Fundamental Group

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Philosophy

Associate groups with manifolds, study the topology by analyzing the group structures.

\[ \mathcal{C}_1 = \{ \text{Topological Spaces, Homeomorphisms} \} \]
\[ \mathcal{C}_2 = \{ \text{Groups, Homomorphisms} \} \]
\[ \mathcal{C}_1 \rightarrow \mathcal{C}_2 \]

Functor between categories.
Suppose $q$ is a base point, all the oriented closed curves (loops) through $q$ can be classified by homotopy. All the homotopy classes form the so-called fundamental group of $S$, or the first homotopy group, denoted as $\pi_1(S, q)$. The group structure of $\pi_1(S, q)$ determines the topology of $S$. 
Let $S$ be a two manifold with a base point $p \in S$,

**Definition (Curve)**

A curve is a continuous mapping $\gamma : [0, 1] \rightarrow S$.

**Definition (Loop)**

A closed curve through $p$ is a curve, such that $\gamma(0) = \gamma(1) = p$.

**Definition (Homotopy)**

Let $\gamma_1, \gamma_2 : [0, 1] \rightarrow S$ be two curves. A homotopy connecting $\gamma_1$ and $\gamma_2$ is a continuous mapping $F : [0, 1] \times [0, 1] \rightarrow S$, such that

$$f(0, t) = \gamma_1(t), f(1, t) = \gamma_2(t).$$

We say $\gamma_1$ is homotopic to $\gamma_2$ if there exists a homotopy between them.
Lemma

Homotopy relation is an equivalence relation.

Proof.

\( \gamma \sim \gamma, F(s, t) = \gamma(t) \). If \( \gamma_1 \sim \gamma_2 \), \( F(s, t) \) is the homotopy, then \( F(1 - s, t) \) is the homotopy from \( \gamma_2 \) to \( \gamma_1 \).

Corollary

All the loops through the base point can be classified by homotopy relation. The homotopy class of a loops \( \gamma \) is denoted as \([\gamma]\).
Definition (Loop product)

Suppose $\gamma_1, \gamma_2$ are two loops through the base point $p$, the product of the two loops is defined as

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t - 1) & \frac{1}{2} \leq t \leq t
\end{cases}$$

Definition (Loop inverse)

$$\gamma^{-1}(t) = \gamma(1 - t).$$
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Figure: Loop inversion

Figure: Loop product
Definition (Fundamental Group)

The homotopy classes of loops through the base point $p$ form a group under the loop product, which is denoted as $\phi_1(S,p)$. 

Let $S = \{s_1, s_2, \cdots, s_n\}$ be $n$ symbols, a word generated by $S$ is a sequence $w = w_1 w_2 \cdots w_m$, where $w_k \in S$. The empty word $\emptyset$ is also treated as a word. The product of two words is the concatenation. The relations $R = \{R_1, R_2, \cdots, R_m\}$ are $m$ words, such that we can replace $R_k$ by the empty word.

**Definition (word equivalence relation)**

Two words are equivalent if we can transform one to the other by finite many steps of the following two operations:

1. Insert a relation word anywhere.
2. If a subword is a relation word, remove it from the word.

**Definition (Word Group)**

All the equivalence classes of the words generated by $S$ form a group under the concatenation, denoted as

$$\langle s_1, s_2, \cdots, s_n | R_1, R_2, \cdots, R_m \rangle$$
Theorem

Suppose $\pi_1(S_1, p_1)$ is isomorphic to $\pi_2(S_2, p_2)$, then $S_1$ is homeomorphic to $S_2$, and vice versa.

The representation of a group is not unique. It is NP hard to verify if two given representations are isomorphic.
Definition (Connected Sum)

Let $S_1$ and $S_2$ be two surfaces, $D_1 \subset S_1$ and $D_2 \subset S_2$ are two topological disks. $f : \partial D_1 \to \partial D_2$ is a homeomorphism between the boundaries of the disks. The connected sum is

$$S_1 \oplus S_2 := S_1 \cup S_2 / \{ p \sim f(p) \}$$

Theorem (Surface Topological Classification)

All the closed surfaces can be represented as

$$S \cong T^2 \oplus T^2 \oplus \ldots \oplus T^2$$

for oriented surfaces, or

$$S \cong \mathbb{RP}^2 \oplus \mathbb{RP}^2 \oplus \ldots \oplus \mathbb{RP}^2.$$

$\mathbb{RP}^2$ is gluing a Möbius band with a disk along its single boundary.
Figure: canonical fundamental group basis

Figure: canonical fundamental domain
For genus $g$ closed surface, one can find *canonical homotopy group generators* $\{a_1, b_1, a_2, b_2, \ldots, a_g, b_g\}$, such that $a_i \cdot a_j = 0$, $b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$, where the operator $r_1 \cdot r_2$ represents the algebraic intersection number between the two loops $\gamma_1$ and $\gamma_2$, and $\delta_{ij}$ is the Kronecker symbol.
Theorem (Fundamental Group, [?] page 136 Proposition 6.12 and page 137 Example 6.13)

For genus $g$ closed surface with a set of canonical basis, the fundamental group is given by

$$\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_a | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = e \rangle$$
Definition (Covering Space)

Suppose a continuous surjective map \( p : \tilde{S} \to S \), such that for each point \( q \in S \) has a neighborhood \( U \), its preimage \( p^{-1}(U) = \bigcup_i \tilde{U}_i \) is a disjoint union of open sets \( \tilde{U}_i \), and the restriction of \( p \) on each \( \tilde{U}_i \) is a local homeomorphism. Then \((\tilde{S}, p)\) is a covering space of \( S \), \( p \) is called a projection map.

Definition (Deck Transformation)

The automorphisms of \( \tilde{S} \), \( \tau : \tilde{S} \to \tilde{S} \), are called deck transformations, if they satisfy \( p \circ \tau = p \). All the deck transformations form a group, covering group, and denoted as \( \text{Deck}(\tilde{S}) \).
Suppose \( \tilde{q} \in \tilde{S}, p(\tilde{q}) = q \). The projection map \( p : \tilde{S} \to S \) induces a homomorphism between their fundamental groups, \( p_* : \pi_1(\tilde{S}, \tilde{q}) \to \pi_1(S, q) \), if \( p_* \pi_1(\tilde{S}, \tilde{q}) \) is a normal subgroup of \( \pi_1(S, q) \) then

**Theorem (Covering Group Structure Theorem, [?] Page 250 Theorem 11.30 and Corollary 11.31)**

The quotient group of \( \frac{\pi_1(S)}{p_* \pi_1(\tilde{S}, \tilde{q})} \) is isomorphic to the deck transformation group of \( \tilde{S} \).

\[
\frac{\pi_1(S, q)}{p_* \pi_1(\tilde{S}, \tilde{q})} \cong \text{Deck}(\tilde{S}).
\]
universal covering space

If a covering space $\tilde{S}$ is simply connected (i.e. $\pi_1(\tilde{S}) = \{e\}$), then $\tilde{S}$ is called a *universal covering space* of $S$. For universal covering space

$$\pi_1(\pi) \cong \text{Deck}(\tilde{S}).$$

The existence of the universal covering space is given by the following theorem,

**Theorem (Existence of the Universal Covering Space, [?] Page 262 Theorem 12.8)**

Every connected and locally simply connected topological space (in particular, every connected manifold) has a universal covering space.
Figure: Universal Covering Space
Lifting to Universal Covering Space

Figure: Universal Covering Space
Lifting to Universal Covering Space

**Figure**: Universal Covering Space
Let $(\tilde{S}, p)$ be the universal covering space of $S$, $q$ be the base point. The orbit of base is $p^{-1}(q) = \{\tilde{q}_k\}$.
Given a loop through $p$, there exists a unique lift of $\gamma \tilde{\gamma} \subset \tilde{S}$, starting from $\tilde{q}_0$.

**Theorem**

$\gamma_1$ and $\gamma_2$ are two loops through the base point, their lifts are $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. $\gamma_1 \sim \gamma_2$ if and only if the end points of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ coincide.
Suppose $S_1$ and $S_2$ are path connected manifolds, $S_1 \cap S_2$ is also path connected.

$$\pi_1(S_1, p) = \langle s_1^1, s_2^1, \ldots, s_1^{n_1} | R_1^1, R_2^1, \ldots, R_{m_1}^1 \rangle,$$
$$\pi_1(S_2, p) = \langle s_2^1, s_2^2, \ldots, s_2^{n_2} | R_1^2, R_2^2, \ldots, R_{m_2}^2 \rangle,$$
$$\pi_1(S_1 \cap S_2, p) = \langle s_3^1, s_3^2, \ldots, s_3^{n_3} | R_1^3, R_2^3, \ldots, R_{m_3}^3 \rangle,$$

then

$$\pi_1(S_1 \cup S_2, p) = \langle S_1 \cup S_2 | R_1 \cup R_2 \cup R_3 \rangle,$$

where $R_3$ is given by the following, for each generator $s_3^k$, it has two representations $w_1 \in \pi_1(S_1, p)$, and $w_2 \in \pi_1(S_2, p)$, then $R_3^k = w_1 w_2^{-1}$. 
Theorem

Show that $\pi_1(S)$ is $\langle a_1, b_1, \cdots, a_g, b_g \rangle$ for a surface $S = \bigoplus_{i=1}^{g} T^2$.

Proof.

By induction. If $g = 1$, obvious. Let $g = 2$,

$$
\begin{align*}
\pi_1(T_1) &= \langle a_1, b_1 | a_1 b_1 a_1^{-1} b_1^{-1} \rangle \\
\pi_1(T_2) &= \langle a_2, b_2 | a_2 b_2 a_2^{-1} b_2^{-1} \rangle \\
\pi_1(T_1 \cap T_2) &= \langle \gamma \rangle
\end{align*}
$$

$[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1}$ in $\pi_1(T_1)$, $[\gamma] = a_2 b_2 a_2^{-1} b_2^{-1}$ in $\pi_1(T_1)$, so

$$
\pi_1(T_1 \cup T_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2]^{-1} \rangle.
$$

where $[a_k, b_k] = a_k b_k a_k^{-1} b_k^{-1}$. 

\square
continued.

Suppose it is true for $g - 1$ case. Then for $g$ case, the intersection is an annulus,

$$
\pi_1(S) = \langle a_1, b_1, \cdots, a_{g-1}, b_{g-1} | \pi_{k=1}^{g-1}[a_k, b_k] \rangle \\
\pi_1(T_g) = \langle a_g, b_g | [a_g, b_g] \rangle \\
\pi_1(S \cap T_g) = \langle \gamma \rangle
$$

$[\gamma] = \pi_{k=1}^{g-1}[a_k, b_k]$ in $\pi_1(S)$ and $[a_g, b_g] \in \pi_1(T_g)$. 
Let $G$ be an unoriented graph, $T$ is a spanning tree of $G$, $G - T = \{ e_1, e_2, \cdots , e_n \}$, where $e_k$ is an edge not in the tree. Then $T \cup e_k$ has a unique loop $\gamma_k$. Choose one orientation of $\gamma_k$.

**Lemma**

The fundamental group of $G$ is $\pi_1(G) = \langle \gamma_1, \gamma_2 \cdots , \gamma_n \rangle$, which is a free group.
Definition (CW-cell decomposition)

A $k$ dimensional cell $D_k$ is a $k$ dimensional topological disk. Suppose $M$ is a $n$-dimensional manifold.

1. 0-skeleton $S_0$ is the union of a set of 0-cells.
2. $k$-skeleton $S_k$

$$S_k = S_{k-1} \cup D_k^1 \cup D_k^2 \cdots \cup D_k^n,$$

such that

$$\partial D_k^i \subset S_{k-1}.$$ 

The $k$-skeleton is constructed by gluing $k$-cells to the $k-1$ skeleton, all the boundaries of the cells are in the $k-1$ skeleton.

3. $S_n = M$.  

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Conformal Geometry
Theorem (CW-cell decomposition)

\[ \pi_1(S_2) = \pi_1(S_3) \cdots \pi_1(S_n) = \pi_1(M) \]

Proof.

using induction. \( S_2 \cap D_3^1 \) is \( \partial D_3^1 \), which is a topological sphere. \( \pi_1(D_3^1) = \langle e \rangle \), \( \pi_1(S^2) \) is \( \langle e \rangle \).