Exterior Calculus

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Simply Connected Domains
Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.
Conformal Module

Multiply Connected Domains
Conformal Module

Multiply Connected Domains
Teichmüller Space

Multiply Connected Domains

Conformal Module: centers and radii, with Möbius ambiguity. The Teichmüller space is $3n - 3$ dimensional, $n$ is the number of holes.
Conformal Module

Torus
Exterior Calculus
Holomorphic 1-form

Figure: Holomorphic 1-form
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In each cohomologous class, there exists a unique harmonic form, which is the smoothest one in the whole class. Each 1-form is dual to a vector field, the harmonic 1-form corresponds to the vector field, which is with zero curl and zero divergence.
Smooth manifold

Figure: manifold
A manifold is a topological space $M$ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps $U_\alpha$ to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \phi_\alpha)$ is called a coordinate chart of $M$. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of $M$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition functions $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.
Tangent Space

Definition (Tangent Vector)

A tangent vector $\xi$ at the point $p$ is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at $p$ an $n$-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p) \xi^j.$$

A smooth vector field $\xi$ assigns a tangent vector for each point of $M$, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n) \frac{\partial}{\partial x_i}.$$  

$\{ \frac{\partial}{\partial x_i} \}$ represents the vector fields of the velocities of iso-parametric curves on $M$. They form a basis of all vector
Definition (Push-forward)

Suppose $\phi : M \to N$ is a differential map from $M$ to $N$, $\gamma : (-\varepsilon, \varepsilon) \to M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = v \in T_p M$, then $\phi \circ \gamma$ is a curve on $N$, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(v) = (\phi \circ \gamma)'(0) \in T_{\phi(p)} N,$$

as the push-forward tangent vector of $v$ induced by $\phi$. 

David Gu
Conformal Geometry
Definition (Differential 1-form)

The tangent space $T_pM$ is an n-dimensional vector space, its dual space $T_p^*M$ is called the cotangent space of $M$ at $p$. Suppose $\omega \in T_p^*M$, then $\omega : T_pM \rightarrow \mathbb{R}$ is a linear function defined on $T_pM$, $\omega$ is called a differential 1-form at $p$.

A differential 1-form field has the local representation

$$\omega(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \omega_i(x^1, x^2, \cdots, x^n)dx_i,$$

where $\{dx_i\}$ are the differential forms dual to $\{\frac{\partial}{\partial x_j}\}$, such that

$$dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}.$$
High order exterior forms

**Definition (Tensor)**

A tensor $\Theta$ of type $(m, n)$ on a manifold $M$ is a correspondence that associates to each point $p \in M$ a multi-linear map

$$\Theta_p : T_p M \times T_p M \times \cdots \times T_p M^* \cdots \times T_p^M \rightarrow \mathbb{R},$$

where the tangent space $T_p M$ appears $m$ times and cotangent space $T_p^* M$ appears $n$ times.

**Definition (exterior $m$-form)**

An exterior $m$-form is a tensor $\omega$ of type $(m, 0)$, which is skew symmetric in its arguments, namely

$$\omega_p(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \cdots, \xi_{\sigma(m)}) = (-1)^\sigma \omega_p(\xi_1, \xi_2, \cdots, \xi_m)$$

for any tangent vectors $\xi_1, \xi_2, \cdots, \xi_m \in T_p M$ and any permutation $\sigma \in S_m$, where $S_m$ is the permutation group.
The local representation of $\omega$ in $(x^1, x^2, \cdots, x^m)$ is

$$\omega = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \omega_{i_1 i_2 \cdots i_m} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_m} = \omega_I dx^I,$$

$\omega_I$ is a function of the reference point $p$, $\omega$ is said to be differentiable, if each $\omega_I$ is differentiable.
Definition (Wedge product)

The wedge product of an $m_1$-form and an $m_2$-form $\omega_2$ is an $m_1 + m_2$-form, which is defined in local coordinates by

$$\omega_{l_1} dx^{l_1} \wedge \omega_{l_2} dx^{l_2} = \omega_{l_1} \omega_{l_2} dx^{l_1} dx^{l_2}.$$ 

A coordinate free representation of wedge product is

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2, \cdots, \xi_{m_1+m_2}) = \sum_{\sigma \in S_{m_1+m_2}} \frac{(-1)^\sigma}{m_1! m_2!} \omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(m_1)}) \omega_2(\cdots).$$
Definition (Pull back)

Suppose $\phi : M \to N$ is a differentiable map from $M$ to $N$, $\omega$ is an $m$-form on $N$, then the pull-back $\phi^* \omega$ is an $m$-form on $M$ defined by

$$(\phi^* \omega)_p(\xi_1, \cdots, \xi_m) = \omega_{\phi(p)}(\phi_* \xi_1, \cdots, \phi_* \xi_m), \ p \in M$$

for $\xi_1, \xi_2, \cdots, \xi_m \in T_p M$, where $\phi_* \xi_j \in T_{\phi(p)} N$ is the push forward of $\xi_j \in T_p M$. 
Suppose that $U \subset \mathbb{R}^n$ is an open set,

$$\omega = f(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n,$$

then

$$\int_U \omega = \int_U f(x)dx^1 dx^2 \cdots dx^n.$$

Suppose $U \subset M$ is an open set of a manifold $M$, a chart $\phi : U \to \Omega \subset \mathbb{R}^n$, then

$$\int_U \omega = \int_{(\phi^{-1})^* \omega}.$$
Integration

Integration is independent of the choice of the charts. Let \( \psi : U \rightarrow \psi(U) \) be another chart, with local coordinates \((u_1, u_2, \cdots, u_n)\)

\[
\int_{\phi(U)} f(x) \, dx^1 \, dx^2 \cdots \, dx^n = \int_{\psi(U)} f(x(u)) \det \left( \frac{\partial x^i}{\partial u^j} \right) \, du^1 \, du^2 \cdots \, du^n.
\]
consider a covering of $M$ by coordinate charts $\{(U_\alpha, \phi_\alpha)\}$ and choose a partition of unity $\{f_i\}, i \in I$, such that $f_i(p) \geq 0$,

$$\sum_i f_i(p) \equiv 1, \forall p \in M.$$ 

Then $\omega_i = f_i \omega$ is an $n$-form on $M$ with compact support in some $U_\alpha$, we can set the integration as

$$\int_M \omega = \sum_i \int_M \omega_i.$$
Suppose $f : M \rightarrow \mathbb{R}$ is a differentiable function, then the exterior derivative of $f$ is a 1-form,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx^i.$$ 

The exterior derivative of an $m$-form on $M$ is an $(m+1)$-form on $M$ defined in local coordinates by

$$d\omega = d(\omega_I dx^I) = (d\omega_I) \wedge dx^I,$$

where $d\omega_I$ is the differential of the function $\omega_I$. 

Conformal Geometry
Theorem (Stokes)

let $M$ be an $n$-manifold with boundary $\partial M$ and $\omega$ be a differentialble $(n - 1)$-form with compact support on $M$, then

$$
\int_{\partial M} \omega = \int_M d\omega.
$$
Let $M$ be a differentiable manifold, $\Omega^n(M)$ represent all the $n$-forms on $M$, $d$ be the exterior derivatives. Then the de Rham complex

$$
\cdots \xrightarrow{d^{q-2}} \Omega^{q-1} \xrightarrow{d^{q-1}} \Omega^{q-1} \xrightarrow{d^{q-1}} \Omega^q \xrightarrow{d^q} \cdots
$$

The exterior differentiation operator

$$
d^m : \Omega^m(M) \to \Omega^{m+1}(M)
$$

is a linear operator with the property

$$
d^m \circ d^{m-1} \equiv 0.
$$
Definition (de Rham cohomology group)

Suppose $M$ is a differential manifold. The $m$-th de Rham cohomology group is defined as

$$H^m_{dR}(M) = \frac{\ker d^m}{\text{im} d^{m-1}}.$$ 

Theorem

The de Rham cohomology group $H^m_{dR}(M)$ is isomorphic to the cohomology group $H^m(M, \mathbb{R})$

$$H^m_{dR}(M) \cong H^m(M, \mathbb{R}).$$
Hodge Star

Suppose $M$ is a Riemannian manifold, we can locally find oriented orthonormal basis of vector fields, and choose parameterization, such that

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\}$$

form an oriented orthonormal basis. Let

$$\{dx_1, dx_2, \ldots, dx_n\}$$

be the dual 1-form basis.

**Definition (Hodge Star Operator)**

The Hodge star operator $*: \Omega^k(M) \to \Omega^{n-k}(M)$ is a linear operator

$$* (dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n.$$
Let $\sigma = (i_1, i_2, \cdots, i_n)$ be a permutation, then the hodge star operator
\[
\star (dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}) = (-1)^\sigma dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n}.
\]
Definition

Let $\eta, \zeta \in \Omega^k(M)$ are two $k$-forms on $M$, then the norm is defined as

$$(\eta, \zeta) = \int_M \eta \wedge^* \zeta.$$

$\Omega^k(M)$ is a Hilbert space.
The codifferential operator $\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ is defined as

$$\delta = (-1)^{k+1+k(n-k)} d^*,$$

where $d$ is the exterior derivative.

The codifferential is the adjoint of the exterior derivative, in that

$$(\delta \zeta, \eta) = (\zeta, d \eta).$$
Laplace Operator

Definition (Laplace Operator)

The Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$,

$$\Delta = d\delta + \delta d.$$ 

Lemma

The Laplace operator is symmetric

$$(\Delta \zeta, \eta) = (\zeta, \Delta \eta)$$

and non-negative

$$(\Delta \eta, \eta) \geq 0.$$ 

Proof.

$$(\Delta \zeta, \eta) = (d\zeta, d\eta) + (\delta \zeta, \delta \eta).$$
Harmonic Forms

Definition (Harmonic forms)
Suppose $\omega \in \Omega^k(M)$, then $\omega$ is called a $k$-harmonic form, if
\[ \Delta \omega = 0. \]

Lemma
$\omega$ is a harmonic form, if and only if
\[ d\omega = 0, \delta \omega = 0. \]

Proof.
\[ 0 = (\Delta \omega, \omega) = (d\omega, d\omega) + (\delta \omega, \delta \omega). \]
Definition (Harmonic form group)
All harmonic $k$-forms form a group, denoted as $H^k_\Delta(M)$.

Theorem (Hodge Decomposition)

\[ \Omega_k = \text{img}d^{k-1} \oplus \text{img} \delta^{k+1} \oplus H^k_\Delta(M). \]

Proof.

\[ (\text{img}d)^\perp = \{ \omega \in \Omega^k(M) | (\omega, d \eta) = 0, \forall \eta \in \Omega^{k-1}(M) \}, \]

because \((\omega, d \eta) = (\delta \omega, \eta)\), so \((\text{img}d^{k-1})^\perp = \text{ker} \delta^k\). Similarly, \((\text{img}\delta^{k+1})^\perp = \text{ker}d^k\).

Because \(\text{img}d^{k-1} \subset \text{ker}d^k\), \(\text{img}\delta^{k+1} \subset \text{ker}\delta^k\), therefore \(\text{img}d^{k-1} \perp \text{img}\delta^{k+1}\),

\[ \Omega^k = \text{img}d^{k-1} \oplus \text{img}\delta^{k+1} \oplus (\text{img}d^{k-1} \oplus \text{img}\delta^{k+1})^\perp \]

\[ H^k_\Delta = \text{ker}d^k \cap \text{ker}\delta^k = (\text{img}d^{k-1} \oplus \text{img}\delta^{k+1})^\perp. \]
suppose $\omega \in \ker d^k$, then $\omega \perp \text{img} \delta^{k+1}$, then $\omega = \alpha + \beta$, 
$\alpha \in \text{img} d^{k-1}$, $\beta \in H^k_\Delta(M)$, define project $h : \ker d^k \to H^k_\Delta(M)$,

**Theorem**

*Suppose $\omega$ is a closed form, its harmonic component is $h(\omega)$, then the map:*

$$h : H^k_{dR}(M) \to H^k_\Delta(M).$$

*is isomorphic.*

Each cohomologous class has a unique harmonic form.