Circle Domain Mapping: Koebe's Theorem

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Motivation

Conformal Module for Poly-annulus

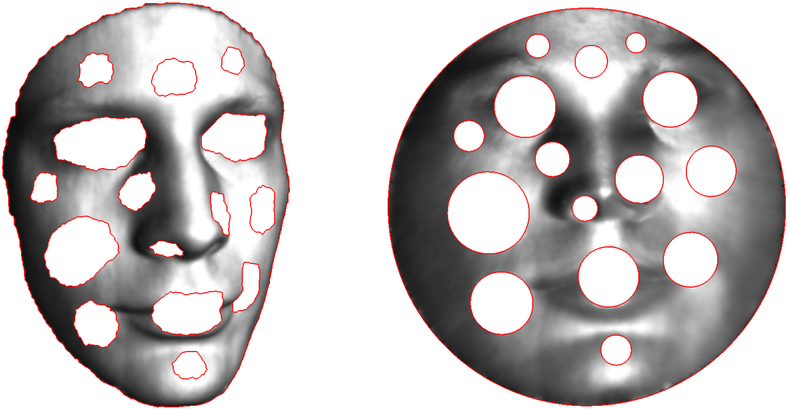


Figure: Conformal mapping from a poly-annulus to a circle domain.

Circle Domain

Definition (Circle Domain)

Suppose $\Omega \subset \hat{\mathbb{C}}$ is a planar domain, if $\partial\Omega$ has finite number of connected components, each of them is either a circle or a point, then Ω is called a circle domain.

Theorem (Koebe)

Suppose S is of genus zero, ∂S has finite number of connected components, then S is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.

Schwartz Reflection Principle

Definition (Mirror Reflection)

Given a circle $\Gamma : |z - z_0| = \rho$, the reflection with respect to Γ is defined as:

$$\varphi_\Gamma : re^{i\theta} + z_0 \mapsto \frac{\rho^2}{r} e^{i\theta} + z_0. \tag{1}$$

Two planar domains S and S' are symmetric about Γ, if $\varphi_\Gamma(S) = S'$.

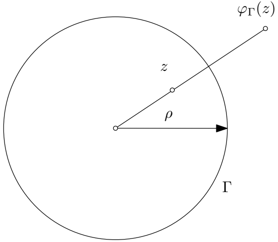


Figure: Reflection about a circle.

Schwartz Reflection Principle

Definition (Reflection)

Suppose Γ is an analytic curve, domain S, S' and Γ are included in a planar domain Ω. There is a conformal map $f : \Omega \to \hat{\mathbb{C}}$, such that $f(\Gamma)$ is a canonical circle, $f(S)$ and $f(S')$ are symmetric about $f(\Gamma)$, then we say S and S' are symmetric about Γ, and denoted as

$$S|S' \quad (\Gamma).$$

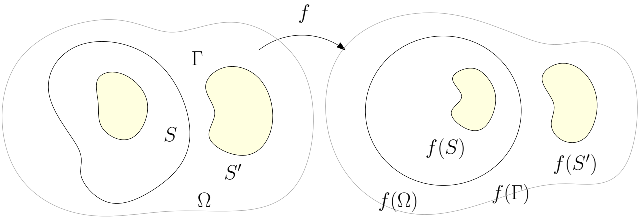


Figure: General symmetry.

Schwartz Reflection Principle

Theorem (Schwartz Reflection Principle)

Assume f is an analytic function, defined on the upper half disk $\{|z| < 1, \Im(z) > 0\}$. If f can be extended to a real continuous function on the real axis, then f can be extended to an analytic function F defined on the whole disk, satisfying

$$F(z) = \begin{cases} f(z), & \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \Im(z) < 0 \end{cases}$$

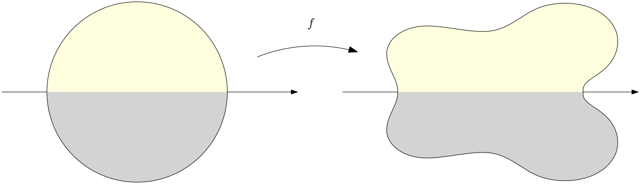
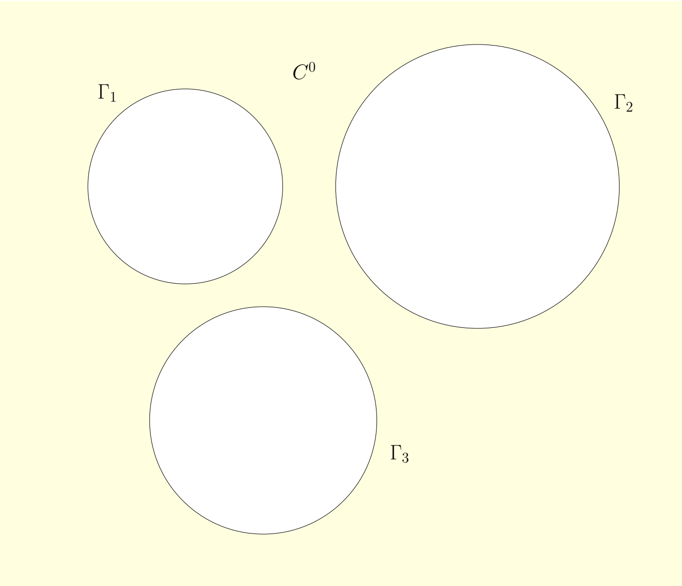
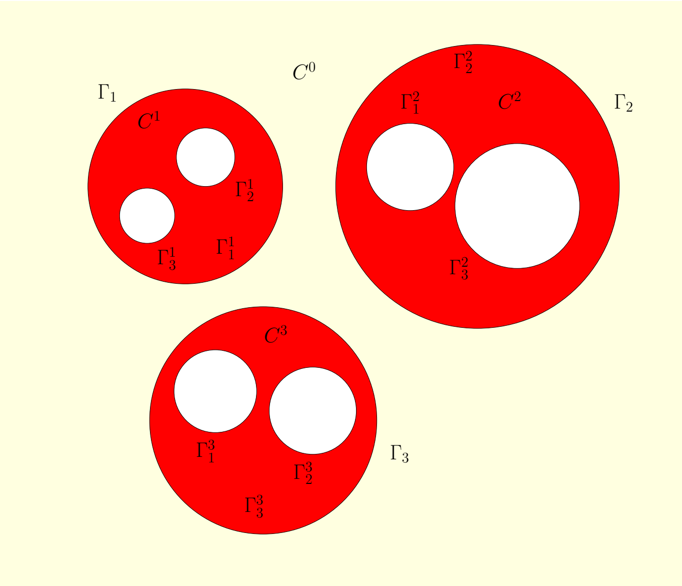


Figure: Schwartz reflection principle.

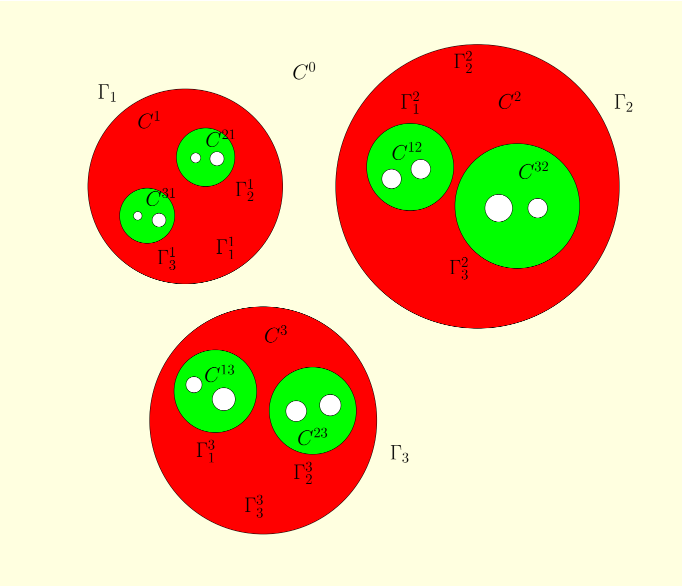
Multiple Reflection



Multiple Reflection



Multiple Reflection



Multiple Reflection

1. Initial circle domain C^0: complex plane remove three disks, its boundary is $\{\Gamma_1, \Gamma_2, \Gamma_3\}$;
2. First level reflection: C^0 is reflected about Γ_{i_1} to C^{i_1}, $i_1 = 1, 2, 3$;
$$\partial C^{i_1} = \Gamma_{i_1}^{i_1} - \sum_{j \neq i_1} \Gamma_j^{i_1},$$
where $\Gamma_{i_1}^{i_1} = \Gamma_{i_1}$.
3. Second level reflection: C^{i_1} is reflected about Γ_{i_2} to $C^{i_1 i_2}$, $i_1 \neq i_2$; the boundary of $C^{i_1 i_2}$ are $\Gamma_j^{i_1 i_2}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$.
$$\partial C^{i_1 i_2} = \Gamma_{i_1}^{i_2} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2}$$
when $j = i_1$, $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$;

Multiple Reflection

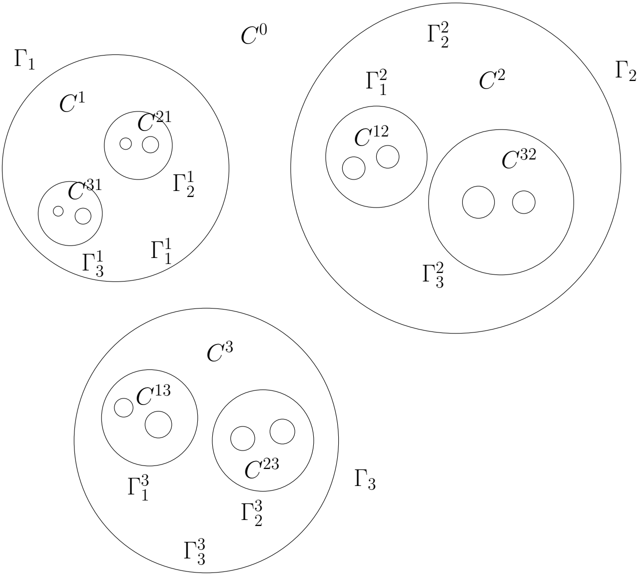
4. Third level reflection: $C^{i_1 i_2}$ is reflected about Γ_{i_3} to $C^{i_1 i_2 i_3}$, $i_1 \neq i_2$, $i_2 \neq i_3$; the boundary of $C^{i_1 i_2 i_3}$ are $\Gamma_j^{i_1 i_2 i_3}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 i_3}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3}$.

$$\partial C^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 i_3}.$$

5. The m-level reflection: $C^{i_1 i_2 \dots i_{m-1}}$ is reflected about Γ_{i_m} to $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$; the boundary of $C^{i_1 i_2 \dots i_{m-1} i_m}$, $i_k \neq i_{k+1}$ are $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$, when $j \neq i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is an interior boundary; when $j = i_1$, $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ is the exterior boundary, $\Gamma_{i_1}^{i_1 i_2 \dots i_{m-1} i_m} = \Gamma_{i_1}^{i_2 \dots i_{m-1} i_m}$ is an interior boundary,

$$\partial C^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 i_3 \dots i_m} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 \dots i_m}.$$

Multiple Reflection



Multiple Reflection

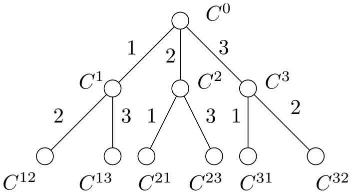


Figure: Reflection tree.

- Each node represents a domain $C^{i_1 i_2 \cdots i_m}$;
- Each edge represents a circle Γ_k, $k = 1, \ldots, n$;
- Father and Son share an edge i_1

$$\Gamma_{i_1}^{i_1 i_2 \cdots i_m} = \Gamma_{i_1}^{i_2 \cdots i_m}.$$

- Each node $C^{(i)}$, $(i) = i_1 i_2 \ldots i_m$ is the path from the root to $C^{(i)}$,

$$C^{(i)} = \varphi_{\Gamma_{i_m}} \circ \varphi_{\Gamma_{i_{m-1}}} \cdots \varphi_{\Gamma_{i_1}}(C^0).$$

Multiple Reflection

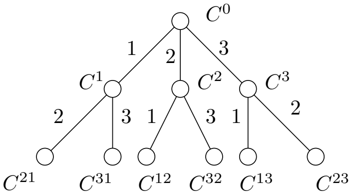


Figure: Embedding tree.

- Father node $C^{i_2\cdots i_m}$ and child node $C^{i_1 i_2\cdots i_m}$ is connected by edge i_1, the exterior boundary of child equals to an interior boundary of the father

$$\Gamma_{i_1}^{i_1 i_2\cdots i_m} = \Gamma_{i_1}^{i_2\cdots i_m}.$$

- From the root C^0 to $C^{i_1\cdots i_m}$, the path is inverse to the index

$$(i)^{-1} = i_m i_{m-1}\cdots i_2 i_1,$$

starting from C^0 crosses Γ^{i_m} to C^{i_m}, crosses $\Gamma_{i_{m-1}}^{i_m}$ to $C^{i_{m-1}i_m}$; when arrives at $C^{i_{k+1}\cdots i_1}$, crosses $\Gamma_{i_k}^{i_{k+1}\cdots i_1}$ to $C^{i_k i_{k+1}\cdots i_1}$; and eventually reach $C^{(i)}$.

Hole Area Estimation

Lemma

Suppose $C^{(i)}$ is an interior node in the reflection tree,

$$(i) = i_1 i_2 \cdots i_m,$$

its exterior boundary is $\Gamma_{i_1}^{(i)}$, interior boundaries are $\Gamma_j^{(i)}$, $j \neq i_1$, we have the estimate:

$$\sum_{j \neq i_1} \alpha(\Gamma_j^{(i)}) \leq \mu^4 \alpha(\Gamma_{i_1}^{(i)}).$$

Hole Area Estimation

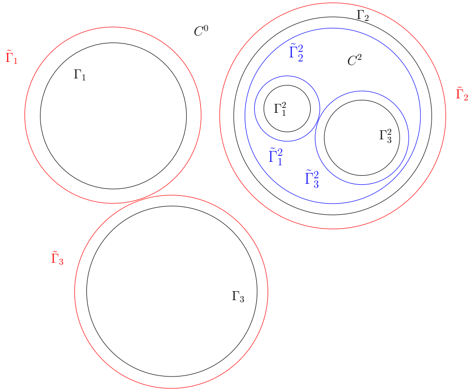


Figure: Hole area estimation.

Enlarge all Γ_k's by factor μ^{-1} to $\tilde\Gamma_k$, $\tilde\Gamma_1$ and $\tilde\Gamma_3$ touch each other; reflect C^0 about Γ_2

- $\Gamma_k | \Gamma_k^2 \quad (\Gamma_2)$.
- $\tilde\Gamma_k | \Gamma_k^2 \quad (\Gamma_2)$.

$$\alpha(\tilde\Gamma_1^2) = \mu^{-2}\alpha(\Gamma_1^2)$$
$$\alpha(\tilde\Gamma_3^2) = \mu^{-2}\alpha(\Gamma_3^2)$$
$$\alpha(\tilde\Gamma_2^2) = \mu^{2}\alpha(\Gamma_2)$$

$$\alpha(\Gamma_1^2)+\alpha(\Gamma_3^2) = \mu^2(\alpha(\tilde\Gamma_1^2)+\alpha(\tilde\Gamma_3^2)) \le \mu^2\alpha(\tilde\Gamma_2^2) = \mu^4\alpha(\Gamma^2).$$

Hole Area Estimation

Lemma

Suppose the boundaries of the initial circle domain C^0 *are* $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$*, consider the reflection tree with* m *layers, then the total area of the holes bounded by the interior boundaries of leaf nodes is no greater than* μ^{4m} *times the area bounded by* Γ_k*'s,*

$$\sum_{(i)=i_1 i_2 \ldots i_m} \sum_{k \neq i_1} \alpha(\Gamma_k^{(i)}) \leq \mu^{4m} \sum_{i=1}^{n} \alpha(\Gamma_i). \qquad (2)$$

Proof.

By induction on m. The area bounded by the exterior boundaries of the nodes in the $k+1$-layer is no greater than μ^4 times that of the k-layer. The total area of the interior boundaries of leaf nodes is no greater than the area bounded by the exterior boundaries of leaf nodes. □

Uniqueness

Theorem (Uniqueness)

Given two circle domains $C_1, C_2 \subset \hat{\mathbb{C}}$, $f : C_1 \to C_2$ is a univalent holomorphic function, then f is a linear rational, namely a Möbus transformation.

Proof.

Assume both C_1 and C_2 include ∞, and $f(\infty) = \infty$. Since f is holomorhic, it maps the boundary circles of C_1 to those of C_2. By Schwartz reflection principle, f can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. 2, the multiple reflected domains cover the whole $\hat{\mathbb{C}}$, hence f can be extended to the whole $\hat{\mathbb{C}}$, since $f(\infty) = \infty$, f is a linear function. If $f(\infty) \neq \infty$, we can use a Möbius map to transform $f(\infty)$ to ∞. □

Existence

Definition (Kernel)

Suppose $\{B_n\}$ is a family of domains on the complex plane, $\infty \in B_k$ for all k. Suppose B is the maximal set: $\infty \in B$, and for any closed set $K \subset B$, there is an N, such that for any $n > N$, $K \subset B_n$. Then B is called the kernel of $\{B_n\}$.

Definition (Domain Convergence)

We say a sequence $\{B_n\}$ converges to its kernel B, if any sub-sequence $\{B_{n_k}\}$ of $\{B_n\}$ has the same kernel B. We denote $B_n \to B$.

Goluzin Theorem

Theorem (Goluzin)

Let $\{A_n\}$ be a sequence of domains on the complex domain. Any domain A_n includes ∞, $n = 1, 2, \cdots ,$. Assume $\{A_n\}$ converges to its kernel A. Let $\{f_n(z)\}$ be a family of analytic function, for all n, $f_n(z)$ maps A_n to B_n surjectively, such that $f_n(\infty) = \infty$, $f'_n(\infty) = 1$. Then $\{f_n(z)\}$ uniformly converges to a univalent analytic function $f(z)$ in the interior of A, if and only if $\{B_n\}$ converges to its kernel B, then the univalent analytic function $f(z)$ maps A to B surjectively.

Existence

Theorem (Existence)

On the z-plane, every n-connected domain Ω can be mapped to a circle domain on the ζ-plane by a univalent holomorphic function. Choose a point $a \in \Omega$, there is a unique map which maps a to $\zeta = \infty$, and in a neighborhood of $z = a$, the map has the power series

$$\frac{1}{z-a} + a_1(z-a) + \cdots \text{ if } a \neq \infty$$

$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$

Existence

Proof.

According to Hilbert theorem, all n-connected domains are conformally equivalent to slit domains. We can assume Ω is a slit domain. We use $\mathcal{S}$ represent all the n-connected slit domains with horizontal slits, and $\mathcal{C}$ the n-connected circle domains. We label all the boundaries of the domains, $\partial\Omega = \bigcup_{k=1}^{n} \gamma_k$. For each slit γ_k, we represent it by the starting point p_k and the length l_k, then we get the coordinates of the slit domain Ω

$$(p_1, l_1, p_2, l_2, \cdots, p_n, l_n).$$

Hence $\mathcal{S}$ is a connected open set in $\mathbb{R}^{3n}$. Similarly, consider a circle domain $\mathcal{D} \in \mathcal{C}$, we use the center and the radius to represent each circle (q_k, r_k), and the coordinates of $\mathcal{D}$ are given by,

$$(q_1, r_1, q_2, r_2, \cdots, q_n, r_n).$$

$\mathcal{C}$ is also a connected open set in $\mathbb{R}^{3n}$. □

Existence

continued

Consider a normalized univalent holomorphic function $f : \Omega \to \mathcal{D}$, $\Omega \in \mathcal{S}$ and $\mathcal{D} \in \mathcal{C}$, f maps the k-th boundary curve γ_k to the k-th circular boundary of $\mathcal{D}$. By the existence of slit mapping and the uniqueness of circle domain mapping, we have

1. Every circle domain $\mathcal{D} \in \mathcal{C}$ corresponds to a unique slit domain $\Omega \in \mathcal{S}$;
2. Every slit domain $\Omega \in \mathcal{S}$ corresponds to at most one circle domain $\mathcal{D} \in \mathcal{C}$.

Then we establish a mapping from circle domains to slit domains $\varphi : \mathcal{C} \to \mathcal{S}$.

Existence

continued

Assume $\{\mathcal{D}_n\}$ is a family of circle domains, converge to the kernel $\mathcal{D}^*$. The domain convergence definition is consistent with the convergence of coordinates, namely, the boundary circles of $\mathcal{D}_n$ converge to the corresponding boundary circles of $\mathcal{D}^*$, denoted as $\lim_{n\to\infty}\mathcal{D}_n = \mathcal{D}^*$. The convergence of slit domains can be similarly defined. By Goluzin's theorem, we obtain the mapping $\varphi : \mathcal{C} \to \mathcal{S}$ is continuous:

$$\varphi(\lim_{n\to\infty}\mathcal{D}_n) = \lim_{n\to\infty}\varphi(\mathcal{D}_n).$$

By the uniqueness of circle domain mapping, we obtain φ is injective. We will prove the mapping φ is surjective.

continued

$\mathcal{C}$ is an open set in Euclidean space $\varphi : \mathcal{C} \to \mathcal{S}$ is injective continuous map. According to invariance of domain theorem, $\varphi(\mathcal{C})$ is an open set, $\varphi : \mathcal{C} \to \varphi(\mathcal{C})$ is a homeomorphism.

Choose a circle domain $\mathcal{D}_0 \in \mathcal{C}$, its corresponding slit domain is $\varphi(\mathcal{D}_0) = \Omega_0 \in \mathcal{S}$, then $\Omega_0 \in \varphi(\mathcal{C})$. Choose another slit map $\Omega_1 \in \mathcal{S}$, we don't know if Ω_1 is in $\varphi(\mathcal{C})$ or not. We draw a path $\Gamma : [0,1] \to \mathcal{S}$, $\Gamma(0) = \Omega_0$ and $\Gamma(1) = \Omega_1$. Let

$$t^* = \sup\{t \in [0,1] | \forall 0 \leq \tau \leq t, \Gamma(\tau) \in \varphi(\mathcal{C})\},$$

namely Γ from starting point to t^* belongs to $\varphi(\mathcal{C})$.

Existence

continued

By the definition of domain convergence,

$$\lim_{n\to\infty} \Gamma(t_n) \to \Gamma(t^*).$$

By $\{\Gamma(t_n)\} \subset \varphi(\mathcal{C})$, there is a family of circle domains $\{\mathcal{D}_n\} \subset \mathcal{C}$, $\varphi(\mathcal{D}_n) = \Gamma(t_n)$. Let $\lim_{n\to\infty} \mathcal{D}_n = \mathcal{D}^*$, by domain limit theorem, we have

$$\varphi(\mathcal{D}^*) = \varphi(\lim_{n\to\infty} \mathcal{D}_n) = \lim_{n\to\infty} \varphi(\mathcal{D}_n) = \lim_{n\to\infty} \Gamma(t_n) = \Gamma(t^*),$$

namely $\varphi(\mathcal{D}^*) = \Gamma(t^*)$, hence $\Gamma(t^*) \in \varphi(\mathcal{C})$. But $\varphi(\mathcal{C})$ is an open set, hence if $t^* < 1$, t^* can be further extended. This contradict to the choice of t^*, hence $t^* = 1$. Therefore $\Omega_1 \in \varphi(\mathcal{C})$. Since Ω_1 is arbitrarily chosen, hence $\varphi : \mathcal{C} \to \mathcal{S}$ is surjective. This proves the existence of the circle domain mapping.

Convergence of Koebe Iteration Method

Koebe Iteration Algorithm

Input: Poly annulus M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;
Output:Conformal map $\varphi : M \to \mathbb{D}$, where $\mathbb{D}$ is a circle domain.

1. Compute a slit map, map the surface to the circular slit domain $f : M \to \mathbb{C}$, γ_0 and γ_k are mapped to the exetior and interior circular boundary of $\mathbb{C}$;
2. Fill the inner circle using Delaunay refinement mesh generation;
3. Repeat step 1 and 2, fill all the holes step by step;

Koebe Iteration Method

Figure: Slit map.

Koebe Iteration Method

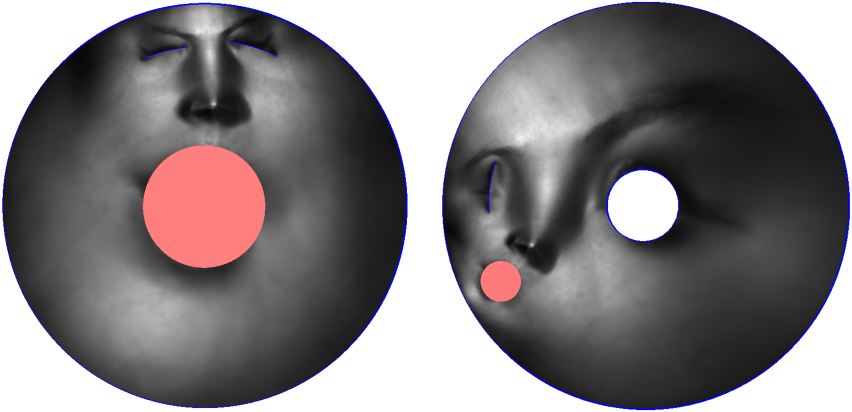


Figure: Hole filling and slit map.

Koebe Iteration Method

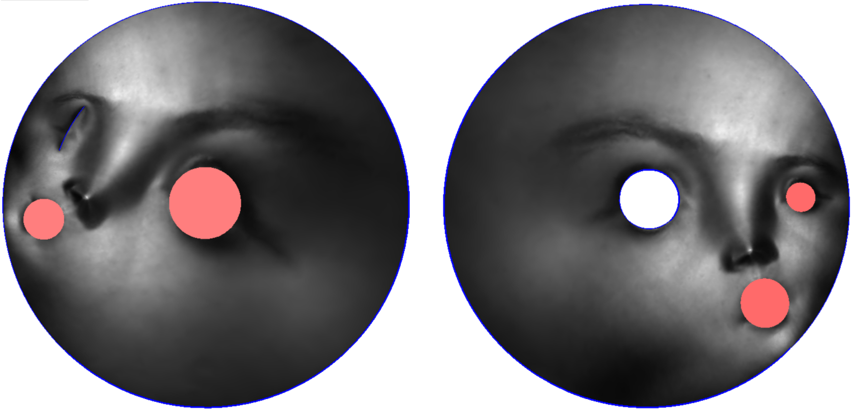


Figure: Hole filling and slit map.

Koebe Iteration Method

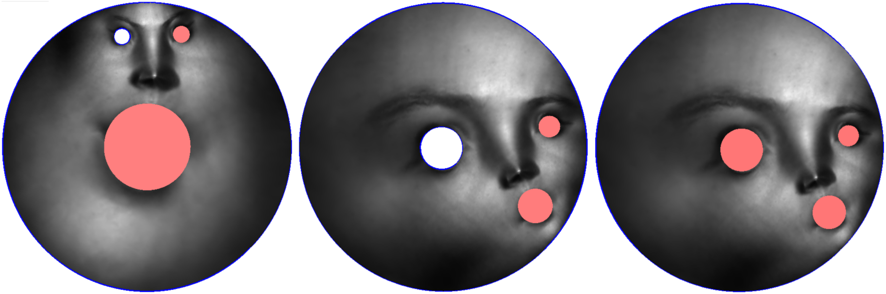
Figure: All holes are filled.

Koebe Iteration Algorithm

4. Puch a hole at the k-th inner boundary;
5. Compute a conformal map, to map the surface onto a canonical planar annulus;
6. Fill the inner circular hole;
7. Repeat step 4 through 6, each time punch a different hole, until the process convergences.

Koebe Iteration Method

Koebe Iteration Method



Koebe Iteration Method

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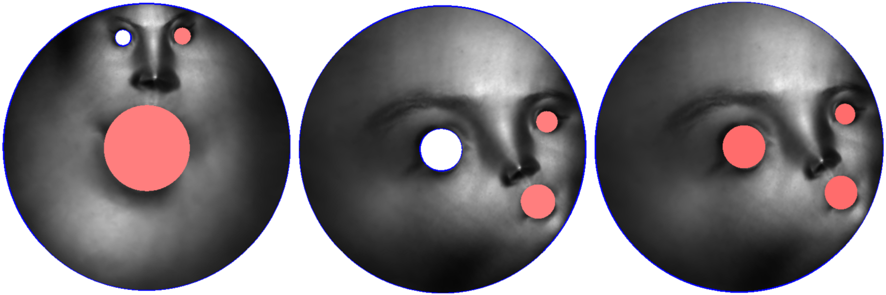
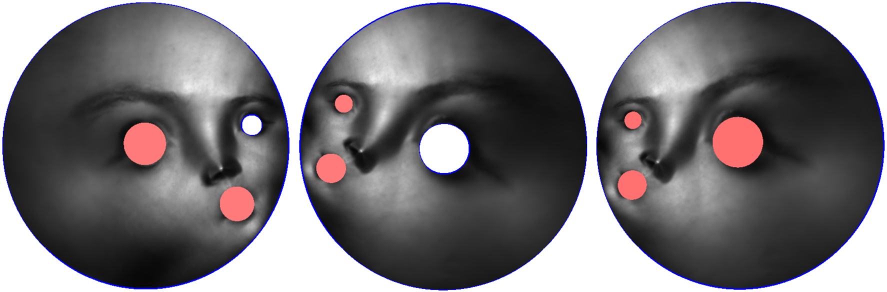


Koebe Iteration Method



Koebe Iteration Method

Koebe Iteration Method

Figure: Final result.

Area, Diameter Estimate

Lemma

Suppose A is a topological annulus on $\mathbb{C}$, the conformal module of A is $\mu^{-1} > 1$, the exterior and interior boundaries of A are Jorgan curves Γ_0 and Γ_1, $\partial A = \Gamma_0 - \Gamma_1$, then we have the area and diameter estimates:

$$\alpha(\Gamma_1) \leq \mu^2 \alpha(\Gamma_0), \tag{3}$$

and

$$[diam\Gamma_1]^2 \leq \frac{\pi}{2\log\mu^{-1}}\alpha(\Gamma_0), \tag{4}$$

where $\alpha(\Gamma_k)$ is the area bounded by Γ_k, $k = 0, 1$.

Area, Diameter Estimate

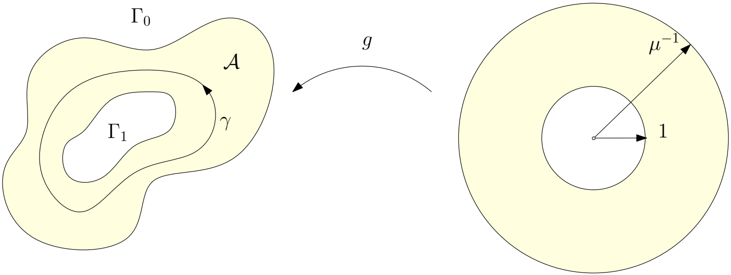


Figure: Topological annulus with conformal module μ^{-1}.

Area, Diameter Estimate

Proof.

Let holomorphic function g maps $\{1 \leq |w| \leq \mu^{-1}\}$ to A,

$$g(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \cdots$$

By Gnowell area estimate, we have

$$\alpha(\Gamma_1) = \pi \left(1 - \sum_{n=1}^{\infty} n|a_n|^2\right)$$
$$\alpha(\Gamma_0) = \pi \left(\mu^{-2} - \sum_{n=1}^{\infty} n|a_n|^2 \mu^{2n}\right)$$

hence, this proves the area inequality (3)

$$\alpha(\Gamma_0) - \mu^{-2}\alpha(\Gamma_1) = \pi \sum^{\infty} n|a_n|^2(\mu^{-2} - \mu^{2n}) \geq 0$$

Area, Diameter Estimate

Continued

The diameter $\text{diam}\Gamma_1$ is determined by $g(\{1 < |w| < \rho\})$, where $\rho \in (1, \mu^{-1})$. The diameter is bounded by half of the boundary length $g(|w| = \rho)$, we have

$$2\text{diam}\Gamma_1 \leq \int_{|w|=\rho} |g'(w)|dw = \int_0^{2\pi} |g'(\rho e^{i\theta})|\rho\theta = \int_0^{2} \pi|g'(\rho e^{i\theta})|\sqrt{\rho}\sqrt{\rho}d\theta,$$

By Schwartz inequality, we have

$$[2\text{diam}\Gamma_1]^2 \leq \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta \int_0^{2\pi} \rho d\theta = 2\pi\rho \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta$$

Area, Diameter Estimate

Continued

Equivalent

$$\frac{2}{\pi\rho}[\text{diam}\Gamma_1]^2 \leq \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta$$

Integrate with respect to ρ,

$$\int_1^{\mu^{-1}} \frac{2}{\pi\rho}[\text{diam}\Gamma_1]^2 d\rho \leq \int_1^{\mu^{-1}} \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta d\rho = \alpha(\Gamma_0) - \alpha(\Gamma_1).$$

Calculate left

$$\frac{2\log\mu^{-1}}{\pi}[\text{diam}\Gamma_1]^2 \leq \alpha(\Gamma_0) - \alpha(\Gamma_1) \leq \alpha(\Gamma_0).$$

This proves inequality (4).

Multiple Reflected Domain

Definition (Multi-reflected circle domain)

Given an m-level embedding relation tree of a circle domain C, the union of all nodes in the tree is called a multiple-reflected circle domain,

$$\Omega_m = \bigcup_{k\leq m} \bigcup_{(i)=i_1 i_2\cdots i_k} C^{(i)} = \hat{\mathbb{C}} \setminus \bigcup_{(i)=i_1 i_2\cdots i_m} \bigcup_{k\neq i_1} \alpha(\Gamma_k^{(i)})$$

where $\alpha(\Gamma)$ is the area bounded by Γ.

Suppose we have a holomorphic univalent map $g_m : \Omega_m \to \hat{\mathbb{C}}$, define

$$C_m = g_m(C^0)$$
$$C_m^{(i)} = g_m(C^{(i)})$$
$$\Gamma_{m,k} = g_m(\Gamma_k)$$
$$\Gamma_{m,k}^{(i)} = g_m(\Gamma_k^{(i)})$$

Symmetric Relation

According to the reflection generation tree, we have the symmetry

$$C^{i_1 i_2 \cdots i_{m-1} i_m} \mid C^{i_1 i_2 \cdots i_{m-1} i_m} \quad (\Gamma_{i_m})$$

this symmetric relation is preserved by the holomorphic map g_m:

$$C_m^{i_1 i_2 \cdots i_{m-1} i_m} \mid C_m^{i_1 i_2 \cdots i_{m-1} i_m} \quad (\Gamma_{m,i_m})$$

therefore g_m maps the embedding relation tree of $\left\{ C^{(i)} \right\}$ to the embedding relation tree of $\left\{ C_m^{(i)} \right\}$.

Hole Area Estimation

Lemma

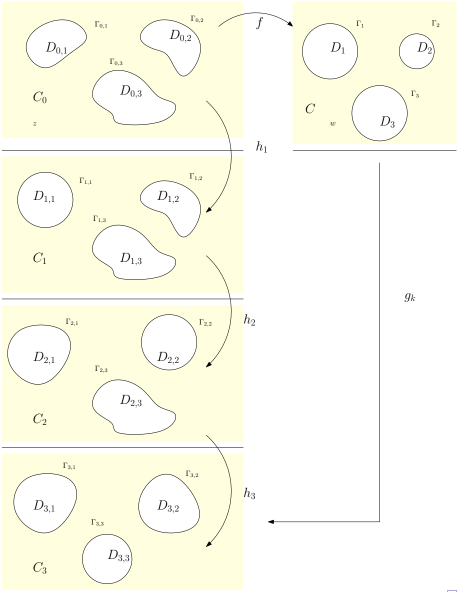
Suppose the boundaries of C_m are $\Gamma_{m,1}, \Gamma_{m,2}, \dots, \Gamma_{m,n}$. In the m-level embedding relation tree of C_m, the total area of the holes bounded by the interior boundaries of leaf nodes is less than μ^{4m} times the total area of holes bounded by $\Gamma_{m,k}$'s,

$$\sum_{(i)=i_1 i_2 \cdots i_m} \sum_{k \neq i_1} \alpha(\Gamma_{m,k^{(i)}}) \leq \mu^{4m} \sum_{i=1}^{n} \alpha(\Gamma_{m,i}). \quad (5)$$

Proof.

Using area estimate (3) and induction on m. □

Koebe's Iteration



Koebe's Iteration

Key Observation

Given a multi-annulus $\mathcal{R}$, there is a bioholomorphic map $g : \mathcal{C} \to \mathcal{R}$ maps a circle domain $\mathcal{C}$ to $\mathcal{R}$. During the process of Koebe's iteration, the domain of the mapping $\mathcal{C}$ can be extended to the image of the multiple reflection, (multiple reflected circle domain), which eventually covers the whole augmented complex plane $\hat{\mathbb{C}}$.

Koebe's Iteration

Lemma

During Koebe's iteration, at the mn-th step, the algorithm generates a univalent holomorphic function g_{mn}, its domain is extended to the m-level reflected circle domain,

$$g_{mn} : \Omega_m \to \hat{\mathbb{C}}.$$

Proof.

Initial domain is C_0, $\infty \in C_0$, the complementary sets are $D_{0,1}, D_{0,2}, \cdots, D_{0,n}$, $\partial D_{0,i} = \Gamma_{0,i}$, $i = 1, 2, \cdots, n$.
There is a biholomorphic function, $f : C_0 \to \mathcal{C}$, the complementary of $\mathcal{C}$ is the set $D_1, D_2, \cdots, D_n$, where D_i's are disks, $\partial D_i = \Gamma_i$ is a canonical circle. In the neighborhood of ∞, $f(z) = z + O(z^{-1})$. □

continued.

By Riemann mapping theorem, there is a Riemann mapping

$$h_1 : \hat{\mathbb{C}} \setminus D_{0,1} \to \hat{\mathbb{C}} \setminus \mathbb{D},$$

maps $\Gamma_{0,1}$ to the unit circle $\Gamma_{1,1}$, C_0 to C_1, satisfying the normalization condition,

$$h_1(\infty) = \infty, \quad h_1'(\infty) = 1,$$

and

$$D_{1,k} = h_1(D_{0,k}),\ k = 2, \cdots, n.$$

Repeat this procedure, at $k \leq n$ step, construct a Riemann mapping,

$$h_k : \hat{\mathbb{C}} \setminus D_{k-1,k} \to \hat{\mathbb{C}} \setminus \mathbb{D},$$

which maps $\Gamma_{k-1,k}$ to the unit circle, C_{k-1} to C_k, $h_k(\infty) = \infty$ and $h'(\infty) = 1$.

continued.

We recursively define the symbols as follows:

$$C_k = h_k(C_{k-1})$$
$$\Gamma_{k,i} = h_k(\Gamma_{k-1,i}), i \neq k$$
$$D_{k,i} = h_k(D_{k-1}, i), i \neq k$$

$D_{k,k}$ is the unit disk $\mathbb{D}$, $\Gamma_{k,k}$ the unit circle. We construct a biholomorphic map $f_k : C_0 \to C_k$:

$$f_k = h_k \circ h_{k-1} \circ \cdots h_1$$

and the biholomorphic map from the circle domain $\mathcal{C}$ to C_k, $g_k : \mathcal{C} \to C_k$,

$$g_k := f_k \circ f^{-1},$$

g_k satisfies normalization condition $g_k(\infty) = \infty$, $g_k'(\infty) = 1$.

Koebe's Iteration

continued.

We generalize the domain of g_k to multiple reflected circle domain. Because $\Gamma_{1,1}$ is a canonical circle, C_1 can be reflected about $\Gamma_{1,1}$ to C_1^1,

$$C_1|C_1^1 \quad (\Gamma_{1,1})$$

$h_2 : \hat{\mathbb{C}} \setminus D_{1,2} \to \hat{\mathbb{C}} \setminus \mathbb{D}$, hence h_2 is well defined on $D_{1,1}$. we denote

$$C_2^1 := h_2(C_1^1), \quad C_2^1|C_2 \quad (\Gamma_{2,1}).$$

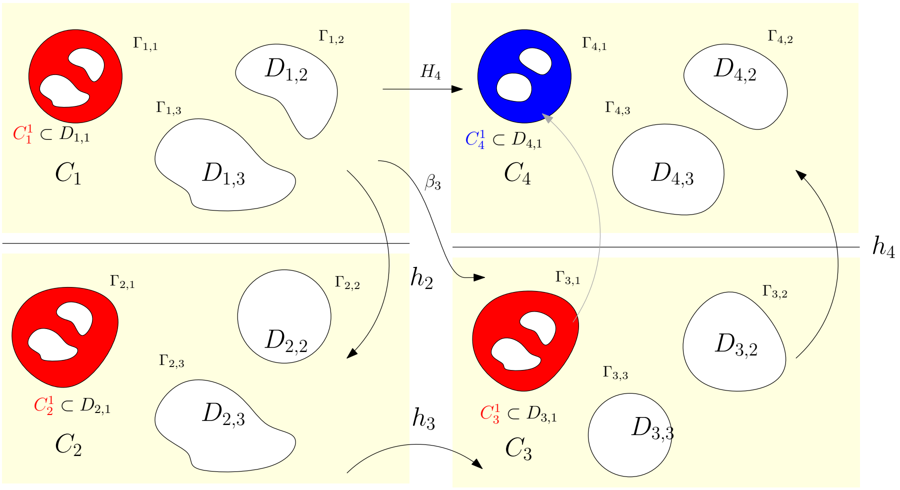
when $k = 2, 3, \cdots, n$, the Riemann mapping h_k is well defined on $C_k \cup D_{k,1}$, domain

$$C_k^1 := h_k \circ h_{k-1} \circ \cdots \circ h_1(C_1^1), \ k = 2, \cdots, n,$$

satisfying

$$C_k^1|C_k \quad (\Gamma_{k,1}).$$

Koebe's Iteration



Koebe's Iteration

continued.

But the map h_{n+1} on $D_{n,1}$ is not defined. We can use Schwartz reflection to define C^1_{n+1}. Consider the composition:

$$\beta_n := h_n \circ h_{n-1} \circ \cdots \circ h_2 : C_1 \to C_n,$$

β_n is well defined on $D_{1,1}$.

$$h_{n+1} \circ \beta_n : C_1 \to C_{n+1},$$

maps the circle $\Gamma_{1,1}$ to the circle $\Gamma_{n+1,1}$, but is not defined on $D_{1,1}$. By Schwartz reflection principle, the map $h_{n+1} \circ \beta_n$ can be extended to

$$H_{n+1} : C_1 \cup C_1^1 \to C_{n+1} \cup C^1_{n+1},$$

where

$$C^1_{n+1} | C_n \quad (\Gamma_{n+1,1}).$$

Koebe's Iteration

Continued.

$$\begin{array}{ccc} C_1 \cup C_1^1 & \xrightarrow{\beta_n} & C_n \cup C_n^1 \\ H_{n+1}\downarrow & & \downarrow H_{n+1}\circ\beta_n^{-1} \\ C_{n+1} \cup C_{n+1}^1 & \xrightarrow{Id} & C_{n+1} \cup C_{n+1}^1 \end{array}$$

we obtain the composition map

$$H_{n+1} \circ \beta_n^{-1} : C_n \cup C_n^1 \to C_{n+1} \cup C_{n+1}^1.$$

for convenience, we still use h_{n+1} to represent $H_{n+1} \circ \beta_n^{-1}$. Hence, we extend the domain of h_{n+1} to C_n^1: $h_{n+1} : C_n \cup C_n^1 \to C_{n+1} \cup C_{n+1}^1$. Repeat this procedure, we conclude: for all $k \geq 1$, C_k^1 and C_k are symmetric,

$$C_k^1 | C_k \quad (\Gamma_{k,1}).$$

Koebe's Iteration

Continued.

Similarly, when $k = 2$, $\Gamma_{2,2}$ is a circle, C_2^2 and C_2 are symmetric about $\Gamma_{2,2}$. When $k > 2$, we define

$$C_k^2 := h_k \circ h_{k-1} \circ \cdots h_3(C_2^2),$$

similarly, for every h_{kn+2} map, we use Schwartz reflection principle to extend analytically. For all $k \geq 2$, $C_k 2$ and C_k are symmetric:

$$C_k^2 | C_k \quad (\Gamma_{k,2}).$$

Similarly, for any $i = 3, \cdots, n$, we use Schwartz reflection principle to extend the domain and define C_k^i symmetric to C_k, for all $k \geq i$,

$$C_k^i | C_k \quad (\Gamma_{k,i}).$$

Koebe's Iteration

Continued.

After the first round of iterations, all C_k^i, $i = 1, 2, \cdots, n$ are defined. Since $\Gamma_{n+1,1}$ is the unit circle, we define C_{n+1}^{i1} to be the mirror image of C_{n+1}^i with respect to $\Gamma_{n+1,1}$, $C_{n+1}^{11} = C_{n+1}$, but all other C_{n+1}^{i1} are newly generated domains $i \neq 1$. Apply the extened Riemann mapping, we get a series of mirror images:

$$C_k^{i1}|C_k^i \quad (\Gamma_{k,1}),\ \forall k \geq n+1, i = 2, 3, \cdots, n.$$

Similarly, we can define mirror image domains:

$$C_k^{ij}|C_k^i \quad (\Gamma_{k,j}), \quad \forall k \geq n+j.$$

Koebe's Iteration

Continued.

After mn iterations, we obtain m-level mirror images $C_k^{i_1 i_2 \cdots i_m}$, satisfying the symmetric relation:

$$C_k^{i_1 i_2 \cdots i_m i_{m+1}} | C_k^{i_1 i_2 \cdots i_m} \quad (\Gamma_k, i_{m+1}), \quad k \geq mn + i_{m+1},$$

Now the j-th boundary of $C_k^{i_1 i_2 \cdots i_m i_{m+1}}$ is denoted as $\Gamma_{k,j}^{i_1 i_2 \cdots i_m i_{m+1}}$,

$$\partial C_k^{i_1 i_2 \cdots i_m i_{m+1}} = \Gamma_{k,i_1}^{i_1 i_2 \cdots i_m i_{m+1}} - \bigcup_{j \neq i_1}^{n} \Gamma_{k,j}^{i_1 i_2 \cdots i_m i_{m+1}}.$$

Koebe's Iteration

Continued.

Consider $g_k^{-1} = f \circ f_k^{-1}$, for all k we have

$$C = g_k^{-1}(C_k)$$

similarly,

$$C^{i_1 i_2 \cdots i_m} = g_k^{-1}(C_k^{i_1 i_2 \cdots i_m})$$

and its boundaries

$$\Gamma_j^{i_1 i_2 \cdots i_m} = g_k^{-1}(\Gamma_{k,j}^{i_1 i_2 \cdots i_m}).$$

Error Estimate

The circle domain $C = C^0$ is reflected about $\Gamma_{i_1}, \Gamma_{i_2}, \cdots, \Gamma_{i_k}$ sequentially, to a k-level mirror relfection image $C^{i_1 i_2 \cdots i_k}$, its interior boundary is

$$\Gamma_j^{i_1 i_2 \cdots i_k} = \Gamma_j \ldots \ldots (i), \quad j \neq i_1,$$

such that i_l and i_{l+1} are not equal. After analytic extension, g_k is defined on the augmented complex plane with $n(n-1)^{k-1}$ disks removed. The boundaries of these disks are

$$\bigcup_{i_1 i_2 \cdots i_k, i_l \neq i_{l+1}} \bigcup_{j \neq i_1} \Gamma_j^{i_1 i_2 \cdots i_k}$$

Error Estimate

We choose a big circle Γ_ρ, enclosing all the initial boundaries Γ_j. For any point $w \in C^0$, by Cauchy formula

$$g(w) - w = \frac{1}{2\pi i}\oint_{\Gamma_\rho} \frac{g_k(s) - w}{s - w} ds - \sum_{(i),j} \frac{1}{2\pi i}\oint_{\Gamma_j^{(i)}} \frac{g_k(s) - w}{s - w} ds$$

at ∞ neighborhood, $g_k(w) - w = O(w^{-1})$, when $\rho \to \infty$

$$\frac{1}{2\pi i}\oint_{\Gamma_\rho} \frac{g_k(s) - w}{s - w} ds = \frac{1}{2\pi i}\oint_{\Gamma_\rho} \frac{g_k(s) - s}{s - w} + \frac{s - w}{s - w} ds \to 0.$$

Error Estimate

Since w is outside all $\Gamma_j^{(i)}$, integration

$$\frac{1}{2\pi i}\oint_{\Gamma_j^{(i)}} \frac{w}{s-w} ds = 0,$$

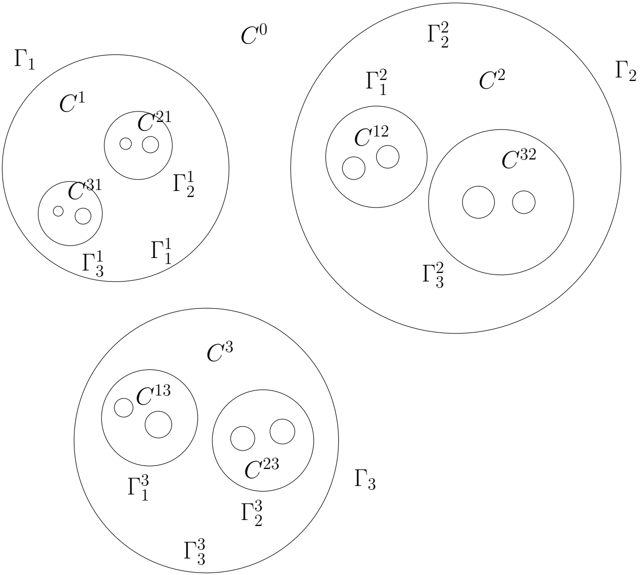
for any complex number $c_j^{(i)}$, integration

$$\frac{1}{2\pi i}\oint_{\Gamma_j^{(i)}} \frac{c_j^{(i)}}{s-w} ds = 0$$

we obtain

$$g_k(w) - w = -\sum_{(i),j} \frac{1}{2\pi i}\oint_{\Gamma_j^{(i)}} \frac{g_k(s) - c_j^{(i)}}{s-w} ds$$

Multiple Reflection



Error Estimate

In the initial circle domain C^0, let distance constant

$$\delta := \min_{i \neq j} \text{dist}(\Gamma_i, \Gamma_j^i),$$

we have $\delta > 0$. Since $\Gamma_j^{(i)} \subset \Gamma_{i_{m-1}}^{i_m}$, $|s - w| > \delta$. Define

$$\delta_{k,j}^{(i)} := \text{diam}(\Gamma_{k,j}^{(i)},$$

the curve $\Gamma_{k,j}^{(i)} = g_k(\Gamma_j^{(i)})$ is enclosed by the circle centered as $c_j^{(i)}$ and diameter $\delta_{k,j}^{(i)}$, then for all $s \in \Gamma_j^{(i)}$,

$$|g_k(s) - c_j^{(i)}| \leq \delta_{k,j}^{(i)},$$

the length of the integration is $\pi\delta_j^{(i)}$, where $\delta_j^{(i)} = \text{diam}(\Gamma_j^{(i)})$.

Error Estimate

$$|g_k(w) - w| \le \sigma_{(i),j} \frac{1}{2\pi} \oint_{\Gamma_j^{(i)}} \frac{|g_k(s) - c_j^{(i)}|}{|s - w|} |ds| \le \sum_{(i),j} \frac{1}{2\pi} \frac{\delta_{k,j}^{(i)}}{\delta} \pi \delta_j^{(i)}$$

$$= \sum_{(i),j} \frac{1}{2\delta} \delta_{k,j}^{(i)} \delta_j^{(i)} \le \sum_{(i),j} \frac{1}{4\delta} \left([\delta_{k,j}^{(i)}]^2 + [\delta_j^{(i)}]^2 \right)$$

For the first term,

$$\sum_{(i),j} [\delta_j^{(i)}]^2 = \frac{4}{\pi} \sum_{(i),j} \alpha(\Gamma_j^{(i)}) \le \mu^{4m} \sum_j \alpha(\Gamma_j) = \frac{4}{\pi} \mu^{4m} \gamma_1,$$

where $\sum_j \alpha(\Gamma_j) = \gamma_1$.

Error Estimate

For the second term, consider the topological annlus bounded by $\tilde{\Gamma}^{(i)}_{k,j}$ and $\Gamma^{(i)}_{k,j}$, by the diameter estimation (4), we obtain

$$[\delta^{(i)}_{j,k}]^2 \leq \frac{\pi}{2\log\mu^{-1}}\alpha(\tilde{\Gamma}^{(i)}_{k,j}),$$

By inequality (5), we obtain

$$\sum_{(i),j}[\delta^{(i)}_{j,k}]^2 \leq \frac{\pi}{2\log\mu^{-1}}\sum_{(i),j}\alpha(\tilde{\Gamma}^{(i)}_{k,j}) \leq \frac{\pi}{2\log\mu^{-1}}\sum_{j}\alpha(\tilde{\Gamma}_{k,j}) = \frac{\pi}{2\log\mu^{-1}}\mu^{4m}\gamma_2,$$

where $\gamma_2 = \sum_j \alpha(\tilde{\Gamma}_{k,j})$.

We estimate γ_1 and γ_2. The circle Γ_ρ enclose all the circles $\tilde{\Gamma}_i$, then $\gamma_1 < \pi\rho^2$. Using $g_k(w)$, we estimate γ_2. g_k is univalent on $|w| > \rho$, in the neighborhood of ∞, $g_k(w) = w + O(w^{-1})$. Perform coordinate change $\zeta = 1/w$, $\eta = 1/z$, construct univalent holomorphic function $\varphi : \zeta \to \eta$,

$$\varphi(\zeta) = \frac{1}{g_k(1/\zeta)},$$

φ is defined on the disk $|\zeta| < \rho^{-1}$, $\varphi(0) = 0$ and $\varphi'(0) = 1$. By Koebe 1/4 theorem,

$$\left\{|\eta| < \frac{1}{4\rho}\right\} \subset \varphi\left(\left\{|\zeta| < \frac{1}{\rho}\right\}\right),$$

equivalently

$$\{|z| > 4\rho\} \subset g_k(\{|w| > \rho\}),$$

hence all $\tilde{\Gamma}_{k,j}$ are included in the interior of $|z| < 4\rho$, hence the total area of all holes

$$\gamma_2 = \sum_j \alpha(\tilde{\Gamma}_{k,j}) < 16\pi\rho^2.$$

Error Estimate

We proved the convergence rate of Koebe's iteration.

Theorem (Convergence Rate of Koebe's Iteration)

In the Koebe's iteration, when $k > mn$,

$$|g_k(w) - w| \leq \frac{1}{4\delta}\left(\frac{4}{\pi}\pi\rho^2 + \frac{\pi}{2\log\mu^{-1}}16\pi\rho^2\right)\mu^{4m}.$$

This shows μ controls the convergence rate.