Infinite Circle Packing

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Definition (Triangulation)

A triangulation of a surface $S$ refers to locally decompose $S$ to a set of topological closed triangles, denoted as $T = \{ t_j \}$, such that either $t_i \cap t_j = \emptyset$ or $t_i$ intersects $t_j$ at a vertex or a whole edge. $T$ can be finite or infinite.

A triangulation is also called a simplical 2-complex.
Definition (Label)

Given a complex $K$ with vertex set $\{v_1, v_2, \ldots, v_j, \ldots\}$, assign each vertex $v_j (j = 1, 2, \cdots)$ a positive number $r_j$ in Euclidean, spherical or hyperbolic geometry, then the set of these positive numbers $R = \{r_1, r_2, \cdots, r_j, \cdots\}$ is called a label of $K$ in Euclidean, spherical or hyperbolic geometry.
Circle Packing

Definition (Circle Packing)

Circle packing refers to a special type of circle pattern, $K$ is a simplical 2-complex, simplicial equivalent to a triangulation of a surface. We say a circle pattern is a circle packing of the complex $K$, if it satisfies the following conditions:

1. For each vertex $v \in K$, there is a circle $C_v$ in $P$ corresponding to it.
2. If $[u, v]$ is an edge of $K$, then $C_u$ and $C_v$ are tangent.
3. If $[u, v, w]$ is a positively oriented face of $K$, then $C_u, C_v, C_w$ form a positively oriented, mutually tangent triple circles.

Definition (Univalent)

A circle packing is called univalent, if there is no overlapping among all the circles. Namely, no two circles intersect at more than one point.
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Definition (Horocircle)

Circles, that are internally tangent to the unit circle, are called horocircles. The centers of horocircles are the tangent points.
Theorem (Andreev-Thurston)

Let $K$ be a combinatorial closed disk, then there exists a circle packing $P_k$ of $K$ in the hyperbolic plane $\mathbb{H}^2$, such that all boundary circles are horocircles. $P_k$ is unique up to hyperbolic rigid motion.
Lemma (Flower)

Suppose $P$ and $\tilde{P}$ are flowers of the same complex $K$, surrounding the center vertex $v$, then

$$\max_{w \sim v} \frac{\tilde{R}(w)}{R(w)} \geq \frac{\tilde{R}(v)}{R(v)}, \min_{w \sim v} \frac{\tilde{R}(w)}{R(w)} \leq \frac{\tilde{R}(v)}{R(v)}$$
Lemma (Max Value Principle)

Suppose $P_K$ and $\tilde{P}_K$ are circle packings of the same complex $K$, then the max and min values of $\frac{\tilde{R}(v)}{R(v)}$ are on boundary circles.
Lemma

Suppose $P_K$ and $\tilde{P}_K$ are circle packings of the same complex $K$, $\tilde{P}_K$ is the maximal circle packing, then $\forall v \in K, R(v) \leq \tilde{R}(v)$. 

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Discrete Riemann mapping

Let $\Omega$ be a simply connected domain contained in the unit disk. Compute a triangulation and construct a mesh. Run hyperbolic ricci flow, such that

$$\forall v_i \in \partial M, u_i \to \infty.$$ 

Then the resulting mapping is the discrete Riemann mapping.

Theorem

Discrete Riemann mapping

Conformal mapping from a simply connected planar domain to the unit disk exists, and unique up to a rigid motion.
Maximal Hyperbolic Ricci flow

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$K$ is a infinite, with boundary, complex of the whole upper half plane.
$K$ is a infinite, with boundary, complex. Get a sequence of simply connected complexes

$$\{K_j, j = 1, 2, 3 \cdots \}$$

1. $v_1 \in K_1$
2. $\forall j, K_j$ is a finite triangulation of a topological disk,
3. $K_j \subset K_{j+1}$
4. $K_j \uparrow K$, namely $\forall L \subset K, L \neq K, \exists j, j$ is big enough, s.t. $L \subset K_j$. 
In general, \( \{K_j | j = 1, 2, 3 \cdots \} \) can be chosen as:

1. \( K_1 = \{v_1\} \).
2. \( K_j = K_{j-1} \cup \{v \in K | \exists w \in K_{j-1}, v \sim w\} \).

Each \( K_j \) has a maximal hyperbolic circle packing in \( H^2 \), denoted as \( P_{K_j} \), the radius function for \( P_{K_j} \) is \( R_j \). Normalize the \( P_{K_j} \), such that the center of the circle of \( v_1 \) is the origin.
Lemma

Suppose $K$ and $L$ are two finite triangulations of closed disk, $L \subseteq K$, $R_K$ and $R_L$ are the radius functions of maximal circle packings of $K$ and $L$ respectively, then

$$\forall v \in L, R_K(v) \leq R_L(v).$$
Infinite Circle Packing
Claim: consider the sequence of radius functions \( \{R_j\} \), because \( K_j \subset K_{j+1} \), for all vertex \( v \in K_j \),

\[
R_{j+1}(v) \leq R_j(v),
\]

\( \{R_j(v)\} \) is monotonously decreasing. Two situations can happen, for a vertex \( v_1 \in K \)

1. \( R_j(v_1) \downarrow r_1, \ r_1 > 0, \) when \( j \to \infty \).
2. \( R_j(v_1) \downarrow 0, \) when \( j \to \infty \).

This is determined by the complex \( K \) itself, independent of the choice of \( v_1 \) and \( K_j \).
1. \( K \) is called the hyperbolic type, there exists a univalent circle packing of \( K \), which fills the whole hyperbolic plane.

2. \( K \) is called the parabolic type, there exists a univalent circle packing of \( K \), which fills the whole Euclidean plane.
Theorem

If $K$ is an infinite complex with boundary, then $K$ is of hyperbolic type.

Lemma (Ring)

For each $k \geq 3$ integer, there exists a constant $c(k) > 0$, such that for a $k$-flower on Euclidean or hyperbolic plane, the center circle radius is $r_0$, then each pedal circle radius

$$r \geq c(k)r_0.$$
Ring lemma holds for interior circles.
Ring lemma also holds for horocircles on the boundary.
Hyperbolic Complex

Fix a boundary vertex $v_1 \in \partial K$, get a sequence finite subcomplexes $\{K_j\}$. Suppose $P_{K_j}$ are the maximal circle packing of $K_j$, with circle of $v_1$ normalized, $R_j(v) = r_1 > 0$, the center of the $c_j(v)$ is at $z_1$.

claim: $\forall v \in K$, $\lim_{j \to \infty} R_j(v) = r_v > 0$.

If $v \in \partial K$, then $c_j(v)$ is a horocircle, $R_j(v) > 0$, $\{R_j(v)\}$ monotonously decreases, the limit exists. If $v \notin \partial K$, find the shortest path from $v_1$ to $v$ in the interior of $K$, $\gamma = [v_1, v_2, \cdots, v_N]$, $v_N = v$. Assume the maximum valence of $K$ is $d$, then by ring lemma

$$c(d)R_j(v_n) \leq R_j(v_{n+1}) \leq \frac{1}{c(d)}R_j(v_n)$$

then

$$(c(d'))^N R_j(v_1) \leq R_j(v) \leq \frac{1}{(c(d'))^N} R_j(v_1)$$

$\{R_j(v)\}$ monotonously decreases, $\lim_{j \to \infty} R_j(v) \to r_v > 0$. $K$ is hyperbolic.
Existence

**Theorem**

Let $K$ be an infinite simply connected complex with boundary, then there exists a univalent circle packing filling the whole hyperbolic plane.

Get the sequence of $\{K_j\}$ and the maximal circle packing sequence $S_1 = \{P_{K_j}\}$, the vertex sequence of $K$ is $\{v_1, v_2, \ldots\}$. Choose a subsequence $S_2 \subset S_1$, such that the centers of the circles of $v_2$ in $S_2$ converges; choose a subsequence $S_3 \subset S_2$, such that the centers of the circles of $v_3$ in $S_3$ converges. Similarly, take $S_{n+1} \subset S_n$, $\lim_{n \to \infty} S_n$ gives the desired circle packing.
Uniqueness

**Theorem**

*Such two circle packings differ by a Möbius transformation.*
Definition (Winding Number)

If $\gamma : [a, b] \rightarrow \mathbb{C}$ is a continuous curve in the plane, and $g : \gamma \rightarrow \mathbb{C}$ is a continuous nonvanishing function on $\gamma$, then the winding number of $g$ on $\gamma$, denoted $P(g; \gamma)$, is the change of the argument of $g$ while transiting $\gamma$, i.e.

$$P(g; \gamma) := \frac{\text{arg}(g(\gamma(b))) - \text{arg}(g(\gamma(a)))}{2\pi}.$$
Definition (Fixed-Point Index)

Let $\gamma$ and $\sigma$ be positively oriented Jordan curves and let $f : \gamma \to \sigma$ be an orientation preserving, fixed point-free homeomorphism. Then the fixed-point index of $f$ denoted as $\eta(f, \gamma)$ is the winding number of $P(f(z) - z; \gamma)$. 
Lemma

Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is an orientation-preserving homeomorphism. If $f : \gamma \to \sigma$ is an orientation preserving, fixed-point free homeomorphism, then same is true for the map $f_1 = \phi \circ f \circ \phi^{-1} : \phi(\gamma) \to \phi(\sigma)$, moreover

$$\eta(\phi \circ f \circ \phi^{-1}; \phi(\gamma)) = \eta(f; \gamma)$$
Lemma

Let $\gamma, \sigma$ be Jordan curves in $\mathbb{C}$, positively oriented; $f : \gamma \to \sigma$ be an orientation preserving homeomorphism with no fixed points. Then

1. $\eta(f; \gamma) = \eta(f^{-1}; f(\gamma))$
2. If $\gamma$ is contained in closure of the domain determined by $\sigma$, or vice versa, then $\eta(f; \gamma) = 1$.
3. If the intersection of $\gamma$ and $\sigma$ contains at most 2 points, then $\eta(f; \gamma) \geq 0$. 
Proof.

1. \( \arg(f(z) - z) = \pi + \arg(f^{-1}(w) - w) \).

2. Homotopically shrink \( \sigma \) to a point. If the point is inside \( \gamma \), then index equals to one; if the point is outside of \( \gamma \), then the index equals to zero.

3. The fixed point index is one. (see the following figure)
Get a maximal circle packing of $K$. Suppose $[u, v, w]$ is a triangle, choose a point $z_0$ inside an interstice of $[c_u, c_v, c_w]$, use

$$z \mapsto \frac{1}{z - z_0}$$

to map the whole circle packing inside the interstice bounded by $c_u, c_v, c_w$. 

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Suppose there are two circle packings $P_K, P'_K$, each interstice determines a unique Möbius transformation, which maps the intersection points to intersection points. The piecewise Möbius transformations can be extended to a diffeomorphism, which is orientation preserving and fix point free. On each interstice, the Möbius map is $f_j : e_j \to e'_j$; on each circle, the map is $g_i : c_i \to c'_i$. The Total map is $F : \mathbb{C} \to \mathbb{C}$. 
Denote the exterior interstice as $\Gamma_P$ and $\Gamma'_P$, and the mappings are defined as above, then

$$\eta(F; \Gamma_P) = \sum_j \eta(f_j, e_j) + \sum_i \eta(g_i, c_i),$$

where the sum is over all interstices and circles inside the exterior interstice.
Key Idea

(a) (b) (c) (d)
Proof.

If $P_K$ and $P_{K'}$ do not differ by a Möbius transformation, then situation (a) will happen. Deformation by Möbius, we get situation (d). Then $\eta(F; \Gamma_P) = -1$. But for any circle or interstice inside $\Gamma_P$, the fixed-point index is non-negative. If the triangulation is finite, then we are done.

Now we are focusing on the case $K$ is infinite.
Proof.

Suppose $P_K$ has countable singularities, denoted as $Singular(P)$. The set $Singular(P)$ is a compact set. By small perturbation,

$$Singular(P) \cap \{ c | circleinP_{K'} \} = \emptyset$$

Denote

1. Circle at $v \in K$ in $P_{K'}$, as $P'_v$
2. Interstices in $P_{K'}$, $\mathcal{J}'$
3. Singularities of $P_{K'}$, $\mathcal{R}'$

$SingularP$ is contained in the union of all disks, interstices and the interior of singularities in $P_{K'}$. 

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Proof.

Because $\text{Singlar}(P)$ is compact, we can choose a finite open cover. Let

$$G = \{(\bigcup_{v \in K} \text{interior}(P_v)) \bigcup (\bigcup_{H \in \mathcal{H}'} H) \bigcup (\bigcup_{L \in \mathcal{L}'} \text{interior}(L))\}$$

which contains $\text{Singlar}(P)$. Let $V_1 \subset V$ be the vertices, whose circles are not contained in $G$. Then each connected component of $V - V_1$ contains at least a singularity, the boundary of each connected component is a Jordan curve, composed by a finite number of circular arcs (of circles in $V_1$), denoted as $\sigma_j \subset P_K$. The corresponding image of $\sigma_j$ is $\gamma_j \subset P_{K'}$. Then $\sigma_j$ is disjoint from $\gamma_j$.

Disregard the interior of $\sigma_j$’s, then the packing is finite.
Replace singularities and their neighboring circles by Jordan curve $\sigma_j$'s (red curves).
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