Conformal Structure

David Gu[1,2]

[1]Computer Science Department
Stony Brook University
[2]Yau Mathematical Sciences Center
Tsinghua University

Tsinghua University

Manifold

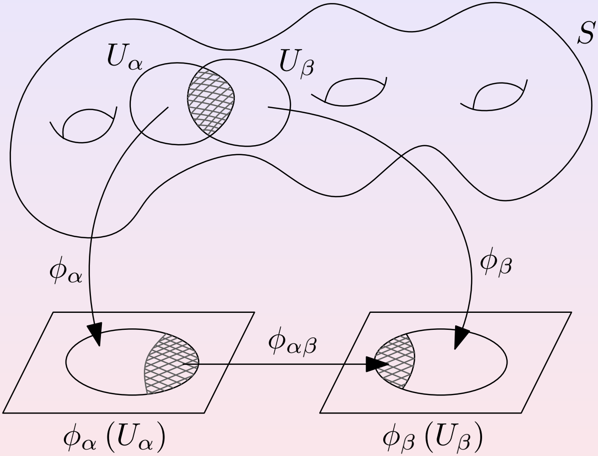
Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then M is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Manifold



S
U_α
U_β
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Holomorphic Function

Definition (Holomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a complex function, $f : x + iy \to u(x,y) + iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

$$dz = dx + idy, d\bar{z} = dx - idy, \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

then if $\frac{\partial f}{\partial \bar{z}} = 0$, then f is holomorphic.

biholomorphic Function

Definition (biholomorphic Function)

Suppose $f: \mathbb{C} \to \mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta} : \mathbb{C} \to \mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose S is a topological surface, consider all the conformal atlases on M, classified by the compatible relation

$$\{all\ conformal\ atlas\}/\sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

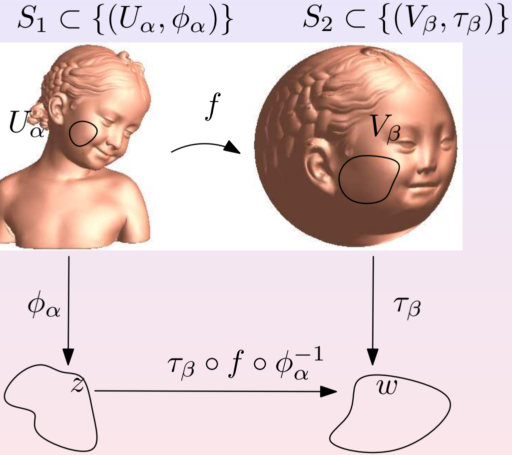
Definition (Smooth map)

Suppose $f : S_1 \to S_2$ is a map between two smooth manifolds. For each point p, choose a chart of S_1, (U_α, ϕ_α), $p \in U_\alpha$). The image $f(U_\alpha) \subset V_\beta$, (V_β, τ_β) is a chart of S_2. The local representation of f

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \tau_\beta(V_\beta)$$

is smooth, then f is a smooth map.

Map between Manifolds



Tangent map

A curve on a manifold is a map $\gamma : [0,1] \to M$, $\gamma(t) \in M$. Choose a local chart (U_α, ϕ_α) with local parameter (x,y), then the curve can be represented as $(x(t), y(t))$. The velocity vector of the curve is represented as

$$\frac{d\gamma(t)}{dt} = \frac{\partial}{\partial x}\frac{dx}{dt} + \frac{\partial}{\partial y}\frac{dy}{dt}.$$

Let $f : M \to N$ be a smooth map, then $f \circ \gamma : [0,1] \to N$ is a curve on N. Choose a local chart on N, (V_β, τ_β) with local parameters (u,v). Then the local representation of the map

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : (x,y) \to (u(x,y), v(x,y)),$$

the local representation of $f \circ \gamma$ is $(u(x(t),y(t)), v(x(t),y(t)))$.

Tangent Map

The velocity vector of $f \circ \gamma$ is

$$\frac{df \circ \gamma}{dt} = \frac{\partial}{\partial u}\frac{du}{dt} + \frac{\partial}{\partial v}\frac{dv}{dt}$$

where

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Definition (Tangent map)

The linear map $df : T_pM \to T_{f(p)}N$, which maps a tangent vector in T_pM to a tangent vector in $T_{f(p)}N$,

$$\frac{d\gamma(t)}{dt} \to \frac{df \circ \gamma(t)}{dt}$$

is call the tangent map of f, or the push-forward map.

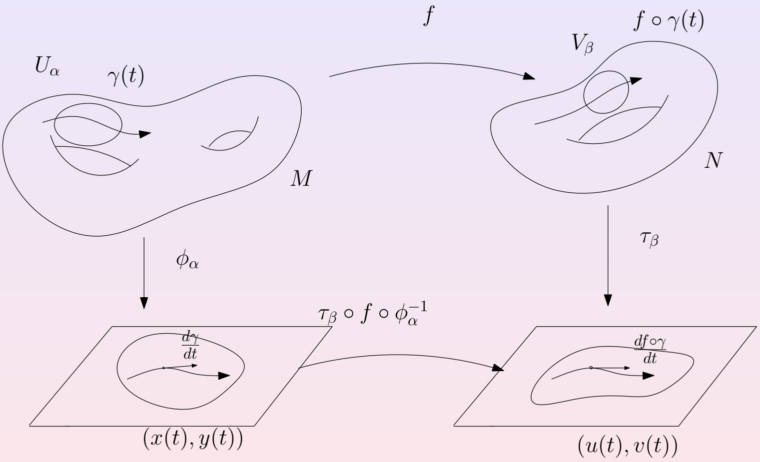
Tangent Map

We say $\frac{df\circ\gamma}{dt}$ is the push-forward of $\frac{d\gamma}{dt}$, and denote it as

$$f_*(\frac{d\gamma}{dt}) = \frac{df\circ\gamma}{dt}.$$

The local representation of the tangent map is the Jacobi matrix.

Tangent Map



f
$f \circ \gamma(t)$
V_β
U_α
$\gamma(t)$
M
N
ϕ_α
τ_β
$\tau_\beta \circ f \circ \phi_\alpha^{-1}$
$\frac{d\gamma}{dt}$
$\frac{df\circ\gamma}{dt}$
$(x(t), y(t))$
$(u(t), v(t))$

Definition (Riemannian Metric)

A Riemannian metric on a smooth manifold M is an assignment of an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$, $\forall p \in M$, such that

1. $g_p(a_1X_1 + a_2X_2, b_1Y_1 + b_2Y_2) = \sum_{i,j=1}^2 a_ib_jg_p(X_i, Y_j)$.
2. $g_p(X,Y) = g_p(Y,X)$
3. g_p is non-degenerate.
4. $\forall p \in M$, there exists local coordinates $\{x^i\}$, such that $g_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are C^∞ functions.

Definition (Pull back Riemannian metric)

Suppose $f:(M,g)\to(N,h)$ is a smooth mapping between two Riemannian manifolds, $\forall p\in M$, $f_*:T_pM\to T_{f(p)}N$ is the tangent map. The pull back metric f^*h induced by the mapping f is given by

$$f^*h(X_1,X_2):=h(f_*X_1,f_*X_2),\forall X_1,X_2\in T_pM.$$

Local representation of the pull back metric is given by

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Conformal Structure

Definition (Conformal equivalent metrics)

Suppose g_1, g_2 are two Riemannian metrics on a manifold M, if

$$g_1 = e^{2u} g_2, u : M \to \mathbb{R}$$

then g_1 and g_2 are conformal equivalent.

Definition (Conformal Structure)

Consider all Riemannian metrics on a topological surface S, which are classified by the conformal equivalence relation,

$$\{Riemannian\ metrics\ on\ S\}/\sim,$$

each equivalence class is called a conformal structure.

Relation between Riemannian metric and Conformal Structure

Definition (Isothermal coordinates)

Suppose (S,g) is a metric surface, (U_α,ϕ_α) is a coordinate chart, (x,y) are local parameters, if

$$g = e^{2u}(dx^2+dy^2),$$

then we say (x,y) are isothermal coordinates.

Theorem

Suppose S is a compact metric surface, for each point p, there exits a local coordinate chart (U,ϕ), such that $p\in U$, and the local coordinates are isothermal.

Riemannian metric and Conformal Structure

Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

All compact metric surfaces are Riemann surfaces.

Problem

Show that holomorphic functions are conformal.

Suppose $f:\mathbb{C}\to\mathbb{C}$ is a holomorphic function, $w=f(z)$, then $dw=\frac{\partial f}{\partial z}dz$, then

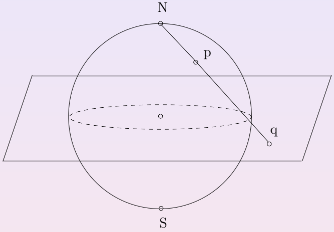
$$dwd\bar{w}=\frac{\partial f}{\partial z}dz\overline{\frac{\partial f}{\partial z}}d\bar{z}$$

$$=\frac{\partial f}{\partial z}\overline{\frac{\partial f}{\partial z}}dzd\bar{z}. \quad (1)$$

Exercise

Problem

Show that stereo-graphic projection is conformal.



$$\begin{cases} u &= \frac{x}{1-z} \\ v &= \frac{y}{1-z} \end{cases} \quad \begin{cases} du &= \frac{dx(1-z)+xdz}{(1-z)^2} \\ dv &= \frac{dy(1-z)+ydz}{(1-z)^2} \end{cases}$$

also $x^2+y^2+z^2=1$, therefore $xdx+ydy+zdz=0$, we get

$$du^2+dv^2=\frac{dx^2+dy^2+dz^2}{(1-z)^2}$$

Exercise

Problem

Show that conformal mapping preserves angles.

Assume $f : (M, g) \to (N, h)$ is conformal, then the pull back metric $f^* h = e^{2u} g$. Let $X_1, X_2 \in T_p M$, the angle between them is θ, then

$$
\begin{aligned}
\cos\theta &= \frac{g(X_1, X_2)}{\sqrt{g(X_1, X_1)}\sqrt{g(X_2, X_2)}} && (2)\\
&= \frac{e^{2u} g(X_1, X_2)}{\sqrt{e^{2u} g(X_1, X_1)}\sqrt{e^{2u} g(X_2, X_2)}} && (3)\\
&= \frac{f^* h(X_1, X_2)}{\sqrt{f^* h(X_1, X_1)}\sqrt{f^* h(X_2, X_2)}} && (4)
\end{aligned}
$$