Topological Algorithm

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Algebraic Topology
Suppose $\sigma \in C_k$, $\sigma = [v_0, v_1, \cdots, c_k]$ Let $p$ be a permutation, then if \{ $p(0), p(1), \cdots, p(k)$ \} differs from \{ $0, 1, \cdots, k$ \} by even number of swaps, then $[v_{\rho(0)}, v_{\rho(1)}, \cdots, v_{\rho(k)}]$ has the same orientation with $\sigma$; if they differ by an odd number of swaps, then $[v_{\rho(0)}, v_{\rho(1)}, \cdots, v_{\rho(k)}] = \sigma^{-1}$. 
Suppose $M$ is a triangle mesh, with vertices, edges and faces $V, E, F$ respectively. Given two simplices $\sigma_i \in C_k$, $\tau_j \in C_{k-1}$, denote the adjacency number,

$$[\sigma_i, \tau_j] = \begin{cases} 
+1 & \tau_j \in \partial \sigma_i \\
-1 & \tau_j^{-1} \in \partial \sigma_i \\
0 & \tau_j \cap \sigma_i = \emptyset 
\end{cases}$$
V = \{v_i\} form the basis of $C_0$, $E = \{e_j\}$ form the basis of $C_1$, $F = \{f_k\}$ form the basis of $C_2$. $\partial_1 : C_1 \rightarrow C_0$ and $\partial_2 : C_2 \rightarrow C_1$ are linear maps, and represented as matrices.

$$\partial_2 = ([f_k, e_j]), \partial_1 = ([e_j, v_i]).$$

construct the following linear operator $\Delta : C_1 \rightarrow C_1$,

$$\Delta = \partial_2 \circ \partial_2^T + \partial_1^T \circ \partial_1,$$

the eigen vectors corresponding to zero eigen values are the basis of $H_1(\Sigma, \mathbb{Z})$. 

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Conformal Geometry
Definition (Cut Graph)

A cut graph $G$ of a mesh $\Sigma$ is a graph formed by non-oriented edges of $\Sigma$, such that $\Sigma/G$ is a topological disk.

Figure: Cut graph
Algorithm: Cut graph

Input: A triangular mesh $\Sigma$.
Output: A cut graph $G$

1. Compute the dual mesh $\tilde{\Sigma}$, each edge $e \in \Sigma$ has a unique dual edge $\tilde{e} \in \tilde{\Sigma}$.
2. Compute a spanning tree $\tilde{T}$ of $\tilde{\Sigma}$.
3. The cut graph is the union of all edges whose dual edges are in $\tilde{T}$.

$$G = \{ e \in \Sigma | \tilde{e} \not\in \tilde{T} \}.$$
Definition (Wedge)

On a face $f$, a corner at vertex $v$ is denoted as a pair $(f, v)$, and represented using halfedge data structure. Given a vertex $v$, the corners adjacent to it are ordered counter-clockwisely. A maximal sequence of adjacent corners without sharp edges form a wedge.

Figure: Cut graph
Fundamental Domain

Algorithm: Fundamental Domain

Input: A mesh $\Sigma$ and a cut graph $G$.
Output: A fundamental domain $\tilde{\Sigma}$.

1. Compute the cut graph $G$ of $\Sigma$, label all the edges in $G$ as sharp edges.
2. Compute the wedges of $\Sigma$ formed by sharp edges.
3. Construct an empty mesh $\tilde{\Sigma}$.
4. For each wedge $w$ insert a vertex into $\tilde{\Sigma}$.
5. For each face $f = [v_0, v_1, v_2]$ in $\Sigma$, insert a face $\tilde{f} = [w_0, w_1, w_2]$ into $\tilde{\Sigma}$, such that the corner $(f, v_k)$ belongs to the wedge $w_k$, $(f, v_k) \in w_k$. 

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Conformal Geometry
Figure: Fundamental domain
**Theorem (Homology basis)**

Suppose $\Sigma$ is a closed mesh, $G$ is a cut graph of $\Sigma$, then the basis of loops of $G$ (assigned with an orientation) is also a homology basis of $\Sigma$.

**Proof.**

The computation of the cut graph in fact find a CW-cell decomposition

$$\Sigma = G \cup D_2,$$

where $D_2$ is a 2-cell. Suppose $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ are the loop basis of $G$, $\partial D_2$ is a loop in $G$, which is represented as a word in $\pi_1(G)$, then

$$\pi_1(\Sigma) = \langle \gamma_1, \gamma_2, \cdots, \gamma_n | \partial D_2 \rangle.$$
Homology basis

Algorithm: Loop basis for a graph $G$

Input: A graph $G$.
Output: A basis of loops of $G$.

1. Compute a spanning tree $T$ of $G$.
2. $G/T = \{e_1, e_2, \cdots, e_n\}$.
3. $e_i \cup T$ has a unique loop, denoted as $\gamma_i$.
4. $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ form a basis of $\pi_1(G)$. 
Figure: Homology basis
Algorithm: Universal covering space

Input: A mesh $\Sigma$.
Output: A finite portion of the universal cover $\tilde{\Sigma}$.

1. Compute a cut graph $G$ of $\Sigma$. We call a vertex on $G$ with valence greater than 2 a knot. The knots divide $G$ to segments, assign an orientation to each segment, label the segments as $\{s_1, s_2, \cdots, s_n\}$.

2. Compute a fundamental domain $\tilde{\Sigma}$, induced by $G$, whose boundary is composed of $\pm s_k$’s.

3. Initialize $\tilde{\Sigma} \leftarrow \tilde{\Sigma}$, $\partial \tilde{\Sigma} \leftarrow \partial \tilde{\Sigma}$, represented using $\pm s_k$’s.

4. Glue one copy of $\tilde{\Sigma}$ to the current $\tilde{\Sigma}$ along only one segment $s_k \in \partial \tilde{\Sigma}$, $-s_k \in \partial \tilde{\Sigma}$, $\tilde{\Sigma} \leftarrow \tilde{\Sigma} \cup s_k \tilde{\Sigma}$.

5. Update $\partial \tilde{\Sigma}$, if $\pm s_k$ are adjacent in $\partial \tilde{\Sigma}$, glue them. Repeat this step, until there is no adjacent $\pm s_k$ in the boundary $\partial \tilde{\Sigma}$.

6. Repeat gluing copies of $\tilde{\Sigma}$ until $\tilde{\Sigma}$ is large enough.
Universal Covering Space

**Figure:** universal covering space
Algorithm: Cohomology Basis

Input: A homology basis \{γ_1, γ_2, \cdots, γ_n\}  
Output: A cohomology basis \{ω_1, ω_2, \cdots, ω_n\}

1. select γ_k, slice the mesh open along γ_k, to get an open mesh Σ_k, \(\partial Σ_k = γ_k^+ - γ_k^-\).
2. Set a 0-form \(f_k: Σ_k \rightarrow \mathbb{R}\), such that \(∀ v \in γ_k^+, f_k(v) = 1; \ ∀ v \in γ_k^-, f_k(v) = 0\).
3. \(δ_0 f_k\) is defined on Σ_k, because it is consistent on the corresponding edges on γ_k^+ and γ_k^-, so it is well defined on Σ as well.
Algorithm: Double Cover

Input: An open mesh with boundaries. Output: The double covering of the mesh, which is a closed symmetric mesh.

1. construct a copy of the input mesh $\Sigma'$
2. reverse the orientation of the copy, each $[v_0, v_1, v_2]$ is converted to $[v_1, v_0, v_2]$.
3. Identify each boundary vertex in $\partial \Sigma$ with the corresponding one in $\partial \Sigma'$,
4. each boundary halfedge $e \in \partial \Sigma$ has a unique corresponding halfedge $e' \in \Sigma'$. glue the corresponding boundary halfedges to the same boundary edge.
Figure: Double Covering