1.4 Semi-Discrete Optimal Transport

Suppose we want to find the optimal transport map \( T : (X, \mu) \rightarrow (Y, \nu) \) with a cost function \( c : X \times Y \rightarrow \mathbb{R} \), where \( \mu \) is a continuous distribution \( d\mu(x) = f(x)dx \) on the compact space \( X \), and \( \nu \) is a discrete distribution \( \nu = \sum_{i=1}^{n} \nu_i \delta(y - y_i) \). In this circumstances, the target space \( Y \) becomes a set of finite distinct points 
\[ Y = \{ y_1, y_2, \ldots, y_n \} \].

Based on the dual problem formulation, we want to find the Kontarovich potential \( \phi : Y \rightarrow \mathbb{R} \) to maximize the functional:
\[
\max_{\varphi} \left\{ F(\varphi) := \int_X \varphi^c(x) d\mu(x) + \int_Y \varphi(y) d\nu(y) \right\}.
\]

Since \( \nu \) is discrete, the above functional can be reformulated as:
\[
\max_{\varphi} \left\{ F(\varphi_1, \ldots, \varphi_n) := \int_X \varphi^c(x) f(x) dx + \sum_{i=1}^{n} \varphi_i \nu_i \right\}. \tag{1.25}
\]

By the definition of \( c \)-transform, we obtain:
\[
\varphi^c(x) = \inf_{y \in Y} c(x, y) - \varphi(y) = \min_{j=1}^{n} c(x, y_j) - \varphi_j
\]

**Definition 1.13 (c-Voronoi Cell Decomposition).** The \( c \)-transform induces a \( c \)-Voronoi cell decomposition of \( X \),
\[
X = \bigcup_{i=1}^{n} W_\varphi(i) \tag{1.26}
\]
where each cell is called a \( c \)-Voronoi cell and defined as
\[
W_\varphi(i) := \{ x \in X | c(x, y_i) - \varphi_i \leq c(x, y_j) - \varphi_j, \forall j = 1, \ldots, n \}. \tag{1.27}
\]

Suppose the intersection between \( W_\varphi(i) \) and \( W_\varphi(j) \) is
\[
\Gamma_\varphi(i, j) = W_\varphi(i) \bigcap W_\varphi(j),
\]
which has zero \( \mu \)-measure. Then the \( c \)-transform \( \varphi^c \) can be written explicitly as
\[
\varphi^c(x) = c(x, y_i) - \varphi_i, \quad \forall x \in W_\varphi(i), \tag{1.28}
\]
plug into (1.25), we obtain
1.4 Semi-Discrete Optimal Transport

\[
F(\phi_1, \ldots, \phi_n) = \sum_{i=1}^{n} \phi_i v_i + \sum_{i=1}^{n} \int_{W_{\phi}(i)} c(x, y_i) - \phi_i \) f(x) \) dx 
\]

(1.29)

by \( \int_{W_{\phi}(i)} \) f(x) \) dx = \( \mu(W_{\phi}(i)) \), the above functional can be simplified to

\[
F(\phi_1, \ldots, \phi_n) = \sum_{i=1}^{n} (v_i - \mu(W_{\phi}(i))) \phi_i + \sum_{i=1}^{n} \int_{W_{\phi}(i)} c(x, y_i) f(x) dx. 
\]

(1.30)

We define an auxiliary function

\[
\lambda(x) = c(x, y_i) - c(x, y_j),
\]

(1.31)

then \( \Gamma_{\phi}(i, j) \) is the level set \( \lambda(x) = \phi_i - \phi_j \). The level sets are orthogonal to the gradient of \( \lambda \),

\[
\nabla \lambda(x) = \nabla_x c(x, y_i) - \nabla_x c(x, y_j).
\]

(1.32)

**Definition 1.14 (Stream Line).** The stream curve along the gradient field of \( \lambda \) can be defined as follows:

\[
\frac{d}{dt} \gamma(x, t) = \frac{\nabla \lambda(x)}{|\nabla \lambda(x)|} \quad \text{and} \quad \gamma(x, 0) = x. 
\]

(1.33)

Along the stream line, we have

\[
\frac{d}{dt} \lambda(\gamma(x, t)) = \langle \nabla \lambda, \dot{\gamma}(\gamma(x, t)) \rangle = |\nabla \lambda|(\gamma(x, t)).
\]

(1.34)

By implicit function theorem, along a stream line \( \gamma(x, t) \), \( \lambda(\gamma(x, t)) \) is invertible, \( (\lambda \circ \gamma)^{-1} \) maps the value of \( \lambda \) to the parameter \( t \).

### 1.4.1 Derivative of Cell Measure

Suppose \( h = (h_1, h_2, \ldots, h_n) \) is a vector with small norm. Some point will change from c-power Voronoi cell \( W_{\phi}(j) \) to \( W_{\phi + h}(i) \), as shown in Figure 1.14. Suppose \( h_i > h_j \), then the \( i \)-th power cell will be enlarged,

\[
h_i > h_j \quad \Rightarrow \quad W_{\phi}(j) \cap W_{\phi + h}(i) \neq \emptyset.
\]

We would like to estimate the \( \mu \)-measure of this set. Each stream line \( \gamma(x, t) \) starting from \( \Gamma_{\phi}(i, j) \) and arriving at \( \Gamma_{\phi + h}(i, j) \) at time \( T(x) \),

\[
h_i - h_j = \lambda(\gamma(x, T)) - \lambda(\gamma(x, 0)) = \int_{0}^{T} \dot{\lambda}(\gamma(x, t)) dt = \int_{0}^{T} |\nabla \lambda|(\gamma(x, t)) dt
\]

\[
\frac{d}{dt} \gamma(x, t) = \frac{\nabla \lambda(x)}{|\nabla \lambda(x)|}
\]

(1.33)
Therefore, we get the length estimate

\[ h_i - h_j = |\nabla \lambda(x)| T(x), \quad \text{for some } \xi \in \gamma(x,t), \quad t \in [0, T(x)]. \]

Since \( \gamma \) has the unit speed, \( T(x) = (h_i - h_j)/|\nabla \lambda(\xi)| \) is the length of the curve. Because \( W_\varphi(j) \cap W_{\varphi+h}(i) \) is compact, from the regularity of \( \lambda \), we have

\[ \|D^2 \lambda(\xi)\| \leq C, \quad \forall \xi \in W_\varphi(j) \cap W_{\varphi+h}(i). \]

Since \( h \) is small enough, from the regularity of \( \lambda \), we have

\[ |\nabla \lambda(x)| - C|h| \leq |\nabla \lambda(\xi)| \leq |\nabla \lambda(x)| + C|h|. \quad (1.35) \]

We obtain the estimate of the length of the curve

\[ T(x) = \frac{h_i - h_j}{|\nabla \lambda(\xi)|} = \frac{h_i - h_j}{|\lambda(x)|} \left( 1 \pm \frac{C}{|\lambda(x)|} |h| + o(|h|^2) \right) = \frac{h_i - h_j}{|\lambda(x)|} + o(|h|^2). \quad (1.36) \]

Since \( \gamma(x,t) \) is perpendicular to \( \Gamma_\varphi(i,j) \), hence the \( \mu \)-measure

\[ \mu(W_\varphi(j) \cap W_{\varphi+h}(i)) = \int_{\Gamma_\varphi(i,j)} f(x) T(x) dx = (h_i - h_j) \int_{\Gamma_\varphi(i,j)} \frac{f(x)}{|\nabla \lambda(x)|} dx + o(|h|^2). \]

Therefore

\[ \mu(W_\varphi(j) \cap W_{\varphi+h}(i)) = (h_i - h_j) \int_{\Gamma_\varphi(i,j)} \frac{f(x)}{|\nabla c(x,y_i) - \nabla c(x,y_j)|} dx + o(|h|^2). \]

This gives the partial derivatives of the \( \mu \)-measure of the cell
\[
\frac{\partial}{\partial \phi_j} \mu(W_\phi(i)) = -\int_{\Gamma_{\phi(i,j)}} f(x) \left| \nabla_x c(x,y_i) - \nabla_x c(x,y_j) \right| dx.
\]

(1.37)

This gives the symmetric relation:

\[
\frac{\partial}{\partial \phi_j} \mu(W_\phi(i)) = \frac{\partial}{\partial \phi_i} \mu(W_\phi(j))
\]

(1.38)

Furthermore,

\[
\frac{\partial}{\partial \phi_i} \mu(W_\phi(i)) = -\sum_{j \neq i} \frac{\partial}{\partial \phi_j} \mu(W_\phi(i))
\]

(1.39)

**Proposition 1.5.** In the linear space orthogonal to \((1,1,\ldots,1)^T\), the matrix

\[
H := \left( \frac{\partial}{\partial \phi_j} \mu(W_\phi(i)) \right)_{i,j}
\]

is positive definite.

**Proof.** From (1.37) we can see that

\[
\frac{\partial}{\partial \phi_j} \mu(W_\phi(i)) \leq 0,
\]

all the elements off diagonal are non-positive. By (1.39), we see that the summation of each row is zero, therefore \((1,1,\ldots,1)^T\) is the eigen vector corresponding to the eigen value zero. The matrix \(H + \varepsilon I\) is diagonal dominant, therefore positive definite, all eigen values are positive. Let \(\varepsilon \to 0\), then all the eigen values of \(H\) are non-negative.

Suppose the zero eigen value is multiple, there is another eigen vector \(v\), \(Hv = 0\) and \(v\) is not equal to \(\alpha(1,1,\ldots,1)^T\) for any real number \(\alpha\). Assume \(v_1 > 0\) and \(|v_1| \geq |v_i|\) for any \(i = 1,\ldots,n\), strict inequality holds for at least some \(i\). Then we get

\[
v_1 h_{11} + v_2 h_{12} + \cdots + v_n h_{1n} \geq v_1 h_{11} - \sum_{i=2}^n |v_i| h_i = \sum_{i=2}^n (v_1 - |v_i|) h_i > 0.
\]

contradiction. This shows \(H\) is positive definite in the subspace orthogonal to \((1,1,\cdots,1)^T\).

\(\square\)

### 1.4.2 Derivatives of Functional

When \(\phi\) is changed to \(\phi + h\), the c-Voronoi cell decomposition is changed as

\[
X = \bigcup_{i=1}^n W_\phi(i) = \bigcup_{j=1}^n W_{\phi+h}(j).
\]

The functional \(F(\phi)\) in (1.29) is changed accordingly.

1.4.2.1 Points Within the Same Cell

One the set of $W_{\phi}(i) \cap W_{\phi+\lambda}(i)$, we have

$$(\phi + h)f(x) - \phi^0(x) = (c(x,y_i) - \phi_i - h_i) - (c(x,y_j) - \phi_j) = -h_i.$$  

This give us that

$$\int_{W_{\phi}(i) \cap W_{\phi+\lambda}(i)} ((\phi + h)f(x) - \phi^0(x)) f(x) dx = -h_i \mu(W_{\phi}(i) \cap W_{\phi+\lambda}(i)).$$  

(1.40)

By (1.37) and (1.39), we know

$$\mu(W_{\phi}(i)) = \mu(W_{\phi}(i) \cap W_{\phi+\lambda}(i)) + \sum_{j \neq i} \mu(W_{\phi}(i) \cap W_{\phi+\lambda}(j))$$

$$= \mu(W_{\phi}(i) \cap W_{\phi+\lambda}(i)) + O(|h|).$$

$$\int_{W_{\phi}(i) \cap W_{\phi+\lambda}(i)} ((\phi + h)f(x) - \phi^0(x)) f(x) dx = -h_i \mu(W_{\phi}(i)) + O(|h|).$$  

(1.41)

1.4.2.2 Points Changing to Different Cells

As shown in Figure 1.15, on the set of $W_{\phi}(j) \cap W_{\phi+\lambda}(i)$, we apply the stream line between $\Gamma_{\phi}(i,j)$ to $\Gamma_{\phi+\lambda}(i,j)$ as defined in (1.33), for $t \in [0, T(x)]$

$$(\phi + h)^{\gamma}(\gamma(x,t)) - \phi^0(\gamma(x,t)) = (c(x,y_i) - \phi_i - h_t) - (c(x,y_j) - \phi_j)$$

$$= \lambda(\gamma(x,t)) - (\phi_i - \phi_j) - h_t$$

$$= \lambda(\gamma(x,t)) - \lambda(\gamma(x,0)) - h_t.$$  

The integration along one stream line

$$\int_{0}^{T(x)} [(\phi + h)^{\gamma}(\gamma(x,t)) - \phi^0(\gamma(x,t))] dt = \int_{0}^{T(x)} [\lambda(\gamma(x,t)) - \lambda(\gamma(x,0)) - h_t] dt.$$
We can define \( \rho(t) = \lambda \circ \gamma(x,t) \), then
\[
\frac{d \rho}{dt}(t) = |\nabla \lambda(\gamma(x,t))| > 0.
\]
by changing variable, \( t \mapsto \rho(t) - \rho(0) \), the above integration can be rewritten as
\[
\int_0^T (\rho(t) - h_i)dt = \int_0^{h_i} \frac{\rho - h_i}{\rho} d\rho
\]
By gradient estimate (1.35), we get
\[
\int_0^{h_i} \frac{\rho - h_i}{\rho} d\rho = \frac{1}{|\nabla \lambda(x)|} \int_0^{h_i} (\rho - h) d\rho + O(|h|^2)
\]
Then we estimate the functional difference on \( W_{\phi(j)} \cap W_{\phi+h(i)} \),
\[
\int_{W_{\phi(j)} \cap W_{\phi+h(i)}} (\phi + h)^c(x) - \phi^c f(x) dx
\]
\[
= \int_{I_{\phi(i,j)}} f(x) \int_0^{T(x)} [(\phi + h)^c(\gamma(x,t) - \phi^c(\gamma(x,t))] dt dx
\]
\[
= \int_{I_{\phi(i,j)}} f(x) \int_0^{T(x)} (\frac{\rho - h_i}{\rho}) d\rho + O(|h|^2)
\]
This shows
\[
\int_{W_{\phi(j)} \cap W_{\phi+h(i)}} (\phi + h)^c - \phi^c f(x) dx = O(|h|^2). \tag{1.42}
\]
Combining the estimate in (1.41) and (1.42), We obtain the following theorem:

**Theorem 1.4 (Semi-discrete Optimal Transport).** Suppose \( X \) is a compact domain in a metric space \( X \), \( \mu \) is a probability measure with continuous density function \( d\mu = f(x)dx \); \( Y \subset X \) is a discrete point set \( Y = \{y_1, \ldots, y_n\} \) with Dirac measure \( \nu = \sum_{i=1}^n \delta(y - y_i) \), \( \nu_i \geq 0 \). The total measures are equal, \( \mu(X) = \sum \nu_i \). Given a \( C^2 \) cost function \( c : X \times Y \to \mathbb{R} \), the Kontarovici dual functional for the semi-discrete optimal transportation problem is defined as
\[
F(\phi) = \sum_{i=1}^n \nu_i \phi_i + \int_X \phi^c(x) f(x) dx.
\]
Then the first order partial derivative of the functional is given by
\[
\frac{\partial F(\phi)}{\partial \phi_i} = \nu_i - \mu(W_{\phi(i)}). \tag{1.43}
\]
the second derivative is given by
\[
\frac{\partial^2 F(\varphi)}{\partial \varphi_i \partial \varphi_j} = \int_{\Gamma_{\varphi(i,j)}} \frac{f(x)}{|V_x c(x,y_j) - V_x c(x,y_i)|} d\mathcal{H}^{d-1}(x). \tag{1.44}
\]

Furthermore, the functional \( F(\varphi) \) is strictly concave in the space

\[ \Phi := \{ \varphi : \varphi_1 + \varphi_2 + \cdots + \varphi_n = 1 \}. \]

**Proof.** By the definition of the functional, we have

\[
F(\varphi + h) = \sum_{i=1}^{n} (\varphi_i + h_i) \nu_i + \int_X ((\varphi + h)^c(x) - \varphi^c(x)) f(x) dx,
\]

Then

\[
F(\varphi + h) - F(\varphi) = \sum_{i=1}^{n} h_i \nu_i + \int_X ((\varphi + h)^c - \varphi^c)(x) f(x) dx.
\]

Consider the c-voronoi cell decomposition,

\[
X = \bigcup_{i=1}^{n} W(\varphi + h)(i) = \bigcup_{i=1}^{n} \left\{ W_{\varphi + h}(i) \cap W_{\varphi}(i) \bigcup_{j \neq i} W_{\varphi + h}(i) \cap W_{\varphi}(j) \right\}
\]

We estimate the second term using this decomposition, by (1.41) and (1.42), we obtain

\[
\int_X ((\varphi + h)^c - \varphi^c)(x) f(x) dx = - \sum_{i=1}^{n} h_i \mu(W_{\varphi}(i)) + O(|h|^2).
\]

This proves the first derivative formula (1.43) of the functional. The partial derivative of the \( \mu \)-measure of the cell (1.37) gives the second derivative formula (1.44) for the functional. The proposition 1.5 proves the strict concavity of the functional. This completes the proof. \( \square \)