Homology Group

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Algebraic Topology
Figure: Surface topological classification
Figure: Topological classification for surfaces with boundaries \((g,b)\).
Definition (connected Sum)

The connected sum $S_1 \oplus S_2$ is formed by deleting the interior of disks $D_i$ and attaching the resulting punctured surfaces $S_i - D_i$ to each other by a homeomorphism $h : \partial D_1 \to \partial D_2$, so

$$S_1 \oplus S_2 = (S_1 - D_2) \cup_h (S_2 - D_2).$$
A Genus eight Surface, constructed by connected sum.
M.c. Escher

Möbius band.
Definition (Projective Plane)

All straight lines through the origin in $\mathbb{R}^3$ form a two dimensional manifold, which is called the projective plane $RP^2$.

A projective plane can be obtained by identifying two antipodal points on the unit sphere. A projective plane with a hole is called a crosscap. $\pi_1(RP^2) = \{\gamma, e\}$.
Theorem (surface Topology)

Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum of tori, or a sphere with a finite number of disjoint discs removed and with cross caps glued in their places. The sphere and connected sums of tori are orientable surfaces, whereas surfaces with crosscaps are unorientable.

Any closed surface is the connected sum

\[ S = S_1 \oplus S_2 \oplus \cdots \oplus S_g, \]

if \( S \) is orientable, then \( S_i \) is a torus. If \( S \) is non-orientable, then \( S_i \) is a projective plane.
Suppose $k + 1$ points in the general positions in $\mathbb{R}^n$, $v_0, v_1, \cdots, v_k$, the standard simplex $[v_0, v_1, \cdots, v_k]$ is the minimal convex set including all of them,

$$\sigma = [v_0, v_1, \cdots, v_k] = \{ x \in \mathbb{R}^n | x = \sum_{i=0}^{k} \lambda_i v_i, \sum_{i=0}^{k} = 1, \lambda_i \geq 0 \},$$

we call $v_0, v_1, \cdots, v_k$ as the vertices of the simplex $\sigma$.

Suppose $\tau \subset \sigma$ is also a simplex, then we say $\tau$ is a facet of $\sigma$. 
A simplicial complex \( \Sigma \) is a union of simplices, such that

1. If a simplex \( \sigma \) belongs to \( \Sigma \), then all its facets also belong to \( \Sigma \).
2. If \( \sigma_1, \sigma_2 \subset \Sigma \), \( \sigma_1 \cap \sigma_2 \neq \emptyset \), then their intersection is also a common facet.
Definition (triangular mesh)

A triangular mesh is a surface $\Sigma$ with a triangulation $T$,

1. Each face is counter-clockwisely oriented with respect to the normal of the surface.
2. Each edge has two opposite half-edges.
Definition (Chain Space)

A $k$ chain is a linear combination of all $k$-simplicies in $\Sigma$, $\sigma = \sum_i \lambda_i \sigma_i, \lambda_i \in \mathbb{Z}$. The $k$ dimensional chain space is the linear space formed by all $k$-chains, denoted as $C_k(\Sigma, \mathbb{Z})$.

A curve on the mesh is a 1-chain, a surface patch is a 2-chain.
Definition (Boundary Operator)

The $n$-th dimensional boundary operator $\partial_n : C_n \rightarrow C_{n-1}$ is a linear operator, such that

$$\partial_n[v_0, v_1, v_2, \cdots, v_n] = \sum_i (-1)^i [v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n].$$

Boundary operator extracts the boundary of a chain.
Definition (closed chain)

A $k$-chain $\gamma \in C_k(\sigma)$ is called a closed $k$-chain, if $\partial_k \gamma = 0$.

A closed 1-chain is a loop. A non-closed 1-chain has boundary vertices.

closed 1-chain  open 1-chain
Definition (Exact Chain)

A $k$-chain $\gamma \in C_k(\sigma)$ is called an exact $k$-chain, if there exists a $(k + 1)$ chain $\sigma$, such that $\partial_{k+1}\sigma = \gamma$.

exact 1-chain  closed, non-exact 1-chain
Theorem (Boundary of Boundary)

The boundary of a boundary is empty

\[ \partial_k \circ \partial_{k+1} \equiv \emptyset. \]

namely, exact chains are closed. But the reverse is not true.
The difference between the closed chains and the exact chains indicates the topology of the surfaces.

1. Any closed 1-chain on genus zero surface is exact.
2. On tori, some closed 1-chains are not exact.
Closed \( k \)-chains form the kernel space of the boundary operator \( \partial_k \). Exact \( k \)-chains form the image space of \( \partial_{k+1} \).

**Definition (Homology Group)**

The \( k \) dimensional homology group \( H_k(\Sigma, \mathbb{Z}) \) is the quotient space of \( \text{ker} \partial_k \) and the image space of \( \text{img} \partial_{k+1} \).

\[
H_k(\Sigma, \mathbb{Z}) = \frac{\text{ker} \partial_k}{\text{img} \partial_{k+1}}.
\]

Two \( k \)-chains \( \gamma_1, \gamma_2 \) are homologous, if they boundary a \( (k+1) \)-chain \( \sigma \),

\[
\gamma_1 - \gamma_2 = \partial_{k+1} \sigma.
\]
Abelianization

The first fundamental group in general is non-abelian. The first homology group is the abelianization of the fundamental group.

\[ H_1(\Sigma) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)]. \]

where \([\pi_1(\Sigma), \pi_1(\Sigma)]\) is the commutator of \(\pi_1\),

\[ [\gamma_1, \gamma_2] = \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}. \]

Fundamental group encodes more information than homology group, but more difficult to compute.
Homotopy group is non-abelian, which encodes more information than homology group.

- In homotopy group, $c \sim [a, b]$,
- In homology group, $c \sim 0$. 
Theorem

Suppose $M$ is a genus $g$ closed surface, then $H_0(M, \mathbb{Z}) \cong \mathbb{Z}$, $H_1(M, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$. 
Definition (Cochain Space)

A $k$-cochain is a linear function

$$\omega : C_k \rightarrow \mathbb{Z}.$$  

The $k$ cochain space $C^k(\Sigma, \mathbb{Z})$ is a linear space formed by all the linear functionals defined on $C_k(\Sigma, \mathbb{Z})$. A $k$-cochain is also called a $k$-form.

Definition (Coboundary)

The coboundary operator $\delta_k : C^k(\Sigma, \mathbb{Z}) \rightarrow C^{k+1}(\Sigma, \mathbb{Z})$ is a linear operator, such that

$$\delta_k \omega := \omega \circ \partial_{k+1}, \omega \in C^k(\Sigma, \mathbb{Z}).$$
Example

$M$ is a 2 dimensional simplicial complex, $\omega$ is a 1-form, then $\delta_1 \omega$ is a 2-form, such that

\[
\delta_1 \omega([v_0, v_1, v_2]) = \omega(\partial_2 [v_0, v_1, v_2]) = \omega([v_0, v_1]) + \omega([v_1, v_2]) + \omega([v_2, v_0])
\]
Coboundary operator is similar to differential operator. $\delta_0$ is the gradient operator, $\delta_1$ is the curl operator.

**Definition (closed forms)**

A $k$-form is closed, if $\delta_k \omega = 0$.

**Definition (Exact forms)**

A $k$-form is exact, if there exists a $k-1$ form $\sigma$, such that

$$\omega = \delta_{k-1} \sigma$$
suppose $\omega \in C^k(\Sigma)$, $\sigma \in C_k(\Sigma)$, we denote the pair

$$< \omega, \sigma > := \omega(\sigma).$$

**Theorem (Stokes)**

$$< d\omega, \sigma > = < \omega, \partial \sigma >.$$

**Theorem**

$$\delta^k \circ \delta^{k-1} \equiv 0.$$ All exact forms are closed. The curl of gradient is zero.
The difference between exact forms and closed forms indicates the topology of the manifold.

**Definition (Cohomology Group)**

The $k$-dimensional cohomology group of $\Sigma$ is defined as

$$H^n(\Sigma, \mathbb{Z}) = \frac{\ker \delta^n}{\operatorname{img} \delta^{n-1}}.$$ 

Two 1-forms $\omega_1, \omega_2$ are cohomologous, if they differ by a gradient of a 0-form $f$,

$$\omega_1 - \omega_2 = \delta_0 f.$$
**Duality**

$H_1(\Sigma)$ and $H^1(\Sigma)$ are dual to each other. Suppose $\omega$ is a closed 1-form, $\sigma$ is a closed 1-chain, then the pair $\langle \omega, \sigma \rangle$ is a bilinear operator.

**Definition (dual cohomology basis)**

Suppose a homology basis of $H_1(\Sigma)$ is $\{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, the dual cohomology basis is $\{\omega_1, \omega_2, \cdots, \omega_n\}$, if and only if

$$\langle \omega_i, \gamma_j \rangle = \delta^j_i.$$
Definition (simplicial mapping)

Suppose $M$ and $N$ are simplicial complexes, $f : M \rightarrow N$ is a continuous map, $\forall \sigma \in M$, $\sigma$ is a simplex, $f(\sigma)$ is a simplex.

For each simplex, we can add its gravity center, and subdivide the simplex to multiple ones. The resulting complex is called the gravity center subdivision.

Theorem

Suppose $M$ and $N$ are simplicial complexes embedded in $\mathbb{R}^n$, $f : M \rightarrow N$ is a continuous mapping. Then for any $\varepsilon > 0$, there exists gravity subdivisions $\tilde{M}$ and $\tilde{N}$, and a simplicial mapping $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$, such that

$$\forall p \in |M|, |f(p) - \tilde{f}(p)| < \varepsilon.$$
If $f : M \rightarrow N$ is a continuous map, then $f$ induces a homomorphism $f_* : H_1(M) \rightarrow H_1(N)$, which push forward the chains of $M$ to the chains in $N$. Similarly, $f$ induces a pull back map $f^* : H^k(N) \rightarrow H^k(M)$. Suppose $\sigma \in C_1(M)$, $\omega \in C^1(N)$,

$$f^* \omega(\sigma) = \omega(f_* \sigma) = \omega(f(\sigma)).$$
Suppose $M$ and $N$ are two closed surfaces. $H_2(M, \mathbb{Z}) = \mathbb{Z}$, $H_2(N, \mathbb{Z}) = \mathbb{Z}$, suppose $[M]$ is the generator of $H_2(M)$, which is the union of all faces. Similarly, $[n]$ is the generator of $H_2(N)$. $f : M \to N$ is a continuous map. Then

$$f_* : \mathbb{Z} \to \mathbb{Z},$$

must have the form $f_*(z) = cz, c \in \mathbb{Z}$.

**Definition**

$f_*([M]) = c[N]$, then the integer $c$ is the degree of the map.

Map degree is the algebraic number of pre-images $f^{-1}(q)$ for arbitrary point $q \in N$, which is independent of the choice of the point $q$.
Degree of a mapping

Figure: Map degree

Example

$G : S \rightarrow S^2$ is the Gauss map, which maps the point $p$ to its normal $\mathbf{n}(p)$, then $\text{deg}(G) = 1 - g$. 