General Discrete Surface Curvature Flows

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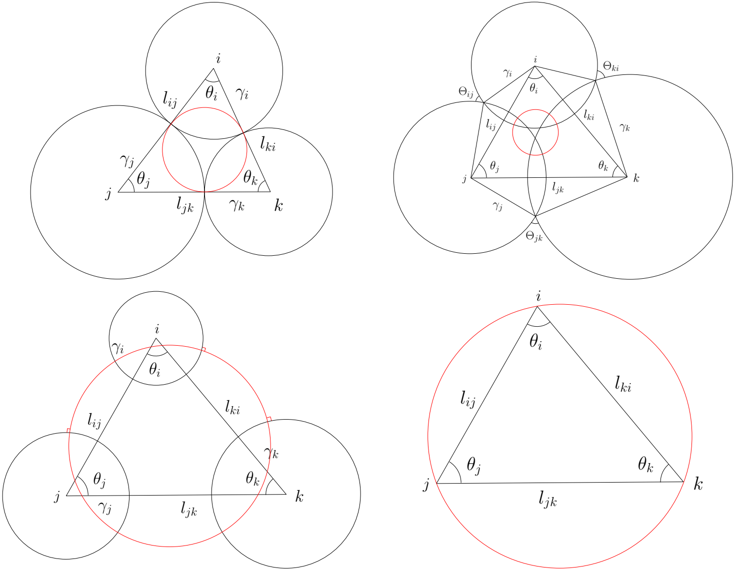
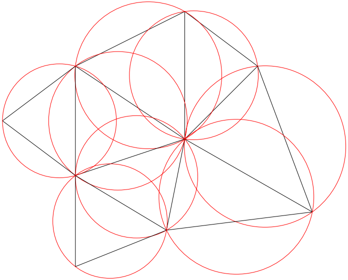
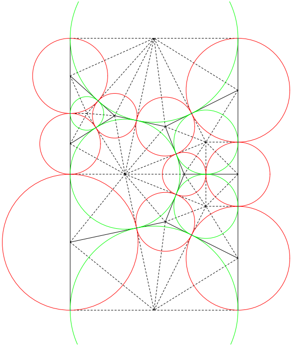
General Derivative Cosine Laws

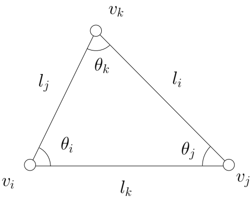
Different Schemes



Different Schemes

Derivative Cosine law



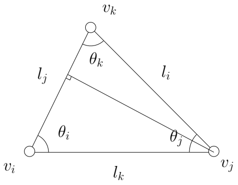
$$A = l_j l_k \sin\theta_i$$

$$\frac{\partial}{\partial l_i}(2 l_j l_k \cos\theta_i) = \frac{\partial}{\partial l_i}(l_j^2 + l_k^2 - l_i^2)$$

$$-2 l_j l_k \sin\theta_i \frac{d\theta_i}{dl_i} = -2 l_i$$

$$\frac{d\theta_i}{dl_i} = \frac{l_i}{A}$$

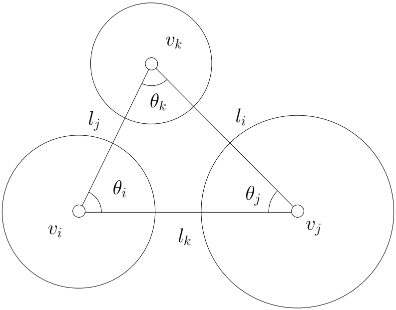
Derivative Cosine law



$$l_j = l_i \cos\theta_k + l_k \cos\theta_i$$

$$
\begin{aligned}
\frac{\partial}{\partial l_j}(2 l_j l_k \cos\theta_i) &= \frac{\partial}{\partial l_j}(l_j^2 + l_k^2 - l_i^2) \\
2 l_j &= 2 l_k \cos\theta_i - 2 l_j l_k \sin\theta_i \frac{d\theta_i}{d l_j} \\
\frac{d\theta_i}{d l_j} &= \frac{l_k \cos\theta_i - l_j}{A} \\
&= -\frac{l_i \cos\theta_k}{A} \\
&= -\frac{d\theta_i}{d l_i} \cos\theta_k
\end{aligned}
$$

Derivative Cosine law



$$l_k^2 = r_i^2 + r_j^2 + 2 r_i r_j I_{ij}$$

I_{ij} inversive distance.

$$\begin{aligned}
\frac{\partial}{\partial r_j} l_i^2 &= \frac{\partial}{\partial r_j}\left(r_j^2 + r_k^2 + 2 r_j r_k I_{jk}\right)\\
2 l_i \frac{dl_i}{dr_j} &= 2 r_j + 2 r_k I_{jk}\\
\frac{dl_i}{dr_j} &= \frac{2 r_j^2 + 2 r_j r_k I_{jk}}{2 l_i r_j}\\
&= \frac{r_j^2 + r_k^2 + 2 r_j r_k I_{jk} + r_j^2 - r_k^2}{2 l_i r_j}\\
&= \frac{l_i^2 + r_j^2 - r_k^2}{2 l_i r_j}
\end{aligned}$$

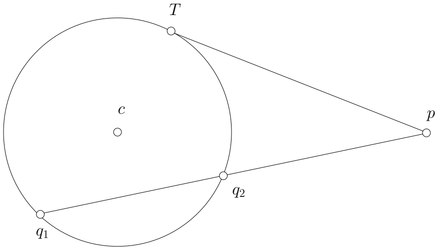
Cosine law

Let $u_i = \log r_i$, then $\frac{d\theta}{du} = \frac{d\theta}{dl}\frac{dl}{dr}\frac{dr}{du}$

$$
\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A}\begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix}\begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix}
$$

$$
\begin{pmatrix} 0 & \frac{l_1^2+r_2^2-r_3^2}{2l_1r_2} & \frac{l_1^2+r_3^2-r_2^2}{2l_1r_3} \\ \frac{l_2^2+r_1^2-r_3^2}{2l_2r_1} & 0 & \frac{l_2^2+r_3^2-r_1^2}{2l_2r_3} \\ \frac{l_3^2+r_1^2-r_2^2}{2l_3r_1} & \frac{l_3^2+r_2^2-r_1^2}{2l_3r_2} & 0 \end{pmatrix}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}
$$

Power

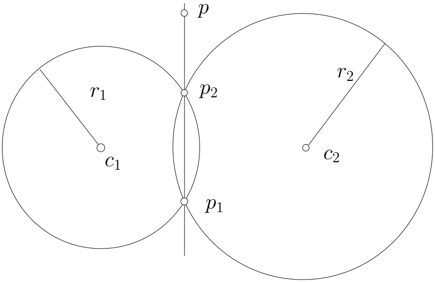


Power

Suppose a point p is not coincident of the center of a circle $\mathbf{c} = (c, r)$ on the plane, the line through p intersects the circle at q_1 and q_2, T is the tangent point, then the power of p with respect to the circle is

$$\begin{aligned} pow(p, \mathbf{c}) &= |pq_1||pq_2| \\ &= |pT|^2 \\ &= |pc|^2 - r^2. \end{aligned}$$

Power



$$pow(p, \mathbf{c_1}) = |pp_1||pp_2| = pow(p, \mathbf{c_2})$$

Equi-Power line

Suppose there are two circles $\mathbf{c_1} = (c_1, r_1)$, $\mathbf{c_2} = (c_2, r_2)$, the equi-power line is the locus

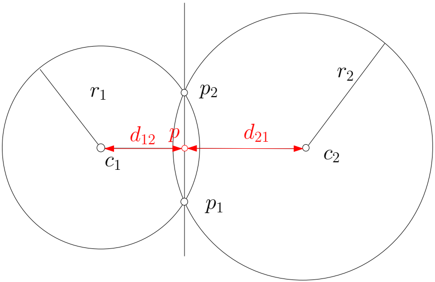
$$pow(p, \mathbf{c_1}) = pow(p, \mathbf{c_2}).$$

The equation of p is

$$|p - c_1|^2 - r_1^2 = |p - c_2|^2 - r_2^2.$$

If two circles intersect at p_1 and p_2, then the line through them is the equi-power line.

Power



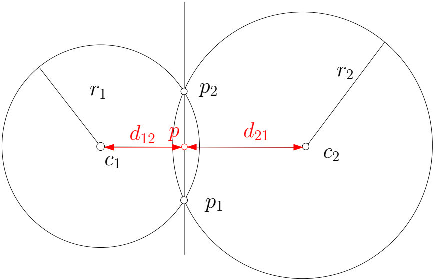
$$pow(p, \mathbf{c_1}) = pow(p, \mathbf{c_2})$$
$$d_{12}^2 - r_1^2 = d_{21}^2 - r_2^2$$

Suppose there are two circles $\mathbf{c_k} = (c_k, r_k)$, the line through c_1 and c_2 intersects the equi-power line at the point p. Assume the length between c_1 and c_2 is l. The distance from p to c_2 is denoted as d_{21}, then

$$\begin{aligned} d_{12} &= \frac{l^2 + r_1^2 - r_2^2}{2l} \\ d_{21} &= \frac{l^2 + r_2^2 - r_1^2}{2l} \end{aligned}$$

obviously, $d_{12} + d_{21} = l$.

Power



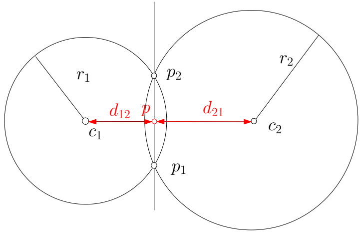
compute the power of p with respect to two circles

$$\begin{aligned} pow(p, \mathbf{c_1}) &= d_{12}^2 - r_1^2 \\ pow(p, \mathbf{c_2}) &= d_{21}^2 - r_2^2 \end{aligned}$$

$$pow(p, \mathbf{c_1}) = -|pp_1||pp_2|$$
$$pow(p, \mathbf{c_2}) = -|pp_1||pp_2|$$
$$pow(p, \mathbf{c_1}) = pow(p, \mathbf{c_2})$$

$$\begin{aligned} d_{12}^2 - d_{21}^2 &= (d_{12} + d_{21})(d_{12} - d_{21}) \\ &= l\frac{r_1^2 - r_2^2}{l} = r_1^2 - r_2^2. \end{aligned}$$

Power



Lemma

The equi-power line is orthogonal to the line connecting the centers.

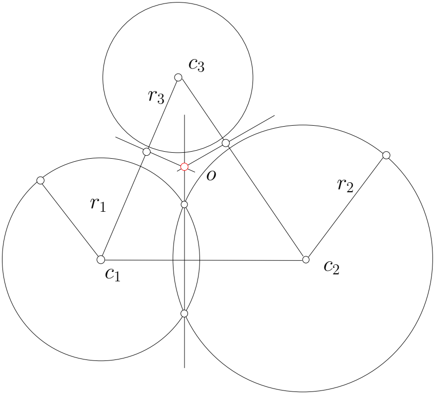
Proof.

Define a function $\phi(p) = pow(p, \mathbf{c_1}) - pow(p, \mathbf{c_2})$,

$$\begin{aligned}\phi(p) &= \langle p - c_1, p - c_1\rangle - r_1^2 - \langle p - c_2, p - c_2\rangle + r_2^2 \\ d\phi(p) &= \langle dp, c_2 - c_1\rangle\end{aligned}$$

so $\nabla\phi = c_2 - c_1$, orthogonal to the level sets of ϕ. The equi-power line is the 0-level set of ϕ. □

Power



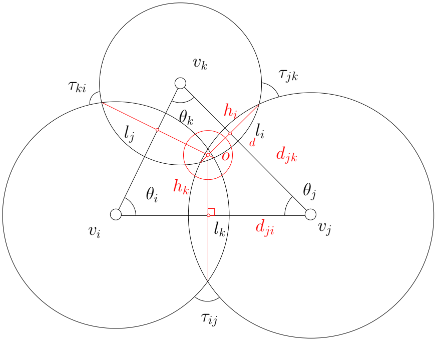
Given three circles $\mathbf{c_k}$, $k = 1, 2, 3$, then three equi-power lines intersect at one point o, which is called the *power center*,

The equi-power lines of $\mathbf{c_1}, \mathbf{c_2}$ and $\mathbf{c_1}, \mathbf{c_3}$ intersects at the point o. Then

$$pow(o, \mathbf{c_1}) = pow(o, \mathbf{c}_2) = pow(o, \mathbf{c_3})$$

so o is also on the equi-power line of $\mathbf{c}_2, \mathbf{c_3}$.

Power Center

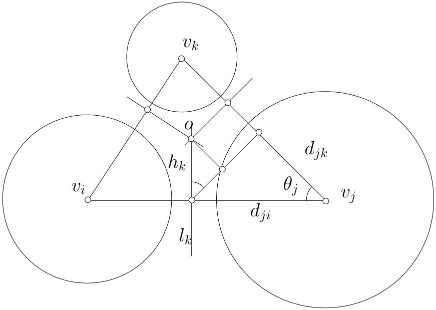


$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$

There are 3 circles $\mathbf{c_k} = (c_k, r_k)$, the power center o is also the center of the unique circle (p, r), which is orthogonal to all 3 circles.

$$pow(o, \mathbf{c_k}) = \langle o - c_k, o - c_k \rangle - r_k^2 = r^2,$$

so the power center is the center of the circle which is orthogonal to the 3 circles.



$$\frac{\partial}{\partial r_j} l_k^2 = \frac{\partial}{\partial r_j}\left(r_i^2 + r_j^2 + 2 l_{ij} r_i r_j\right)$$

$$pow(o, \mathbf{c_i}) = pow(o, \mathbf{c_j})$$

$$|ov_i|^2 - r_i^2 = |ov_j|^2 - r_j^2$$

$$\boxed{|ov_i|^2 - |ov_j|^2 = r_i^2 - r_j^2}$$

$$2l_k \frac{dl_k}{dr_j} = 2r_j + 2r_i l_{ij}$$

$$r_j \frac{dl_k}{dr_j} = \frac{2r_j^2 + 2r_i r_j l_{ij}}{2l_k}$$

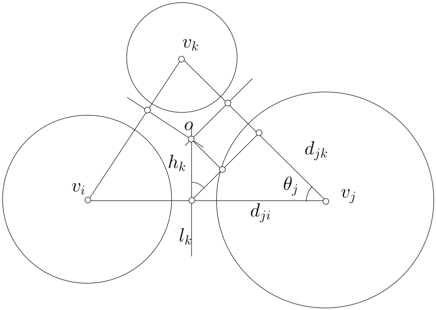
$$= \frac{r_j^2 + 2r_i r_j l_{ij} + r_i^2 - r_i^2 + r_j^2}{2l_k}$$

$$= \frac{l_k^2 + r_j^2 - r_i^2}{2l_k}$$

$$= \frac{l_k^2 + |ov_j|^2 - |ov_i|^2}{2l_k} = d_{ji}$$

Therefore in $\Delta v_i v_j o$, $\frac{dl_k}{du_j} = d_{ji}$.

Derivative Cosine law



The distance from o to edge $[v_i, v_j]$ is h_k.

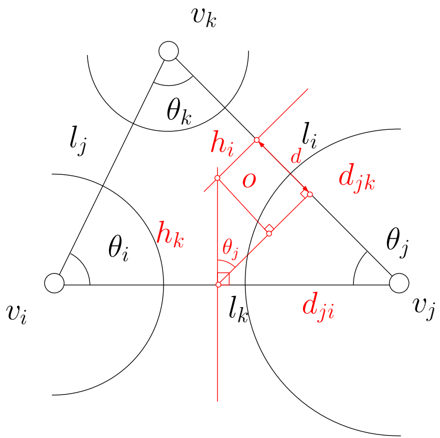
Theorem (Symmetry)

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$

$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$

$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Derivative Cosine law



$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

Proof.

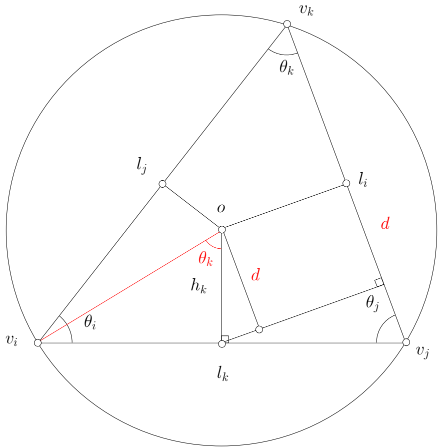
$$\begin{aligned}
\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\partial \theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(d_{jk} - d_{ji}\cos\theta_j) \\
&= \frac{dl_i}{l_i l_k \sin\theta_j} \\
&= \frac{h_k \sin\theta_j}{l_k \sin\theta_j} \\
&= \frac{h_k}{l_k}
\end{aligned}$$

□

Inversive Distance CP Metric - Local Rigidity

The Discrete Ricci energy of Inversive distance CP metric is convex, but the conformal factor space is non-convex. Therefore it has local rigidity, not global rigidity.

Yamabe Flow



$$l_k \leftarrow e^{u_i} l_k e^{u_j}$$

Shrink three circles to vertices, then the power center o becomes the circum-center.

$$\begin{aligned}
\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\partial \theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(l_i - l_k\cos\theta_j) \\
&= \frac{2l_i d}{l_i l_k \sin\theta_j} \\
&= \frac{2h_k}{l_k} \\
&= \cot\theta_k
\end{aligned}$$

Discrete Yamabe flow - Local Rigidity

The Discrete Ricci energy of discrete Yamabe flow is convex, but the conformal factor space is non-convex. Therefore it has local rigidity, not global rigidity.

Extremal Length

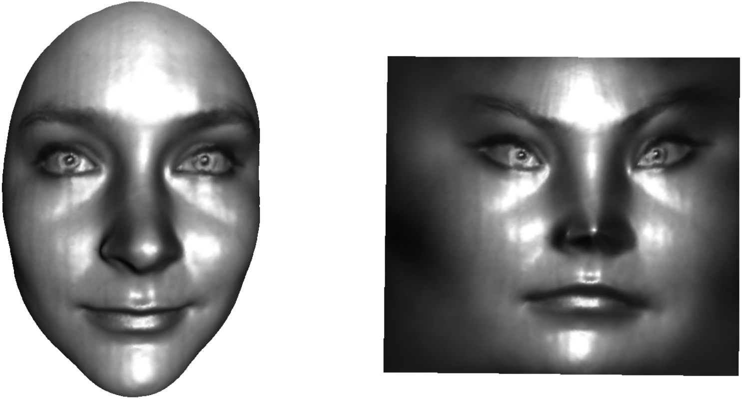


Figure: The conformal module of a topological quadrilateral.

Topological Annulus

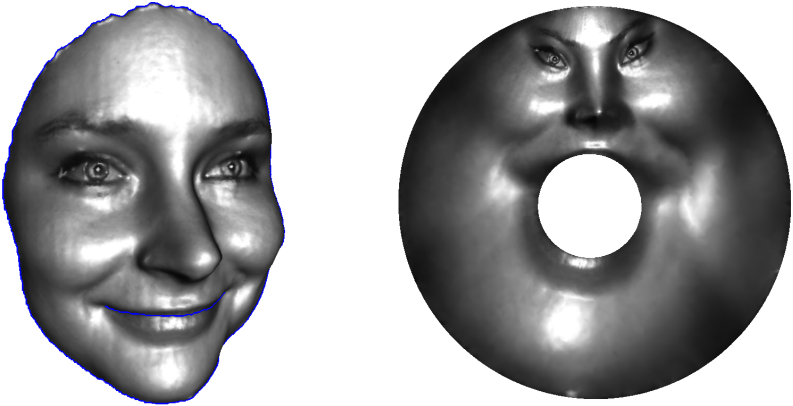


Figure: The conformal module of a topological annulus.

Costa Minimal Surface

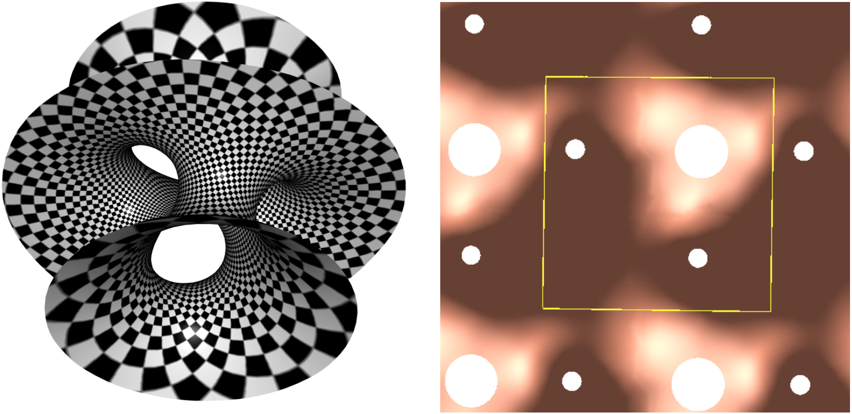


Figure: Costa minimal surface.

Circle Packing and Square Packing

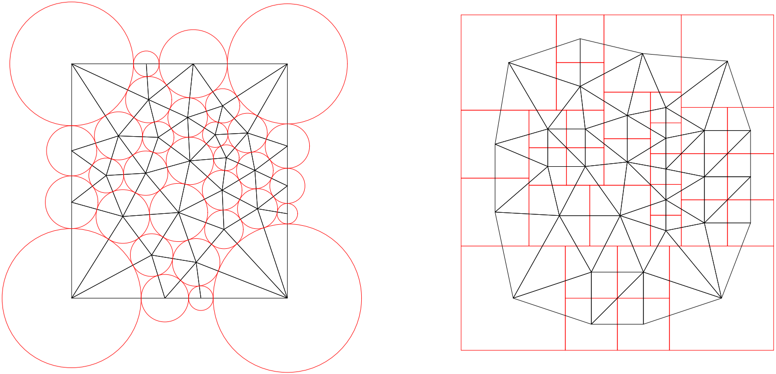


Figure: Circle packing and square packing.

Circle Packing Art

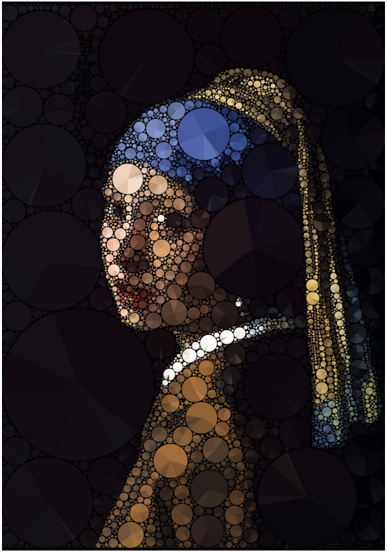


Figure: Girl with a Pearl Earring. (by Mario Klingemann)

Circle Packing Art

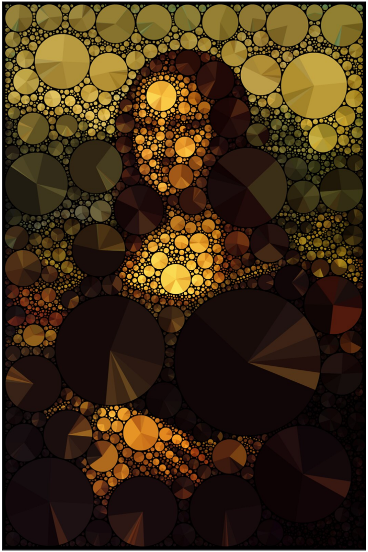


Figure: Mona Lisa. (by Mario Klingemann)

Circle Packing Art

Figure: The Starry Night. (by Mario Klingemann)

Hyperbolic Surface Ricci Flow

Polyhedral Surface

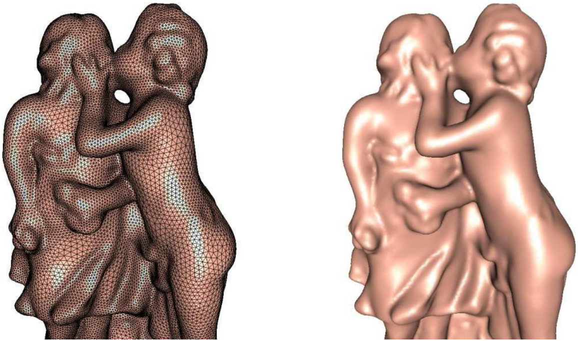


Figure: Polyhedral surface.

Background Geometries

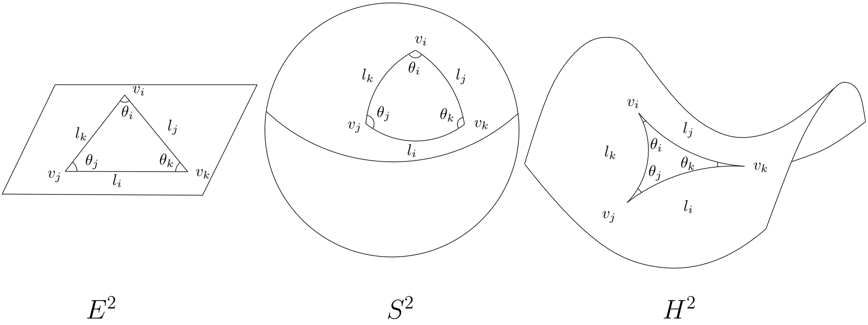


Figure: Constant curvature triangle.

We can glue hyperbolic or spherical triangles isometrically along the common edges to construct the triangle mesh. Then we say the surface is with hyperbolic or spherical background geometry.

Hyperbolic Triangle

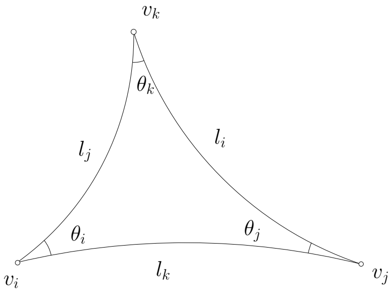


Figure: Hyperbolic triangle.

Cosine law:

$$\cos\theta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}$$

Sine law:

$$\frac{\sinh l_i}{\sin\theta_i} = \frac{\sinh l_j}{\sin\theta_j} = \frac{\sinh l_k}{\sin\theta_k}$$

Area

$$A = \frac{1}{2}\sinh l_j \sinh l_k \sin\theta_i$$

Hyperbolic Derivative Cosine Law

Lemma

The hyperbolic derivative cosine law is represented as:

$$\frac{\partial \theta_i}{\partial l_i} = \frac{\sinh l_i}{A}, \frac{\partial \theta_i}{\partial l_j} = -\frac{\sinh l_i}{A}\cos\theta_k$$

Compared with Euclidean cosine law, we replace the edge lengths l_i by $\sinh l_i$.

Curvature

Definition (Discrete Curvature)

Given a discrete surface with hyperbolic background geometry $(S, V, \mathcal{T}, l)$, every triangle is a hyperbolic geodesic triangle, the vertex discrete curvature is defined as the angle deficit

$$K(v) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk}, & v \notin \partial S \\ \pi - \sum_{jk} \theta_i^{jk}, & v \in \partial S \end{cases}$$

Theorem (Gauss-Bonnet)

The discrete Gauss-Bonnet theorem is represented as:

$$\sum_{v \notin \partial S} K(v) + \sum_{v \in \partial S} K(v) - Area(S) = 2\pi\chi(S)$$

Discrete Conformal Metric Deformation

Definition (Conformal Deformation)

Given discrete conformal factor function $u : V(\mathcal{T}) \to \mathbb{R}$, hyperbolic vertex scaling is defined as $y := u * l$,

$$\sinh\frac{y_k}{2} = e^{\frac{u_i}{2}}\sinh\frac{l_k}{2}e^{\frac{u_j}{2}}$$

Lemma (Symmetry)

The symmetric relations holds:

$$\frac{\partial\theta_i}{\partial u_j} = \frac{\partial\theta_j}{\partial u_i} = \frac{C_i + C_j - C_k - 1}{A(C_k + 1)}$$

where $S_k = \sinh y_k, C_k = \cosh y_k$.

Definition (Hyperbolic Entropy Energy)

$$E_f(u_i, u_j, u_k) = \int^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

The Hessian matrix of the entropy energy is:

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix} \begin{pmatrix} 0 & \frac{S_1}{C_1+1} & \frac{S_1}{C_1+1} \\ \frac{S_2}{C_2+1} & 0 & \frac{S_2}{C_2+1} \\ \frac{S_3}{C_3+1} & \frac{S_3}{C_3+1} & 0 \end{pmatrix}$$

which is strictly convex.

Discrete Entropy Energy on a Mesh

Definition (Entropy Energy)

The entropy energy on a triangle mesh with hyperbolic background geometry equals to

$$E(\mathbf{u}) = \int^{\mathbf{u}} \sum_i (\bar{K}_i - K_i) du_i$$

Definition (Hyperbolic Ricci Flow)

Hence the discrete hyperbolic surface Ricci flow is defined as:

$$\frac{du_i(t)}{dt} = \bar{K}_i - K_i(t),$$

which is the gradient flow of the discrete hyperbolic entropy energy. The strict concavity of the discrete entropy ensures the uniqueness of the solution to the flow. The existence is given by Gu-Luo-Sun using Teichmüller theory and hyperbolic geometry.

Uniformizaton of High Genus Surface

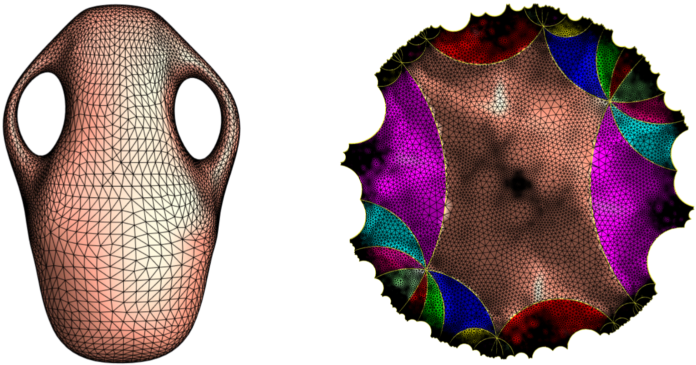


Figure: Uniformization of a genus two surface.

Uniformization

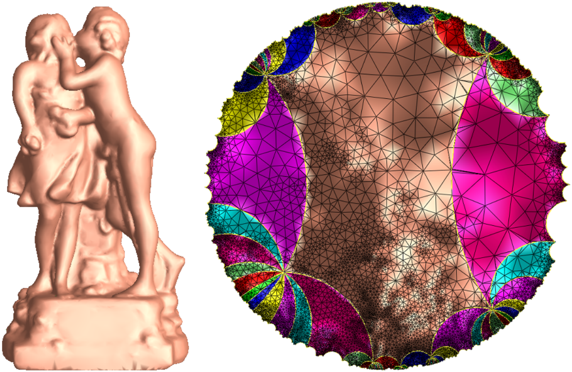


Figure: Uniformization of a genus three surface.

Uniformization

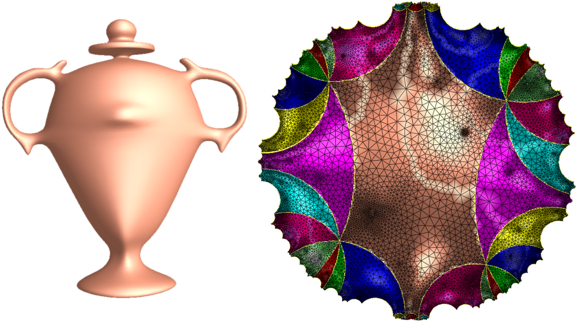


Figure: Uniformization of a genus two surface.

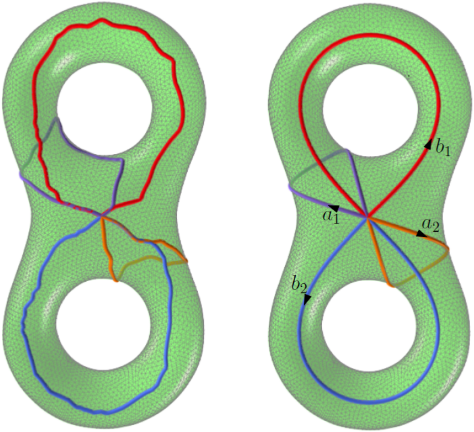
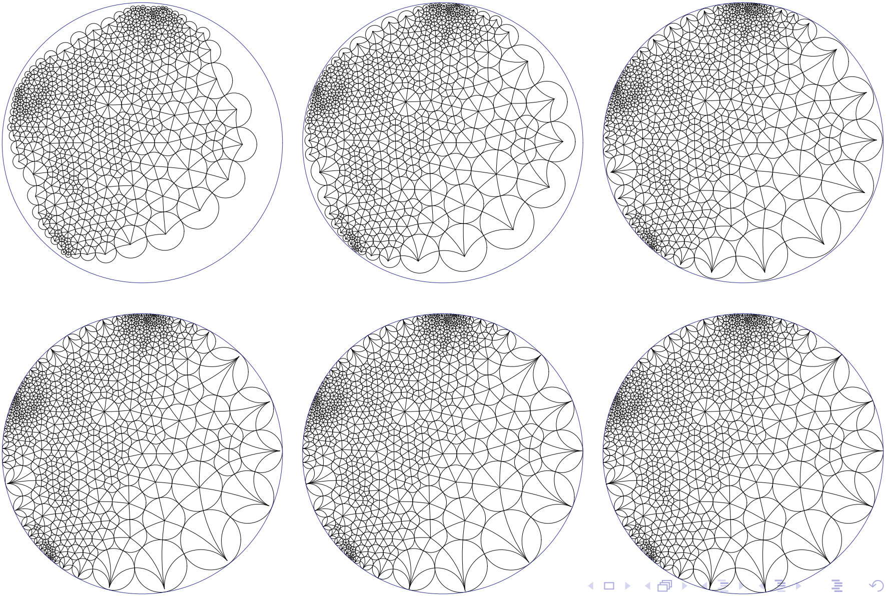


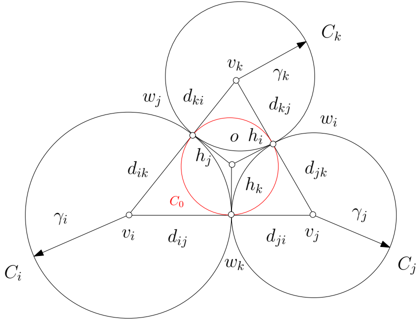
Figure: Shortest word problem.

Discrete Riemann Mapping

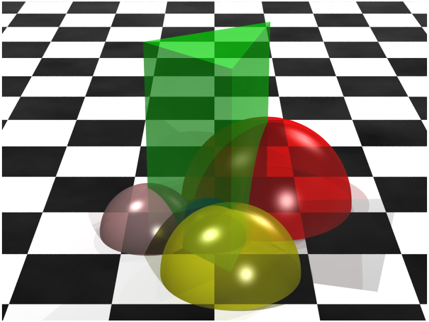


Unified Discrete Surface Ricci Flow

Unified Ricci Flow



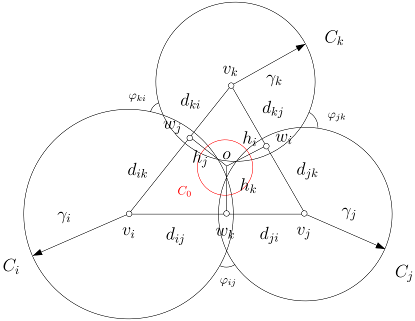
(a)Tangential CP



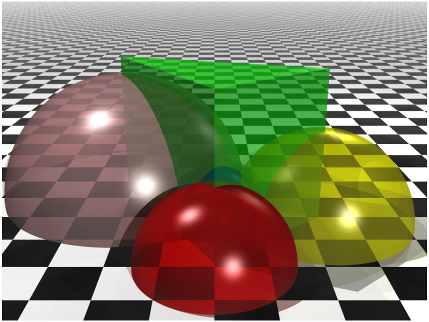
(b) Generalized Hyperbolic Tetrahedron, $(\eta, \epsilon) = (1, 1)$

Figure: Tangential circle packing.

Thurston's Circle Packing

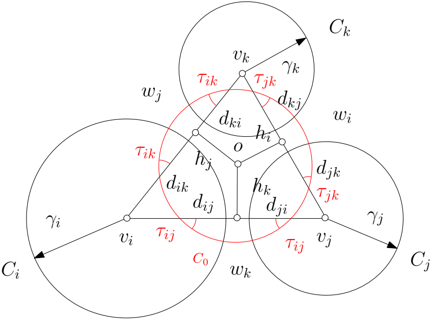


(a)Thurston's Circle packing

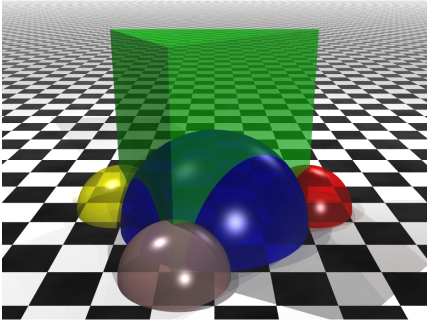


(b)Generalized Hyperbolic Tetrahedron, $0 \leq \eta < 1, \epsilon = 1$

Figure: Thurston's circle packing.



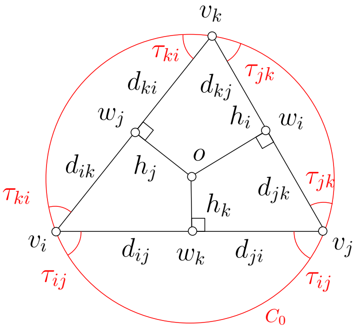
(c)Inversive distance CP



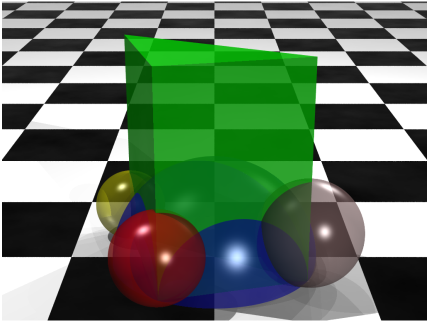
(d)Generalized Hyperbolic Tetrahedron, $\eta > 1, \epsilon = 1$

Figure: Inversive distance circle packing.

Yamabe Flow



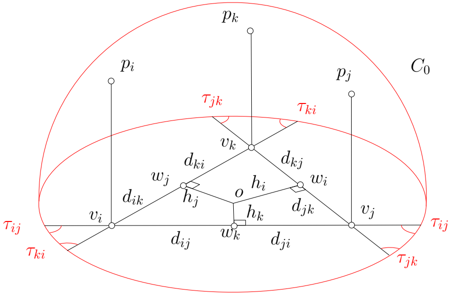
(d)Yamabe flow



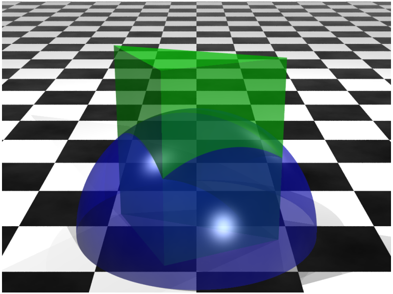
(e)Generalized Hyperbolic Tetrahedron, $\eta > 0, \epsilon = 0$

Figure: Yamabe flow.

Virtual Radius Circle Packing



(e)Virtual radius CP

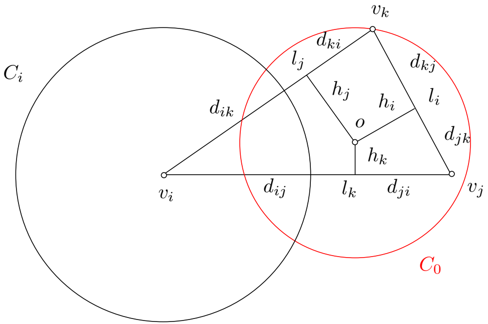


(f)Generalized Hyperbolic Tetrahedron, $\eta > 0, \epsilon = -1$

Figure: virtual radius circle packing.

$$l_k^2 = -r_i^2 - r_j^2 + 2\eta_{ij} r_i r_j.$$

Mixed Type



(f)mixed type

Figure: Mixed typed circle packing.

Conformal Factor

Definition (Discrete Conformal Factor)

The discrete conformal factor is defined as $u : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{E}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Edge Length

Definition (Edge Length)

The edge lengths are given by

$$u_i = \begin{cases} l_{ij}^2 &= 2\eta_{ij} e^{u_i+u_j} + \varepsilon_i e^{2u_i} + \varepsilon_j e^{2u_j} & \mathbb{E}^2 \\ \cosh l_{ij} &= \frac{4\eta_{ij} e^{u_i+u_j} + (1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})}{(1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})} & \mathbb{H}^2 \\ \cos l_{ij} &= \frac{-4\eta_{ij} e^{u_i+u_j} + (1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})}{(1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})} & \mathbb{S}^2 \end{cases}$$

Edge Length

Scheme	ε_i	ε_j	η_{ij}
Tangential Circle Packing	$+1$	$+1$	$+1$
Thurston's Circle Packing	$+1$	$+1$	$[0,1]$
Inversive Distance Circle Packing	$+1$	$+1$	$(0,\infty)$
Yamabe Flow	0	0	$(0,\infty)$
Virtual Distance Circle Packing	-1	-1	$(0,\infty)$
Mixed Type	$\{-1,0,+1\}$	$\{-1,0,+1\}$	$(0,\infty)$

Table: Parameters for schemes.

Entropy Energy

Definition (Entroy on a Face)

A discrete surface with $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ background geometry, and a circle packing metric $(\Sigma, \gamma, \eta, \varepsilon)$. For each triangle $[v_i, v_j, v_k]$ with inner angle $(\theta_i, \theta_j, \theta_k)$, the entropy energy for the face is given by

$$E_f(u_i, u_j, u_k) = \int^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k.$$

Entropy Energy

Definition (Entroy on a mesh)

A discrete surface with $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ background geometry, and a circle packing metric $(\Sigma, \gamma, \eta, \varepsilon)$. The discrete entropy energy for the whole mesh is defined as

$$E_= \int^{(u_1,u_2,\cdots,u_n)} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

The mesh entropy can be represented as the face energies

$$E_\sigma = \sum_{i=1}^{n} (\bar{K}_i - 2\pi) u_i + \sum_{f \in F} E_f.$$

Symmetry

Suppose a triangle $[v_i, v_j, v_k]$ is with background geometry $\mathbb{S}^2$,$\mathbb{E}^2$, $\mathbb{H}^2$, conformal factor (u_i, u_j, u_k), edge length (l_i, l_j, l_k), inner angles $(\theta_i, \theta_j, \theta_k)$, entropy energy is

$$E(u_i, u_j, u_k) = \int^{(u_i,u_j,u_k)} \theta_i du_i + \theta_j du_j + \theta_k du_k. \tag{1}$$

Then the Hessian matrix is given by

$$\frac{\partial(\theta_i, \theta_j, \theta_k)}{\partial(u_i, u_j, u_k)} = -\frac{1}{2A} L\Theta L^{-1} D, \tag{2}$$

where, A is the triangle area

$$A = \frac{1}{2}\sin\theta_i s(l_j) s(l_k), \tag{3}$$

Symmetry

The matrix L is

$$L = \begin{pmatrix} s(l_i) & 0 & 0 \\ 0 & s(l_j) & 0 \\ 0 & 0 & s(l_k) \end{pmatrix} \tag{4}$$

Θ

$$\Theta = \begin{pmatrix} -1 & \cos\theta_k & \cos\theta_j \\ \cos\theta_k & -1 & \cos\theta_i \\ \cos\theta_j & \cos\theta_i & -1 \end{pmatrix} \tag{5}$$

matrix D is

$$D = \begin{pmatrix} 0 & \tau(i,j,k) & \tau(i,k,j) \\ \tau(j,i,k) & 0 & \tau(j,k,i) \\ \tau(k,i,j) & \tau(k,j,i) & 0 \end{pmatrix} \tag{6}$$

Hessian Matrix

where

$$s(x) = \begin{cases} x & \mathbb{E}^2 \\ \sinh x & \mathbb{H}^2 \\ \sin x & \mathbb{S}^2 \end{cases}$$

and

$$\tau(i,j,k) = \begin{cases} \frac{1}{2}(l_i^2 + \epsilon_j \gamma_j^2 - \epsilon_k \gamma_k^2) & \mathbb{E}^2 \\ \cosh l_i \cosh^{\epsilon_j} \gamma_j - \cosh^{\epsilon_k} \gamma_k & \mathbb{H}^2 \\ \cos l_i \cos^{\epsilon_j} \gamma_j - \cos^{\epsilon_k} \gamma_k & \mathbb{S}^2 \end{cases}$$

Geometric Interpretation

For each triangle, there is a power circle, orthogonal to three vertex circles. The distance from the power center to each edge is h_i, h_j, h_k. Then we have the geometric interpretation to the Hessian matrix: with $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$ background geometry,

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{h_3}{l_3}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tanh h_3}{\sinh^2 l_3}\sqrt{2\cosh^{\epsilon_1} r_1 \cosh^{\varepsilon_2} r_2 \cosh l_3 - \cosh^{2\varepsilon_1} r_1 - \cosh^{2\varepsilon_2} r_2}$$

$$\frac{\partial \theta_1}{\partial u_2} = \frac{\partial \theta_2}{\partial u_1} = \frac{\tan h_3}{\sin^2 l_3}\sqrt{-2\cos^{\varepsilon_1} r_1 \cos^{\varepsilon_2} r_2 \cos l_3 + \cos^{2\varepsilon_1} r_1 + \cos^{2\varepsilon_2} r_2}$$