{\mathcal{T}}^{-1} : D_{pl}(\mathcal{T}) \to D(\mathcal{T}), \quad x(e) \mapsto 2\ln x(e)$$

Penner's λ-length maps Euclidean Delaunay triangulation to decorated hyperbolic Delaunay triangulation. Furthermore Delaunay property implies triangle inequality, hence $A_{\mathcal{T}}$ is a diffeomorphism.

Diffeomorphisms Among Teichmüller Spaces

Suppose triangulations $\mathcal{T}$ and $\mathcal{T}'$ differ by an edge swap, consider a polyhedral metric $[d] \in D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')$, then under d, there are four co-circle vertices in (T) and $(T)'$. By Ptolemy equality, we obtain for any $x \in \Phi_{\mathcal{T}}^{-1}(D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}'))$,

$$\Phi_{\mathcal{T}}^{-1} \circ \Phi_{\mathcal{T}'}(x) = \Psi_{\mathcal{T}}^{-1} \circ \Psi_{\mathcal{T}'}(x)$$

this is equivalent to

$$A_{\mathcal{T}}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')} = A_{\mathcal{T}'}|_{D_{pl}(\mathcal{T}) \cap D_{pl}(\mathcal{T}')}$$

In this way, we glue the piecewise diffeomorphisms $A_{\mathcal{T}}$ to form a global diffeomorphism:

$$A : T_{pl}(S, V) \to T_D(S, V), \quad A|_{D_{pl}(\mathcal{T})} = A_{\mathcal{T}}|_{D_{pl}(\mathcal{T})}$$

Further proof shows this mapping is globally C^1 diffeomorphic.

Existence Proof

First, we construct a map: $F : \Omega_u \to \Omega_K$,

$$\Omega_u \xrightarrow{\exp} \{p\} \times \mathbb{R}^{|V|}_{>0} \to T_D(S, V) \xrightarrow{A^{-1}} T_{pl}(S, V) \xrightarrow{K} \Omega_K \tag{10}$$

where the domain Ω_u is the intersection between the discrete conformal factor space and the Euclidean hyperplane

$$\Omega_u = \mathbb{R}^n \cap \left\{ \mathbf{u} \middle| \sum_{i=1}^{n} u_i = 0 \right\} \tag{11}$$

the range Ω_K is the discrete curvature space,

$$\Omega_K = \left\{ \mathbf{K} \in (-\infty, 2\pi)^n \middle| \sum_{i=1}^{n} K_i = 2\pi\chi(S) \right\} \tag{12}$$

both of them are open sets in the Euclidean space $\mathbb{R}^{n-1}$. Because $A : T_{pl}(S, V) \to T_D(S, V)$ is C^1, $K : T_{pl}(S, V) \to \mathbb{R}^n$ is real analytic, hence F is C^1.

Existence Proof

We show that the map $F : \Omega_u \to \Omega_K$ is injective. Consider the convexity of the entropy energy

$$\mathcal{E}(\mathbf{u}) = \int^{\mathbf{u}} \sum_{i=1}^{n} K_i du_i.$$

The Hessian Matrix is the discrete Laplace-Beltrami operator, hence the entropy is strictly convex on the domain Ω_u. Furthermore, the domain Ω_u is convex, the gradient of the entropy is the current discrete curvature. Hence, the map $\mathbf{u} \mapsto \nabla \mathcal{E}(\mathbf{u}) = \mathbf{K}(\mathbf{u})$ is injective.

Existence Proof

We then show that the map $F : \Omega_u \to \Omega_K$ is surjective. This requires domain inviarance theorem.

Theorem (Invariance of Domain)

Suppose U is a domain (connected open set) in $\mathbb{R}^n$, if $f : U \to \mathbb{R}^n$ is continuous and injective, then $V = f(U)$ is open, and f is a homeomorphism between U and V.

Because both Ω_u and Ω_K are all $n-1$ dimensional open sets, F is continuous and injective, hence $F(\Omega_u)$ is an open set. And $F : \Omega_u \to F(\Omega_u)$ is homeomorphic. We need to show $\Omega_K = F(\Omega_u)$.

Existence Proof

Since $F(\Omega_u)$ is open, we need to show $F(\Omega_u)$ is closed in Ω_K. We take a sequence $\{x_k\} \subset \Omega_u$, such that x_k leaves all the compact sets in Ω_u. We need to show $F(x_k)$ leaves all the compact sets in Ω_K. We need the Akiyoshi theorem:

Theorem (Akiyoshi(2001))

For any complete hyperbolic metric d on $S - V$ with finite area, there exists finite number of isotopy classes of triangulations $\mathcal{T}$, such that

$$[d] \times \mathbb{R}^n_{>0} \bigcap D(\mathcal{T}) \neq \emptyset.$$

Furthermore, there is finite number of triangulations $\{\mathcal{T}_1, \dots, \mathcal{T}_k\}$, such that for any decoration $\mathbf{w} \in \mathbb{R}^n_{>0}$, the Delaunay triangulation of (d, w) is isotopic to one of such $\mathcal{T}_i$.

By Akiyoshi theorem, $\{p\} \times \mathbb{R}^n_{>0}$ intersects $T_D(S, V)$ at a finite number of cells, hence we can assume the Delaunay triangulation $\mathcal{T}$ is fixed.

Existence Proof

$\{x_k\}$ leaves all the compact sets in Ω_u. By taking subsequences, we may assume that for each vertex v_i, $\lim_k x_i^{(k)} = t_i$ exists in $[-\infty, +\infty]$. Due to the normalization that $\sum_i x_i^{(k)} = 0$ and $x^{(k)}$ doesn't converge to any vector in Ω_u, there exists $t_i = \infty$ and $t_j = -\infty$. We label vertices by black and white. The vertex v_i is black if and only if $t_i = -\infty$ and white otherwise.

Lemma (Coloring)

1. *There doesn't exist a triangle $\tau \in \mathcal{T}$ with exactly two white vertices.*
2. *If $\Delta v_1 v_2 v_3$ is a triangle in $\mathcal{T}$ with exactly one white vertex at v_1, then the inner angle at v_1 converges to 0 as $k \to \infty$ in the metric d_k.*

Existence Proof

Proof.

To see (1), suppose otherwise, there exists a Euclidean triangle of lengths $a_i e^{u_j^{(n)}+u_k^{(n)}}$, $\{i,j,k\}=\{1,2,3\}$, where $\lim_n u_i^{(n)} > -\infty$ for $i=2,3$ and $\lim_n u_1^{(n)} = -\infty$. By the triangle inequality, we have

$$a_2 e^{u_1^{(n)}+u_3^{(n)}} + a_3 e^{u_1^{(n)}+u_2^{(n)}} > a_1^{u_2^{(n)}+u_3^{(n)}}.$$

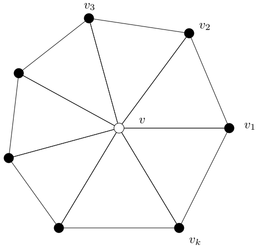
This is the same as

$$a_2 e^{-u_2^{(n)}} + a_3 e^{-u_3^{(n)}} > a_1^{-u_1^{(n)}}.$$

However, the left-hand-side is bounded, the right-hand-side tends to ∞. The contradiction shows (1) holds.

To see (2), the triangle is similar to one with edge lengths, $\{a_1 e^{-u_1^{(n)}}, a_2 e^{-u_2^{(n)}}, a_3 e^{-u_3^{(n)}}\}$, converge to $\{c,\infty,\infty\}$, hence the angle α_1 tends to 0. □

Existence Proof



We now finish the proof of $F(\Omega_u) = \Omega_k$ as follows. Since the surface S is connected, there exists an edge e whose end points v, v_1 have different colors. Assume v is white and v_1 is black. Let $v_1, \ldots, v_k$ be the set of all vertices adjacent to v so that v, v_i, v_{i+1} form vertices of a triangle and let $v_{k+1} = v_1$. Now apply above lemma to triangle $\Delta v v_1 v_2$ with v white and $v1$ black, we conclude that v_2 must be black. Inductively, we conclude that all v_i's, for $i = 1, 2, \ldots, k$, are black. By part (2) of the above lemma, we conclude that the curvature of d_n at v tends to 2π. This shows that $F(\Omega_u^{(n)})$ tends to ∞ of Ω_k. Therefore $F(\Omega_u) = \Omega_k$. $\square$