

# Conformal Structure

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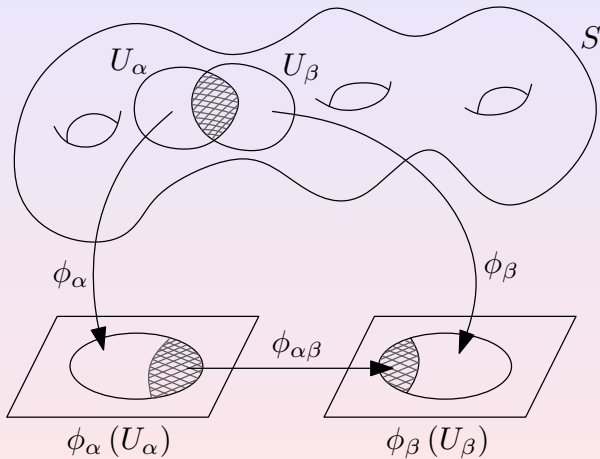
## Definition (Manifold)

$M$  is a topological space,  $\{U_\alpha\} \alpha \in I$  is an open covering of  $M$ ,  $M \subset \cup_\alpha U_\alpha$ . For each  $U_\alpha$ ,  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$  is a homeomorphism. The pair  $(U_\alpha, \phi_\alpha)$  is a chart. Suppose  $U_\alpha \cap U_\beta \neq \emptyset$ , the transition function  $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$  is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then  $M$  is called a smooth manifold,  $\{(U_\alpha, \phi_\alpha)\}$  is called an atlas.

# Manifold



## Definition (Holomorphic Function)

Suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a complex function,  
 $f : x + iy \rightarrow u(x, y) + iv(x, y)$ , if  $f$  satisfies Riemann-Cauchy  
equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

then  $f$  is a holomorphic function.

Denote

$$dz = dx + idy, d\bar{z} = dx - idy, \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

then if  $\frac{\partial f}{\partial \bar{z}} = 0$ , then  $f$  is holomorphic.

# biholomorphic Function

## Definition (biholomorphic Function)

Suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is invertible, both  $f$  and  $f^{-1}$  are holomorphic, then then  $f$  is a biholomorphic function.



## Definition (Conformal Atlas)

Suppose  $S$  is a topological surface, (2 dimensional manifold),  $\mathfrak{A}$  is an atlas, such that all the chart transition functions  $\phi_{\alpha\beta} : \mathbb{C} \rightarrow \mathbb{C}$  are bi-holomorphic, then  $A$  is called a conformal atlas.

## Definition (Compatible Conformal Atlas)

Suppose  $S$  is a topological surface, (2 dimensional manifold),  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are two conformal atlases. If their union  $A_1 \cup A_2$  is still a conformal atlas, we say  $\mathfrak{A}_1$  and  $\mathfrak{A}_2$  are compatible.

# Conformal Structure

The compatible relation among conformal atlases is an equivalence relation.

## Definition (Conformal Structure)

Suppose  $S$  is a topological surface, consider all the conformal atlases on  $M$ , classified by the compatible relation

$$\{\text{all conformal atlas}\} / \sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

## Definition (Smooth map)

Suppose  $f : S_1 \rightarrow S_2$  is a map between two smooth manifolds. For each point  $p$ , choose a chart of  $S_1$ ,  $(U_\alpha, \phi_\alpha)$ ,  $p \in U_\alpha$ . The image  $f(U_\alpha) \subset V_\beta$ ,  $(V_\beta, \tau_\beta)$  is a chart of  $S_2$ . The local representation of  $f$

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \rightarrow \tau_\beta(V_\beta)$$

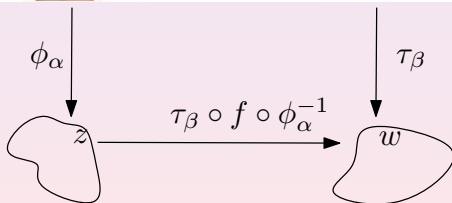
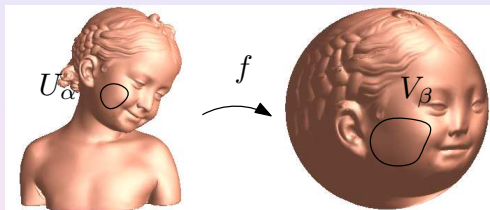
is smooth, then  $f$  is a smooth map.



# Map between Manifolds

$$S_1 \subset \{(U_\alpha, \phi_\alpha)\}$$

$$S_2 \subset \{(V_\beta, \tau_\beta)\}$$



# Tangent map

A curve on a manifold is a map  $\gamma: [0, 1] \rightarrow M$ ,  $\gamma(t) \in M$ . Choose a local chart  $(U_\alpha, \phi_\alpha)$  with local parameter  $(x, y)$ , then the curve can be represented as  $(x(t), y(t))$ . The velocity vector of the curve is represented as

$$\frac{d\gamma(t)}{dt} = \frac{\partial}{\partial x} \frac{dx}{dt} + \frac{\partial}{\partial y} \frac{dy}{dt}.$$

Let  $f: M \rightarrow N$  be a smooth map, then  $f \circ \gamma: [0, 1] \rightarrow N$  is a curve on  $N$ . Choose a local chart on  $N$ ,  $(V_\beta, \tau_\beta)$  with local parameters  $(u, v)$ . Then the local representation of the map

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : (x, y) \rightarrow (u(x, y), v(x, y)),$$

the local representation of  $f \circ \gamma$  is  $(u(x(t), y(t)), v(x(t), y(t)))$ .

# Tangent Map

The velocity vector of  $f \circ \gamma$  is

$$\frac{df \circ \gamma}{dt} = \frac{\partial}{\partial u} \frac{du}{dt} + \frac{\partial}{\partial v} \frac{dv}{dt}$$

where

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

## Definition (Tangent map)

The linear map  $df : T_p M \rightarrow T_{f(p)} N$ , which maps a tangent vector in  $T_p M$  to a tangent vector in  $T_{f(p)} N$ ,

$$\frac{d\gamma(t)}{dt} \rightarrow \frac{df \circ \gamma(t)}{dt}$$

is called the tangent map of  $f$ , or the push-forward map.

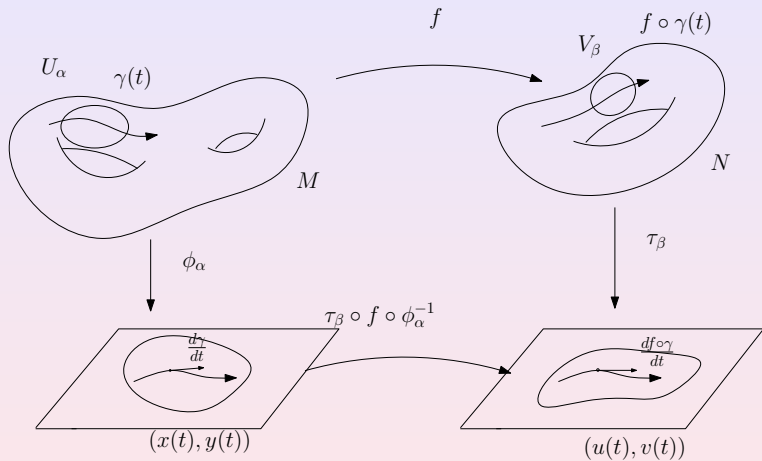
# Tangent Map

We say  $\frac{df \circ \gamma}{dt}$  is the push-forward of  $\frac{d\gamma}{dt}$ , and denote it as

$$f_*\left(\frac{d\gamma}{dt}\right) = \frac{df \circ \gamma}{dt}.$$

The local representation of the tangent map is the Jacobi matrix.

# Tangent Map



## Definition (Riemannian Metric)

A Riemannian metric on a smooth manifold  $M$  is an assignment of an inner product  $g_p : T_p M \times T_p M \rightarrow \mathbb{R}, \forall p \in M$ , such that

- 1  $g_p(a_1 X_1 + a_2 X_2, b_1 Y_1 + b_2 Y_2) = \sum_{i,j=1}^2 a_i b_j g_p(X_i, Y_j)$ .
- 2  $g_p(X, Y) = g_p(Y, X)$
- 3  $g_p$  is non-degenerate.
- 4  $\forall p \in M$ , there exists local coordinates  $\{x^i\}$ , such that  $g_{ij} = g_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$  are  $C^\infty$  functions.

# Pull back Riemannian Metric

## Definition (Pull back Riemannian metric)

Suppose  $f : (M, g) \rightarrow (N, h)$  is a smooth mapping between two Riemannian manifolds,  $\forall p \in M$ ,  $f_* : T_p M \rightarrow T_{f(p)} N$  is the tangent map. The pull back metric  $f^* h$  induced by the mapping  $f$  is given by

$$f^* h(X_1, X_2) := h(f_* X_1, f_* X_2), \forall X_1, X_2 \in T_p M.$$

Local representation of the pull back metric is given by

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

# Conformal Structure

## Definition (Conformal equivalent metrics)

Suppose  $g_1, g_2$  are two Riemannian metrics on a manifold  $M$ , if

$$g_1 = e^{2u}g_2, u : M \rightarrow \mathbb{R}$$

then  $g_1$  and  $g_2$  are conformal equivalent.

## Definition (Conformal Structure)

Consider all Riemannian metrics on a topological surface  $S$ , which are classified by the conformal equivalence relation,

$$\{\text{Riemannian metrics on } S\} / \sim,$$

each equivalence class is called a conformal structure.



# Relation between Riemannian metric and Conformal Structure

## Definition (Isothermal coordinates)

Suppose  $(S, g)$  is a metric surface,  $(U_\alpha, \phi_\alpha)$  is a coordinate chart,  $(x, y)$  are local parameters, if

$$g = e^{2u}(dx^2 + dy^2),$$

then we say  $(x, y)$  are isothermal coordinates.

## Theorem

*Suppose  $S$  is a compact metric surface, for each point  $p$ , there exists a local coordinate chart  $(U, \phi)$ , such that  $p \in U$ , and the local coordinates are isothermal.*

## Corollary

*For any compact metric surface, there exists a natural conformal structure.*

## Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

## Theorem

*All compact metric surfaces are Riemann surfaces.*

## Problem

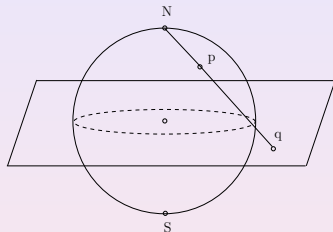
*Show that holomorphic functions are conformal.*

Suppose  $f : \mathbb{C} \rightarrow \mathbb{C}$  is a holomorphic function,  $w = f(z)$ , then  $dw = \frac{\partial f}{\partial z} dz$ , then

$$\begin{aligned} dw d\bar{w} &= \frac{\partial f}{\partial z} dz \overline{\frac{\partial f}{\partial z}} d\bar{z} \\ &= \frac{\partial f}{\partial z} \overline{\frac{\partial f}{\partial z}} dz d\bar{z}. \quad (1) \end{aligned}$$

## Problem

Show that stereo-graphic projection is conformal.



$$\begin{cases} u = \frac{x}{1-z} \\ v = \frac{y}{1-z} \end{cases} \quad \begin{cases} du = \frac{dx(1-z) + xdz}{(1-z)^2} \\ dv = \frac{dy(1-z) + ydz}{(1-z)^2} \end{cases}$$

also  $x^2 + y^2 + z^2 = 1$ , therefore  $xdx + ydy + zdz = 0$ , we get

$$du^2 + dv^2 = \frac{dx^2 + dy^2 + dz^2}{(1-z)^2}$$

## Problem

*Show that conformal mapping preserves angles.*

Assume  $f : (M, g) \rightarrow (N, h)$  is conformal, then the pull back metric  $f^*h = e^{2u}g$ . Let  $X_1, X_2 \in T_pM$ , the angle between them is  $\theta$ , then

$$\cos \theta = \frac{g(X_1, X_2)}{\sqrt{g(X_1, X_1)}\sqrt{g(X_2, X_2)}} \quad (2)$$

$$= \frac{e^{2u}g(X_1, X_2)}{\sqrt{e^{2u}g(X_1, X_1)}\sqrt{e^{2u}g(X_2, X_2)}} \quad (3)$$

$$= \frac{f^*h(X_1, X_2)}{\sqrt{f^*h(X_1, X_1)}\sqrt{f^*h(X_2, X_2)}} \quad (4)$$