Discrete Euclidean Curvature Flow

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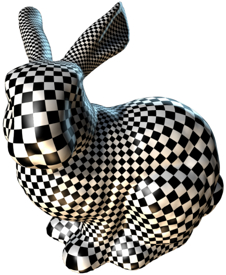
Isothermal Coordinates

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Gaussian Curvature

Under the isothermal coordinates, the Riemannian metric is $\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2)$, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

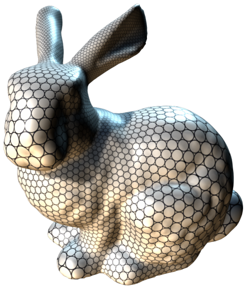
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ *be a compact 2-dimensional Riemannian manifold. Then there is a metric* $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ *conformal to* $\mathbf{g}$ *which has constant Gauss curvature.*

Surface Uniformization

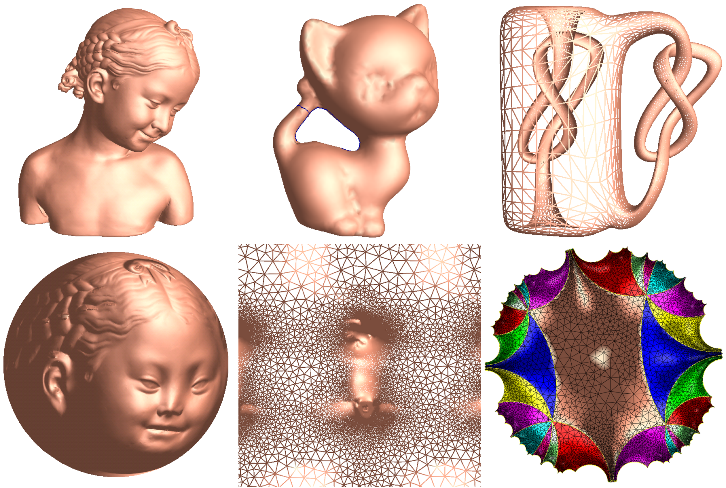


Figure: Closed surface uniformization.

Surface Uniformization

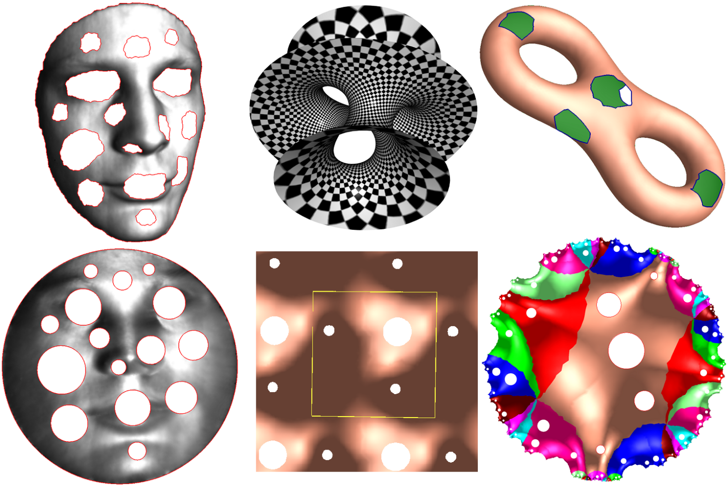


Figure: Open surface uniformization.

Surface Ricci Flow

Proposition

During the curvature flow $\frac{d\lambda}{dt} = -K$, then

$$\frac{d}{dt}K = 2K^2 + \Delta_{\mathbf{g}} K.$$

$$\begin{aligned}
\frac{d}{dt}K &= \frac{d}{dt}(-e^{-2\lambda}\Delta\lambda) \\
&= -\left(-2\frac{d\lambda}{dt}\right)e^{-2\lambda}\Delta\lambda - e^{-2\lambda}\Delta\frac{d\lambda}{dt} \\
&= \left(-2\frac{d\lambda}{dt}\right)\boxed{-e^{-2\lambda}\Delta\lambda} - \boxed{e^{-2\lambda}\Delta}\frac{d\lambda}{dt} \\
&= \left(-2\frac{d\lambda}{dt}\right)K - \Delta_{\mathbf{g}}\frac{d\lambda}{dt} \\
&= 2K^2 + \Delta_{\mathbf{g}}K
\end{aligned}$$

Key Idea

$$K = -\Delta_{\mathbf{g}}\lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = \frac{d}{dt}\Delta_{\mathbf{g}}\lambda$$

Let $\frac{d\lambda}{dt} = -K$,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

Diffusion and reaction equation!

Surface Ricci Flow

Definition (**Hamilton's Surface Ricci Flow**)

A closed surface with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

The normalized surface Ricci flow,

$$\frac{dg_{ij}}{dt} = \frac{2\pi\chi(S)}{A(0)} - 2Kg_{ij},$$

where $A(0)$ is the initial surface area.

The normalized surface Ricci flow is area-preserving, the Ricci flow will converge to a metric such that the Gaussian curvature is constant $\frac{2\pi\chi(S)}{A(0)}$ every where.

Ricci Flow

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

- Ricci flow

$$\frac{du}{dt} = \bar{K} - K.$$

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.

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- Isometric gluing of triangles in $\mathbb{E}^2$.

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- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

Discrete Generalization

Concepts

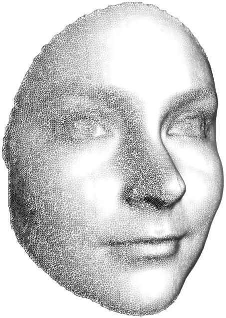
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

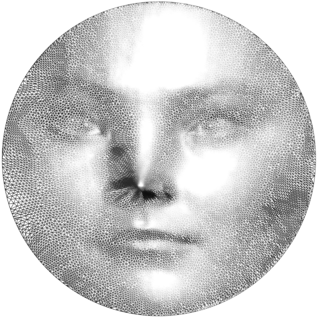
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ \ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

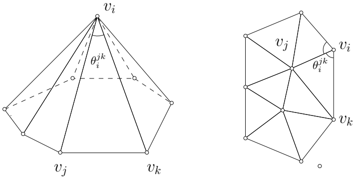
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

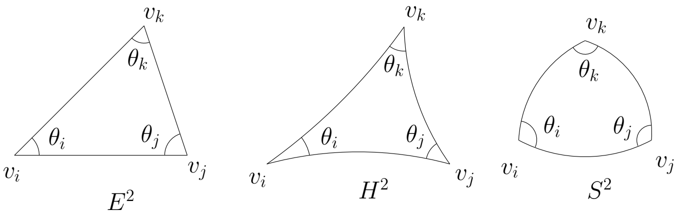
$$K(v_i) = 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M; K(v_i) = \pi - \sum_{jk} \theta_{jk}, v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



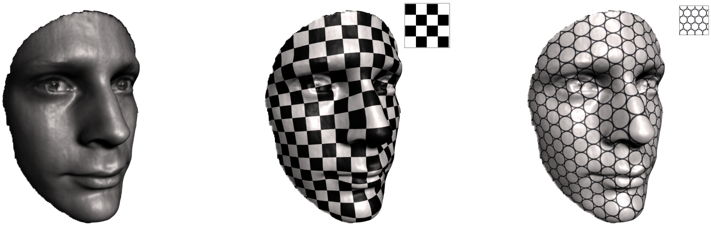
cosine laws

$$
\begin{aligned}
\cos l_i &= \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} && \mathbb{S}^2 \\
\cosh l_i &= \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} && \mathbb{H}^2 \\
1 &= \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} && \mathbb{E}^2
\end{aligned}
$$

Discrete Conformal Metric Deformation

Conformal maps Properties

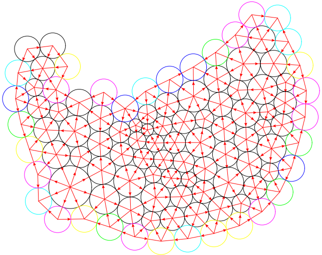
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

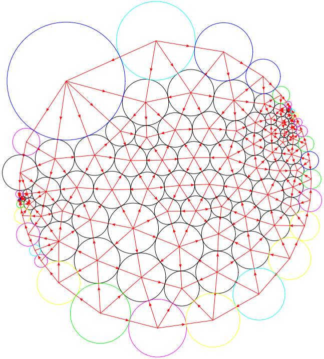


Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Discrete Conformal Metric Deformation vs CP





Circle Packing Metric

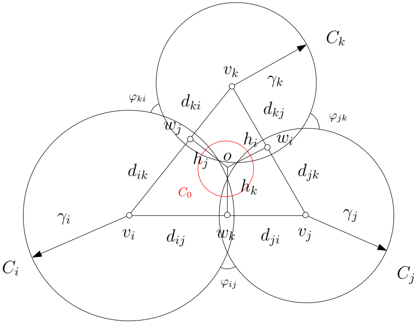
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\varphi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\varphi_{ij} | \forall e_{ij}\}$$



Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u}: V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

Unified Framework of Discrete Curvature Flow

Analogy

- Curvature flow

$$\frac{du}{dt} = \bar{K} - K,$$

- Energy

$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$

- Hessian of E denoted as Δ,

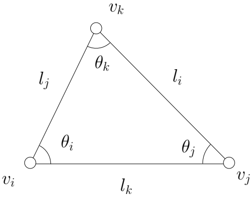
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Criteria for Discretization

Key Points

- Convexity of the energy $E(\mathbf{u})$
- Convexity of the metric space ($\mathbf{u}$-space)
- Admissible curvature space ($\mathbf{K}$-space)
- Preserving or reflecting richer structures
- Conformality

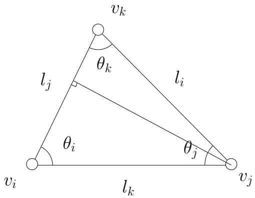
Derivative Cosine law



$$A = l_j l_k \sin\theta_i$$

$$\begin{aligned}
\frac{\partial}{\partial l_i}(2 l_j l_k \cos\theta_i) &= \frac{\partial}{\partial l_i}(l_j^2 + l_k^2 - l_i^2) \\
-2 l_j l_k \sin\theta_i \frac{d\theta_i}{dl_i} &= -2 l_i \\
\frac{d\theta_i}{dl_i} &= \frac{l_i}{A}
\end{aligned}$$

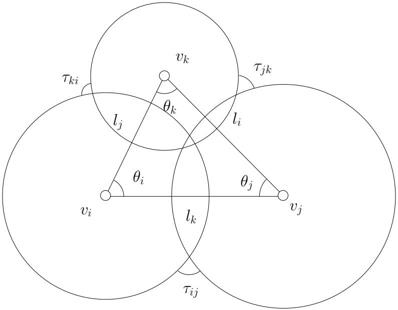
Derivative Cosine law



$$l_j = l_i \cos\theta_k + l_k \cos\theta_i$$

$$\begin{aligned}
\frac{\partial}{\partial l_j}(2 l_j l_k \cos\theta_i) &= \frac{\partial}{\partial l_j}(l_j^2 + l_k^2 - l_i^2)\\
2 l_j &= 2 l_k \cos\theta_i - 2 l_j l_k \sin\theta_i \frac{d\theta_i}{dl_j}\\
\frac{d\theta_i}{dl_j} &= \frac{l_k \cos\theta_i - l_j}{A}\\
&= -\frac{l_i \cos\theta_k}{A}\\
&= -\frac{d\theta_i}{dl_i}\cos\theta_k
\end{aligned}$$

Derivative Cosine law



$$l_k^2 = r_i^2 + r_j^2 + 2\cos\tau_{ij} r_i r_j$$

$$\frac{\partial}{\partial r_j} l_i^2 = \frac{\partial}{\partial r_j}\left(r_j^2 + r_k^2 + 2 r_j r_k \cos\tau_{jk}\right)$$

$$2 l_i \frac{d l_i}{d r_j} = 2 r_j + 2 r_k \cos\tau_{jk}$$

$$\frac{d l_i}{d r_j} = \frac{2 r_j^2 + 2 r_j r_k \cos\tau_{jk}}{2 l_i r_j}$$

$$= \frac{r_j^2 + r_k^2 + 2 r_j r_k \cos\tau_{jk} + r_j^2 - r_k^2}{2 l_i r_j}$$

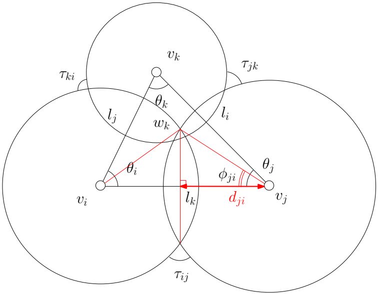
$$= \frac{l_i^2 + r_j^2 - r_k^2}{2 l_i r_j}$$

Derivative Cosine law

Let $u_i = \log r_i$, then $\frac{d\theta}{du} = \frac{d\theta}{dl}\frac{dl}{dr}\frac{dr}{du}$

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A}\begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix}\begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{l_1^2+r_2^2-r_3^2}{2l_1r_2} & \frac{l_1^2+r_3^2-r_2^2}{2l_1r_3} \\ \frac{l_2^2+r_1^2-r_3^2}{2l_2r_1} & 0 & \frac{l_2^2+r_3^2-r_1^2}{2l_2r_3} \\ \frac{l_3^2+r_1^2-r_2^2}{2l_3r_1} & \frac{l_3^2+r_2^2-r_1^2}{2l_3r_2} & 0 \end{pmatrix}\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}\begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

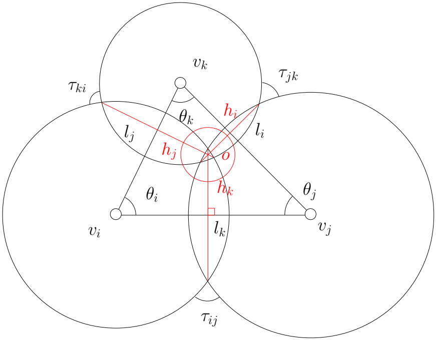


$$l_k^2 = r_i^2 + r_j^2 + 2\cos\tau_{ij} r_i r_j$$

$$\begin{aligned} 2l_k \frac{dl_k}{dr_j} &= 2r_j + 2r_i \cos\tau_{ij} \\ r_j \frac{dl_k}{dr_j} &= \frac{2r_j^2 + 2r_i r_j \cos\tau_{ij}}{2l_k} \\ &= \frac{r_j^2 + r_i^2 + 2r_i r_j \cos\tau_{ij} + r_j^2 - r_i^2}{2l_k} \\ &= \frac{l_k^2 + r_j^2 - r_i^2}{2l_k} \end{aligned}$$

In triangle $[v_i, v_j, w_k]$,

$$\frac{dl_k}{du_j} = 2\frac{l_k r_j \cos\phi_{ji}}{2l_k} = r_j \cos\phi_{ji} = d_{ji}$$

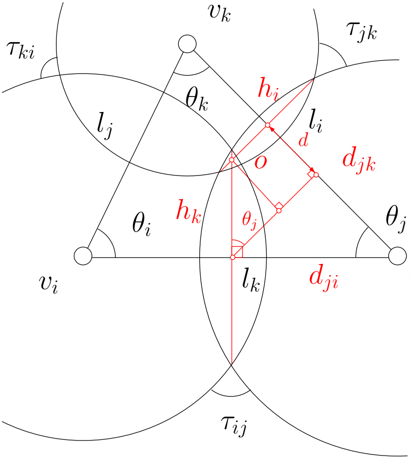


There is a unique circle orthogonal to three circles (v_i, r_i), the center is o, the distance from o to edge $[v_i, v_j]$ is h_k.

Theorem (Derivative Cosine Law)

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$
$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$
$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Derivative Cosine law



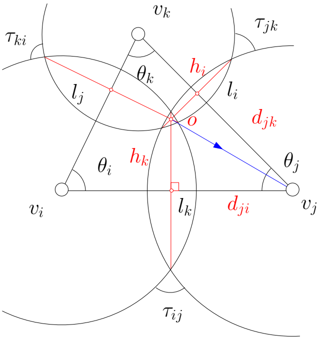
$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

Proof.

$$\begin{aligned}
\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\partial \theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(d_{jk} - d_{ji}\cos\theta_j) \\
&= \frac{dl_i}{l_i l_k \sin\theta_j} \\
&= \frac{h_k \sin\theta_j}{l_k \sin\theta_j} \\
&= \frac{h_k}{l_k}
\end{aligned}$$

□

Derivative Cosine law



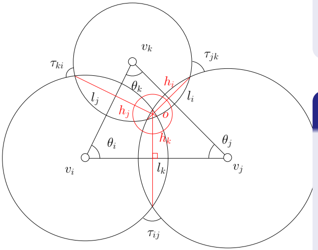
$$
\begin{aligned}
\frac{\partial \langle v_j - v_i, v_j - v_i \rangle}{\partial u_j} &= 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \rangle \\
\frac{\partial l_k^2}{\partial u_j} &= 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \rangle \\
\frac{\partial l_k}{\partial u_j} &= \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \rangle \\
d_{ji} &= \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \rangle
\end{aligned}
$$

Similarly

$$d_{jk} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_k}{l_i} \rangle$$

$$\frac{\partial v_j}{\partial u_j} = v_j - o$$

So $\frac{\partial v_j}{\partial u_j} = v_j - 0$.



Lemma

For any three non-obtuse angles $\tau_{ij}, \tau_{jk}, \tau_{ki} \in [0, \frac{\pi}{2})$ and any three positive numbers r_1, r_2 and r_3, there is a configuration of 3 circles in Euclidean geometry, unique upto isometry, having radii r_i and meeting in angles τ_{ij}.

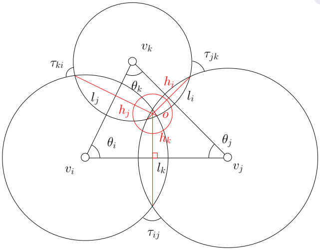
Proof.

$$max\{r_i^2, r_j^2\} < r_i^2 + r_j^2 + 2r_ir_j\cos\tau_{ij} \leq (r_i + r_j)^2$$

$$max\{r_i^2, r_j^2\} < l_k \leq r_i + r_j$$

so

$$l_k \leq r_i + r_j < l_i + l_j.$$



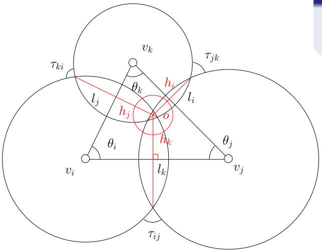
$\omega = \theta_i du_i + \theta_j du_j + \theta_k du_j$

Lemma

ω is closed 1-form in $\Omega := \{(u_1, u_2, u_3) \in \mathbb{R}^3\}$.

Because $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$, so

$$\begin{aligned} d\omega &= \left(\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_j}{\partial u_i}\right) du_j \wedge du_i + \\ &\quad \left(\frac{\partial \theta_j}{\partial u_k} - \frac{\partial \theta_k}{\partial u_j}\right) du_k \wedge du_j + \\ &\quad \left(\frac{\partial \theta_k}{\partial u_i} - \frac{\partial \theta_i}{\partial u_k}\right) du_i \wedge du_k \\ &= 0. \end{aligned}$$



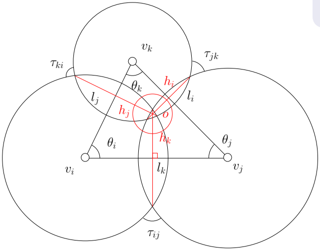
Lemma

The Ricci energy $E(u_1, u_2, u_3)$ is well defined.

Because $\Omega = \mathbb{R}^3$ is convex, closed 1-form is exact, therefore $E(u_1, u_2, u_3)$ is well defined,

$$E(u_1, u_2, u_3) = \int_{(0,0,0)}^{(u_1,u_2,u_3)} \omega.$$

Discrete Ricci Energy



Lemma

The Ricci energy $E(u_1, u_2, u_3)$ is strictly concave on the subspace $u_1 + u_2 + u_3 = 0$.

The gradient $\nabla E = (\theta_1, \theta_2, \theta_3)$, the Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial\theta_1}{\partial u_1} & \frac{\partial\theta_1}{\partial u_2} & \frac{\partial\theta_1}{\partial u_3} \\ \frac{\partial\theta_2}{\partial u_1} & \frac{\partial\theta_2}{\partial u_2} & \frac{\partial\theta_2}{\partial u_3} \\ \frac{\partial\theta_3}{\partial u_1} & \frac{\partial\theta_3}{\partial u_2} & \frac{\partial\theta_3}{\partial u_3} \end{pmatrix}$$

$$E(u_1, u_2, u_3) = \int_{(0,0,0)}^{(u_1,u_2,u_3)} \omega$$

because of $\theta_1 + \theta_2 + \theta_3 = \pi$,

$$\frac{\partial\theta_i}{\partial u_i} = -\frac{\partial\theta_i}{\partial u_j} - \frac{\partial\theta_i}{\partial u_k} = -\frac{\partial\theta_j}{\partial u_i} - \frac{\partial\theta_k}{\partial u_i}$$

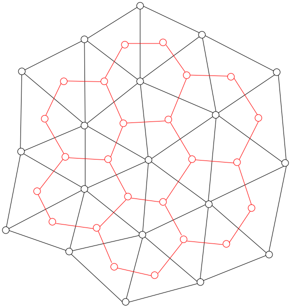
Ricci energy

Proof.

$$H = -\begin{pmatrix} \frac{h_3}{l_3} + \frac{h_2}{l_2} & -\frac{h_3}{l_3} & -\frac{h_2}{l_2} \\ -\frac{h_3}{l_3} & \frac{h_3}{l_3} + \frac{h_1}{l_1} & -\frac{h_1}{l_1} \\ -\frac{h_2}{l_2} & -\frac{h_1}{l_1} & \frac{h_2}{l_2} + \frac{h_1}{l_1} \end{pmatrix}$$

$-H$ is diagonal dominant, it has null space $(1,1,1)$, on the subspace $u_1 + u_2 + u_3 = 0$, it is strictly negative definite. Therefore the discrete Ricci energy $E(u_1, u_2, u_3)$ is strictly concave. □

Discrete Ricci Energy



$$\omega = \sum_{v_i \in M} K_i du_i$$

$$E(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{u}} \omega.$$

Lemma

The Ricci energy $E(\mathbf{u})$ is strictly convex on the subspace $\sum_{v_i \in M} u_i = 0$.

The gradient $\nabla E = (K_1, K_2, \cdots, K_n)$. The Ricci energy

$$E(\mathbf{u}) = 2\pi \sum_{v_i \in M} u_i - \sum_{[v_i, v_j, v_k] \in M} E_{ijk}(u_i, u_j, u_k)$$

where E_{ijk} is the ricci energy defined on the face $[v_i, v_j, v_k]$. The linear term won't affect the convexity of the energy. The null space of the Hessian is $(1, 1, \cdots, 1)$. In the subspace $\sum u_i = 0$, the energy is strictly convex.

Uniqueness

Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \to \mathbb{R}$ is a strictly convex function, then the map

$$\mathbf{x} \to \nabla f(\mathbf{x})$$

is one-to-one.

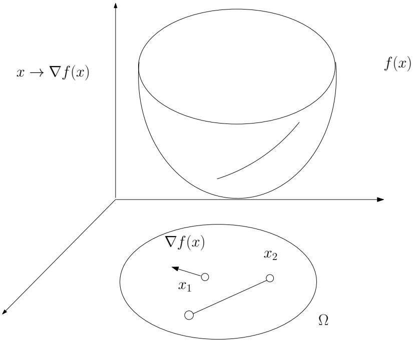
Proof.

Suppose $x_1 \neq x_2$, $\nabla f(x_1) = \nabla f(x_2)$. Because Ω is convex, the line segment $(1-t)x_1 + tx_2$ is contained in Ω. construct a convex function $g(t) = f((1-t)x_1 + tx_2)$, then $g'(t)$ is monotonous. But

$$g'(0) = \langle \nabla f(x_1), x_2 - x_1 \rangle = \langle \nabla f(x_2), x_2 - x_1 \rangle = g'(1),$$

contradiction. □

Uniqueness



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Uniqueness

Theorem (Global Rigidity)

Suppose M is a mesh, with circle packing metric, all edge intersection angles are non-obtuse. Given the target curvature $(K_1, K_2, \cdots, K_n)$, $\sum_i K_i = 2\pi\chi(M)$. If the solution $(u_1, u_2, \cdots, u_n) \in \Omega(M), \sum_i u_i = 0$ exists, then it is unique.

Proof.

The discrete Ricci energy E on $\Omega \cap \{\sum_i u_i = 0\}$ is convex,

$$\nabla E(u_1, u_2, \cdots, u_n) = (K_1, K_2, \cdots K_n).$$

Use previous lemma. □

Existence

Theorem (Thurston)

Suppose (T, Φ) *is a weighted generalized triangulation of a closed surface* M *and* I *is a proper subset of vertices of* V*, here the weight is a map* $\Phi : E \to [0, \frac{\pi}{2})$*. Then for any circle packing metric based on* (T, Φ)*, we have*

$$\sum_{i \in I} K_i(u) > - \sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi\chi(F_I),$$

where F_I *is the CW-subcomplex of cells whose vertices are in* I *and*

$$Lk(I) = \{(e, v) | v \in I, e \cap I = \emptyset, (e, v) \text{ form a triangle}\}$$