1.3 Oliker-Prussner Method

Oliker-Prussner method solves the Monge-Ampère equation based on comparison principle in convex geometry.

\[
\det D^2 u = f \quad \text{in } \Omega \\
u = \phi \quad \text{on } \partial \Omega
\]

1.3.1 Discretization

As shown in Figure 1.5 left frame, the domain $\Omega$ is triangulated. There are many algorithms to triangulate a domain in the Euclidean space. For example, we can use Delaunay refinement algorithm to compute high quality meshes for convex domains.

We denote the vertices of $\partial \Omega$ by $b_1, b_2, \ldots, b_N$, ($b_{N+i} = b_i$). The function $\phi : \partial \Omega \rightarrow \mathbb{R}$ is assumed to be continuous and piecewise linear so that over each edge $[b_i, b_{i+1}]$ it is a linear function. Then the curve $C = \{(u, \phi(u)) | u \in \partial \Omega\}$ is a piecewise linear curve in $\mathbb{R}^3$ and its vertices project (one-to-one) into $b_1, \ldots, b_N$.

Let $\Omega \subset \mathbb{R}^2$ and $a_1, \ldots, a_n$ arbitrary (but distinct) points inside $\Omega$. The class of continuous convex functions over $\Omega$ is denoted by $\mathcal{W}$. Let $u \in \mathcal{W}$ and suppose that its graph $S_u$ is a polyhedral surface, that is, it can be subdivided into a finite number of convex planar regions (faces). Assume further that $u|_{\partial \Omega} = \phi$, the only possible vertices of $S_u$ where there is a strictly supporting plane (intersecting $S_u$ only in the vertex) and which project into interior points of $\Omega$ lie on vertical rays at the points $a_1, \ldots, a_n$. The collection of such functions in $\mathcal{W}$ is denoted as $\mathcal{W}_n$. Figure 1.5 shows...
one piecewise linear convex function \( u \in \mathcal{W}_n \), whose graph is the lower convex hull of the points \( \{(a_i, u(a_i))\} \) and \( \{(b_j, u(b_j))\} \).

Suppose \( \Omega \) is triangulated as shown in the Figure 1.5, suppose the triangulation is Delaunay, then the corresponding Voronoi diagram is given by

\[
\mathbb{R}^2 = \bigcup_{i=1}^n W_i,
\]

each Voronoi cell \( W_i \) is centered at \( a_i \). We define the discrete measure

\[
\nu := \sum_{i=1}^n \nu_i \delta(y - a_i), \quad \nu_i = \int_{W_i} f \, dx_1 \wedge dx_2.
\]

The following problem is posed:

**Problem 1.1 (Alexandrov Solution to Monge-Ampère Equation).** Find a function \( u \in \mathcal{W}_n \) such that

\[
\mu_u(a_i) = \nu_i \quad i = 1, 2, \ldots, n.
\]

where \( \mu_u \) is the Monge-Ampère measure induced by the function \( u \in \mathcal{W}_n \).

Namely, \( \mu_u = \nu \), \( u \) is the Alexandrov solution to the original Monge-Ampère equation (1.16).

### 1.3.2 Legendre Duality

Suppose \( \mathbb{R}^d \) be a Euclidean space, its dual space \( (\mathbb{R}^d)^* \) is the linear space of all affine functions defined on \( \mathbb{R}^d \). Let \( l_{p,q} : \mathbb{R}^d \to \mathbb{R} \) be an affine function \( l_{p,q}(x) := \langle p, x \rangle - q \), its dual is a point in \( (\mathbb{R}^d)^* \) with coordinates \( l_{p,q}^* := (p, q) \).

\[
l_{p,q}(x) = \langle p, x \rangle - q \iff l_{p,q}^* = (p, q).
\]

**Definition 1.10 (Supporting Plane).** Suppose \( u : \mathbb{R}^d \to \mathbb{R} \) is a continuous function. A hyper-plane \( l_{p,q} \) is a supporting plane if \( u(x) \geq l_{p,q}(x), \forall x \in \mathbb{R}^d \). If at some point \( x \in \mathbb{R}^d, l_{p,q}(x) = u(x) \), the \( l_{p,q} \) is called a strictly supporting hyper-plane of \( u \).

Intuitively, if the hyper-plane of \( l_{p,q} \) is beneath the graph of \( u, S_u \), then \( l_{p,q} \) supports \( u \). If furthermore, a supporting hyper-plane \( l_{p,q} \) touches \( S_u \) at some point \((x, u(x))\), the \( l_{p,q} \) is a strict supporting hyper-plane.

**Definition 1.11 (Supporting Plane Set).** Suppose \( u : \mathbb{R}^d \to \mathbb{R} \) is a continuous function with graph \( S_u \). The supporting plane set is defined as

\[
\mathcal{B}_u := \{l_{p,q} : u(x) \geq l_{p,q}(x), \forall x\}.
\]
The strict supporting plane set is defined as

$$\mathcal{S}_u := \{ l_{p,q} \in \mathcal{R}_u : u(x) = l_{p,q}(x), \text{for some } x \}.$$  

Figure 1.6 shows some supporting planes of $u$. The hyper-plane in red color $l_{p,q}$ is a strict supporting hyper-plane.

**Lemma 1.3.** If $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, then its supporting hyper-plane set $\mathcal{R}_u$ is convex.

**Proof.** Suppose $l_{p_1,q_1}$ and $l_{p_2,q_2}$ are in $\mathcal{R}_u$, then for each $x$

$$((1-\lambda)p_1+\lambda p_2,x) - ((1-\lambda)q_1+\lambda q_2) = (1-\lambda) ((p_1,x)-q_1) + \lambda ((p_2,x)-q_1) \leq u(x).$$

hence $l_{(1-\lambda)p_1+\lambda p_2,(1-\lambda)q_1+\lambda q_2} \in \mathcal{R}_u$. This shows $\mathcal{R}_u$ is convex. Note that the convexity of $\mathcal{R}_u$ doesn’t rely on the convexity of $u$. \qed

**Definition 1.12 (Epigraph).** Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, the epigraph is the point set in $\mathbb{R}^{d+1}$.

$$\text{epi}(u) := \{(x, \xi) | \xi \geq u(x)\}.$$  

In the left frame of the Figure 1.7, all the lines under the graph of $u$ form $\mathcal{R}_u$. The epigraph of $u^*$ is the shadow region in the right frame of Figure 1.7.

**Lemma 1.4.** Let $u : \mathbb{R}^d \to \mathbb{R}$ is continuous, $u^* : (\mathbb{R}^d)^* \to \mathbb{R}$ is the Legendre dual of $u$, then there is a one-to-one correspondence between $\mathcal{R}_u$ and $\text{epi}(u^*)$, $l_{p,q} \in \mathcal{R}_u$ if and only if $l_{p,q}^* \in \text{epi}(u^*)$. 

![Figure 1.6 supporting plane set of $u$.](image-url)
Fig. 1.7 Duality between the supporting planes of $u$ and the epigraph of $u^\ast$.

**Proof.** Suppose $l_{p,q} \in \mathcal{B}_u$, then for any $x \in \mathbb{R}^d$, $u(x) \geq \langle p, x \rangle - q$, then

$$\forall x \quad q \geq \langle p, x \rangle - u(x) \implies q \geq \sup_x \langle p, x \rangle - u(x) = u^\ast(p).$$

Therefore $(p, q) \in \text{epi}(u^\ast)$. Conversely, suppose $(p, q) \in \text{epi}(u^\ast)$, then

$$q \geq u^\ast(p) = \sup_x \langle p, x \rangle - u(x) \geq \langle p, x \rangle - u(x), \forall x,$$

hence $u(x) \geq \langle p, x \rangle - q, \forall x \in \mathbb{R}^d$. Namely, $l_{p,q} \in \mathcal{B}_u$. \hfill \Box

**Lemma 1.5.** Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, $u^\ast$ is the Legendre dual of $u$. The strict supporting plane set of $u$ $\mathcal{S}_u$ and the graph of $u^\ast$ $\mathcal{S}_{u^\ast}$ has one-to-one correspondence. $l_{p,q} \in \mathcal{S}_u$ if and only if $(p, q) \in \mathcal{S}_{u^\ast}$.

**Proof.** Suppose $l_{p,q} \in \mathcal{S}_u$, then $\forall x$, $u(x) \geq \langle p, x \rangle - q$, namely $\forall x$, we have $q \geq \langle p, x \rangle - u(x)$, hence

$$q \geq \sup_x \langle p, x \rangle - u(x) = u^\ast(p).$$

Also, there is a point $\bar{x}$, such that $u(\bar{x}) = \langle p, \bar{x} \rangle - q$.

$$q = \langle p, \bar{x} \rangle - u(\bar{x}) \leq \sup_x \langle p, x \rangle - u(x) = u^\ast(p),$$

therefore $u^\ast(p) = q, (p, q) \in \mathcal{S}_{u^\ast}$.

Conversely, if $(p, q) \in \mathcal{S}_{u^\ast}$, then $q = u^\ast(p)$, hence $l_{p,u^\ast(p)}$ is a strict supporting hyper-plane of $u$, $l_{p,u^\ast(p)} \in \mathcal{S}_u$. \hfill \Box

**Lemma 1.6.** Suppose $u : \mathbb{R}^d \to \mathbb{R}$ is a continuous function, $u^\ast : (\mathbb{R}^d)^* \to \mathbb{R}$ is the Legendre dual of $u$. Then $u^\ast$ is a convex function.

**Proof.** The supporting hyper-plane set $\mathcal{B}_u$ is convex, which has one-to-one correspondence to $\text{epi}(u^\ast)$, hence $\text{epi}(u^\ast)$ is convex. The epigraph is convex, so $u^\ast$ is convex. Note that even if $u$ is non-convex, its Legendre dual $u^\ast$ is still convex. \hfill \Box
Furthermore, if \( u \) is convex then
\begin{equation}
\text{Equation (1.19)}
\end{equation}
\[
p \in \partial u(x) \iff u(x) + u^*(x) = \langle p, x \rangle,
\]
and
\begin{equation}
\text{Equation (1.20)}
\end{equation}
\[
p \in \partial u(x) \implies x \in \partial u^*(p).
\]

**Proof.** Equation (1.19) Suppose \( p \in \partial u(x) \) then there is a supporting hyper-plane \( l_{p,q} \) touches \( S_u \) at \( (x, u(x)) \), namely
\[
\forall y \ u(y) \geq l_{p,q}(y) = \langle p, y \rangle - q \implies \forall y \ q \geq \langle p, y \rangle - u(y) \implies q \geq u^*(p)
\]
\[
u(x) = l_{p,q}(x) = \langle p, x \rangle - q \implies q = \langle p, x \rangle - u(x) \implies q \leq u^*(p)
\]
This shows \( q = u^*(p) \). From \( u(x) = \langle p, x \rangle - q \), we obtain
\[
\langle p, x \rangle = u(x) + q = u(x) + u^*(p).
\]
Conversely, if \( \langle p, x \rangle = u(x) + u^*(p) \), then
\[
u^*(p) = \langle p, x \rangle - u(x) = \sup_y \langle p, y \rangle - u(y) \geq \langle p, y \rangle - u(y) \quad \forall y
\]
this implies \( \forall y \ u(y) \geq \langle p, y - x \rangle + u(x) \). This shows \( l_{p,u^*(p)} \) is a supporting hyper-plane of \( u \) at \( x \). Therefore \( p \in \partial u(x) \).

Equation (1.20) Suppose \( p \in \partial u(x) \), then we obtain \( u(x) + u^*(p) = \langle x, p \rangle \). By definition,
\[
u^*(q) = \sup_x \langle x, q \rangle - u(x) \geq \langle x, q \rangle - u(x) = \langle x, q \rangle - (\langle p, x \rangle - u^*(p)) = \langle x, q - p \rangle + u^*(p),
\]
this shows \( x \in \partial u^*(p) \).

Equation (1.21) If \( x \in \partial u^*(p) \), then
\[
u^*(p) + u^{**}(x) = \langle x, p \rangle,
\]
since \( u \) is convex, \( u^{**} = u \), hence \( u^*(p) + u(x) = \langle x, p \rangle \), by (1.19) we obtain \( p \in \partial u(x) \).

**1.3 Oliker-Prussner Method**

Figure 1.8 shows the Legendre transformation. A piecewise linear convex function \( u \in \mathcal{W}_a \) is shown in the left frame, its Legendre dual \( u^* \) shown in the right frame. Fix a point \( x \), by the definition of subdifferential
\[ \partial u(x) = \{ p : u(y) \geq \langle p, y-x \rangle + u(x) \}. \]

It is easy to see that \( \partial u(x) \) is convex. Every point \( p \in \partial u(x) \) is on the hyper-plane \( u^*(p) = \langle p, x \rangle - u(x) \).

**Vertex Case:** If \((x, u(x))\) is a vertex \( v \) on the graph \( S_u \), in this situation \( |\partial u(x)| > 0 \). Then we define
\[ v^* := \{(p, u^*(p)) | p \in \partial u(x)\}, \]
which is a linear face on the graph of \( u^* \). \( v^* \subset S_{u^*} \).

**Face Case:** Suppose \((x, u(x))\) is an interior point of a face \( f \in S_u \), and the gradient of the linear function of \( f \) is \( p \), then \( \partial u(x) = \{p\} \). Then all the interior points inside \( f \) corresponding to the same point \((p, u^*(p)) \in S_{u^*}\). This shows \( |\partial u^*(p)| > 0 \), hence \((p, u^*(p)) \) is vertex of \( S_{u^*} \), denoted as \( f^* \).

**Edge Case:** Suppose \((x, u(x))\) is an interior point of an edge \( e \in S_u \). Then \( \partial u(x) \) is line segment with measure zero, its dual \( e^* \) is an edge on \( S_{u^*} \).

We use \( V(S_u), E(S_u) \) and \( F(S_u) \) to represent the vertex, edge and face set of the polyhedron \( S_u \) respectively. Each vertex \( v \in V(S_u) \) on the graph \( S_u \) is denoted as \((x_v, y_v, z_v)\) for both interior and boundary vertices; each face \( f \in S_u \) is a triangle \( f := [v_i, v_j, v_k] \) has a supporting plane \( \pi_f \) with linear equation \( z = p_f x + q_f y - r_f \). The coefficients \((p_f, q_f, r_f)\) can be calculated from the coordinates of \( v_i, v_j \) and \( v_k \) directly: the volume of the tetrahedron formed by the face and any point on the plane \( \pi_f \) is zero, which gives the linear equation
\[
\begin{vmatrix}
  x & y & z & 1 \\
  x_i & y_i & z_i & 1 \\
  x_j & y_j & z_j & 1 \\
  x_k & y_k & z_k & 1 \\
\end{vmatrix} = 0, \quad (1.22)
\]
which in turn gives \((p_f, q_f, r_f)\). By the geometric property of subdifferential
With the class $W$, we can obtain the following duality relation between $S_u$ and $S_{u^*}$ directly.

**Proposition 1.4 (Piecewise Linear Convex Function Duality).** Given a piecewise linear convex function $u \in W_0$, its Legendre dual is $u^*$. Suppose the graph $S_u \subset \mathbb{R}^3$ with coordinates $(x,y,z)$, and the graph of its Legendre dual $S_{u^*} \subset (\mathbb{R}^3)^*$ with coordinates $(p,q,r)$.

1. Each vertex $v \in V(S_u)$ with coordinates $(x_r,y_r,z_r)$ corresponds to a face in $S_{u^*}$, whose supporting plane is given by $r = x_v p + y_v q - z_v$, denoted as $v^*$;
2. Each face $f \in F(S_u)$ with the supporting plane $z = p_f x + q_f y - r_f$ corresponds to a vertex in $S_{u^*}$ with coordinates $(p_f,q_f,r_f)$, denoted as $f^*$;
3. $S_u$ is the lower convex hull of its vertices $S_u = \text{conv}\{(v_i)\}, v_i \in V(S_u);
4. $S_u$ is the upper envelope of the supporting planes through their faces $S_u = \text{env}\{(r_f)\}, f \in F(S_u);
5. $S_{u^*}$ is the lower convex hull of its vertices $S_{u^*} = \text{conv}\{(f^*)\}, f \in F(S_u);
6. $S_{u^*}$ is the upper envelope of its faces $S_{u^*} = \text{env}\{(v^*)\}, v \in V(S_u)$.

### 1.3.4 Iterative Algorithm

With the class $W_0$ we associate a map $g : W_0 \rightarrow \mathbb{R}^n$, where

$$g(u) = (g_1(u), \ldots, g_n(u)), \quad g_i(u) = \mu_i(u). \quad (1.23)$$

This is the discrete version of the Monge-Ampère measure on $W_0$. The problem (1.18) becomes now equivalent to the problem of finding a $u^* \in W_0$ for which $g(u^*) = v$, where $v = (v_1, v_2, \ldots, v_n)$ is a vector with non-negative components prescribed in advance.

**Lemma 1.7 (Piecewise Quadratic).** Each component of the mapping $g : W_0 \rightarrow \mathbb{R}$ is a piecewise polynomial of $u = (u_1, u_2, \ldots, u_2)$. In particular, $g_i(u)$ is a piecewise quadratic of $u$, globally continuous.

**Proof.** Suppose $u \in W_0$ be a piecewise linear convex function. Let $u^*$ be the Legendre dual of $u$. Fix a vertex $v_i = (a_i, u_i)$ on the graph of $u$, its dual $v_i^*$ is a face on the graph of $u^*$. The projection of $v_i^*$ is a planar convex polygon, denoted as $a_i^*$. Then $g_i(u)$ is the area of $a_i^*$. Now we compute the area of this polygon.

Suppose on the graph $S_u$, the one-ring neighbor of $v_i$ is denoted as $N_i$. Assume there are $n_j$ vertices in the neighbor, sorted counter clockwise and denoted as $v_j, j = 1, \ldots, n_i$. Consider the $j$-th face adjacent to $v_i$, the triangle $[v_i, v_j, v_{j+1}]$, whose linear equation is given by (1.22). After expansion, we obtain

$$
\begin{align*}
x & \mid y_i & u_i & 1 \\
y & \mid y_j & u_j & 1 \\
y & \mid y_{j+1} & u_{j+1} & 1 \\
+ & \mid z & x_i & y_i & 1 \\
+ & \mid z & x_j & y_j & 1 \\
+ & \mid z & x_{j+1} & y_{j+1} & 1 \\
\end{align*}
\begin{align*}
x & \mid y_i & u_i & 1 \\
y & \mid y_j & u_j & 1 \\
y & \mid y_{j+1} & u_{j+1} & 1 \\
+ & \mid z & x_i & y_i & 1 \\
+ & \mid z & x_j & y_j & 1 \\
+ & \mid z & x_{j+1} & y_{j+1} & 1 \\
\end{align*} = 0.
$$
We can rewrite it as \( z = p_j x + q_j y - r_j \), it is easy to see that \( p_j, q_j, r_j \) are linear functions of \( u_i, u_j, u_{j+1} \). Next, we compute the area of \( a_i^r \),
\[
g_i(u) = \sum_{j=1}^{n} p_j(u) q_{j+1}(u) - p_{j+1}(u) q_j(u),
\]

It is quadratic polynomial of \((u_1, \ldots, u_n)\), in particular, it is quadratic of \( u_i \).

Fix a \( u, S_u \) is the convex hull of \( \{(a_i, u_i)\}_{i=1}^n \). The projection of \( S_u \) induces a triangulation of \( \Omega \) with vertices \( \{a_i\}_{i=1}^n \) and \( \{b_j\}_{j=1}^N \). The number of all possible triangulations is finite. We can decompose \( \mathbb{R}^n \) into a finite number of cells,
\[
\mathbb{R}^n = \bigcup_{k=1}^{N} U_k,
\]
if \( u_1 \) and \( u_2 \) are in the same cell, \( u_1, u_2 \in U_k \), then \( S_{u_1} \) and \( S_{u_2} \) share the same combinatorial structure. Within each cell, \( g_i(u) \) has the same polynomial representation. Therefore, globally \( g_i(u) \) is a piecewise quadratic polynomial.

It is easy to see that the convex hull \( S_u \) continuously depends on \( u \), so is the Legendre dual \( S_{u^*} \). Therefore \( g(u) \) is globally continuous. \( \square \)

**Lemma 1.8.** Let \( u \) and \( u' \in \mathcal{W}_n \), and \( \bar{u} \) and \( \bar{u}' \) be the corresponding vectors in \( \mathbb{R}^n \). Suppose \( g(u) - g(u') \geq 0 \), namely for any \( i = 1, \ldots, n \), \( \mu_u(a_i) - \mu_{u'}(a_i) \geq 0 \). Then \( \bar{u}' - \bar{u} \geq 0 \), therefore \( u(p) \leq u'(p) \) for \( p \in \overline{\Omega} \). If \( g(u) \equiv g(u') \), then \( u(p) \equiv u'(p) \), \( \forall p \in \overline{\Omega} \).

**Proof.** This can be proven using comparison principle of the weak solution to Monge-Ampère equation. \( \square \)

**Lemma 1.9.** There exists a unique \( u \in \mathcal{W}_n \) such that \( g(u) = 0 \), namely \( \mu_u(a_i) = 0 \), for \( i = 1, \ldots, n \).

**Proof.** Let \( q_i = (b_i, \varphi(b_i)) \) where \( b_i \) are vertices on \( \partial \Omega \), and consider the convex hull of these points \( \{q_1, \ldots, q_N \} \). Then the lower convex hull defines a piecewise linear function \( u \in \mathcal{W}_n \). For every interior point \( a_i \), the Monge-Ampère measure \( \mu_u(a_i) \) is zero, so \( g(u) = 0 \). \( \square \)

**Theorem 1.1 (Oliker-Prussner One Step).** Let \( u \in \mathcal{W}_n, g_i(u) \leq v_i, i = 1, \ldots, n \), and at least one of the inequalities be strict. Then there is a finite algorithm for defining a unique \( u' \in \mathcal{W}_n \) for which \( u' \leq u, g_i(u') \leq v_i, \), \( \sum_{i=1}^{n} g_i(u) < \sum_{i=1}^{n} g_i(u') \).

**Proof.** Let \( S_u \) be the graph of \( u \in \mathcal{W}_n \). We fix all the other vertices, and only change the height of \( v_i \), namely change the position of \( v_i \) from \((a_i, u_i)\) to \((a_i, \xi)\), \( \xi \in [-\infty, u_i] \).

The interval \( (-\infty, u_i] \) can be partitioned into a finite number of intervals \( \Lambda_s = [\alpha_s, \beta_s] \), \( s = 1, 2, \ldots, k \). The connectivity of the convex hull of the graph remains fixed within each interval \( \Lambda_s \), and changed when \( \xi \) crosses different intervals. At the critical value \( \beta_s \), two faces adjacent to \( v_i \) becomes co-planar.

For convenience, we define the measure of the partial differential at \( a_i \) as
1.3 Oliker-Prussner Method

\[ h_i(\xi) = g_i(u_1, \ldots, u_{i-1}, \xi, u_{i+1}, \ldots, u_n). \]

By lemma 1.7, \( h_i(\xi) \) is a piecewise quadratic polynomial. Furthermore \( h_i(\xi) \) is monotonous by comparison principle if \( \xi_1 < \xi_2 \), then \( h_i(\xi_1) > h_i(\xi_2) \). Therefore \( h(\xi) \) is a strictly monotone. It is easy to see that when \( \xi \to -\infty, h(\xi) \to \infty \). The strict monotonocity of \( h(\xi) \) shows that the equation

\[ h_i(\xi) = v_i \quad (1.24) \]

has a unique solution, denoted as \( \xi_i \).

In the same way we find solutions \( \xi_i \) for all Eqn. (1.24) corresponding to the system \( h_i(\xi) = v_i \). The equations are processed consecutively starting with \( i = 1 \). Thus, for each interior point \( a_i \in \Omega \), we have the point \( (a_i, \xi_i) \). The boundary points \( (b_j, \varphi(b_j)) \) remain fixed in the process of finding \( \xi_i, i = 1, \ldots, n \). We take the lower convex hull of \( (a_i, \xi_i) \) and \( (b_j, \varphi(b_j)) \). The resulting graph we denote by \( S_\tau \) corresponding to an updated function \( u' \). Clearly \( u'(p) \leq u(p) \) for all \( p \in \Omega, u'(p) = u(p) \) for all \( p \in \partial \Omega \), and the only possible vertices of \( S_u \) are among the points \( (a_i, u'(a_i)) \). Thus \( u' \in \mathcal{H}'_n \).

Let us show that \( g_i(u') \leq g_i(\tau) = v_i \).

We claim that there exists a constant \( \nu \neq 0 \) depending only on \( \nu \) and boundary values of \( \chi \) and \( \partial \Omega \).

Finally, we note that for any two functions \( u \) and \( u' \in \mathcal{H}'_n \), such that \( u(p) \geq u'(p) \) for all \( p \in \Omega \) and \( u \neq u' \), the property \( \sum_{i=1}^{n} g_i(u) < \sum_{i=1}^{n} g_i(u') \) holds. Indeed, since on \( \partial \Omega \), \( u'(p) = u'(p) \), for any plane supporting to \( S_u \), there exists a parallel plane supporting to \( S_{u'} \). But, obviously there are planes supporting to \( S_u \) for which there are no parallel planes supporting to \( S_u \). This completes the proof.

\[ \square \]

**Theorem 1.2 (Oliker Prussner - Whole Algorithm).** With the use of the algorithm in theorem 1.1 one can construct a monotone sequence of functions \( u_t \in \mathcal{H}'_n, t = 1, 2, \ldots, \) and \( u_t(p) \geq u_{t+1}(p) \), for all \( p \in \Omega \), converging in \( C(\Omega) \)-norm to a unique function \( u^* \in \mathcal{H}'_n \) such that \( g(u^*) = v \).

**Proof.** As the function \( u_0 \) one can take the one constructed in Lemma 1.9. The elements of the sequence \( \{u_t\} \) are generated as in theorem 1.1, and if for some \( t, \|g_t(u_t) - v_t\| < \varepsilon, i = 1, \ldots, n \) for a prescribed threshold \( \varepsilon \), the procedure stops; otherwise it continues. The result is a monotone sequence \( u_t \in \mathcal{H}'_n, t = 0, 1, \ldots, \) for which \( g_t(u_t) \leq v_t, i = 1, \ldots, n \). We show now that this sequence is convergent.

Put \( \sum_{t=1}^{n} v_t = \chi \). We claim that there exists a constant \( C > 0 \), depending only on \( \chi \) and boundary values of \( u_t|\partial \Omega = \varphi \) for all such that
\[ |u_t|_{L^\infty} = \max_{\Omega} |u_t| \leq C \quad \forall t = 1, 2, \ldots \]

First of all, since elements of \( \mathcal{W}'_n \) are piecewise linear functions, if
\[ \max_{\Omega} |u_t| = \max_{\partial \Omega} |u_t| \]
the estimate is obvious.

Suppose \( \max_{\Omega} |u_t| = |u_t(a_i)| \) for some \( i \). If \( u_t(a_i) \to -\infty \), let \( u'_t \) be the function in \( \mathcal{W}'_n \) whose graph is the lower convex hull of
\[ \text{conv}\{(b_1, \varphi(b_1)), \ldots, (b_m, \varphi(b_m)), (a_i, u_t(a_i))\} \]
Since the plane passing through one boundary edge and \( (a_i, u_t(a_i)) \) approaches a vertical plane as \( t \) approaches \( \infty \), we may conclude that \( \mu u'_t(\Omega) = \sum_{i=1}^{n} g_i(u'_t) \) approaches \( \infty \). However, as \( u'_t \geq u_t \), we know that \( \sum_{i=1}^{n} g_i(u'_t) \leq \sum_{i=1}^{n} g_i(u_t) \), which contradicts the assumption that \( \sum_{i=1}^{n} g_i(u_t) \leq \chi \).

Under such circumstances the monotone sequence of vectors \( ((u_1), (u_2), \ldots, (u_n)) = u_t(a_i) \) is bounded and converges to a vector \( (u^*_1, u^*_2, \ldots, u^*_n) \). Taking the lower convex hull of the points \( (a_i, u^*_i) \) and \( (b_j, \varphi(b_j)) \), we obtain a graph \( S_{u^*} \) of some function \( u^* \in \mathcal{W}'_n \).

Obviously, the convergence of this sequence of vectors implies convergence in \( C(\Omega) \)-norm of \( u_t \) to \( u^* \).

Let us show that for all \( i \), \( g_i(u^*) = v_i \). We treat \( g_i \) as the function of the height vector \( g_i(\xi_1, \ldots, \xi_n) \), which is a piecewise quadratic polynomial. The height vector is in a compact set \( \mathcal{X} := \{(\xi_1, \ldots, \xi_n) : u^* \leq \xi \leq u_0\} \)
By lemma 1.7, \( g_i(\xi) \) is \( C^1 \), therefore the gradient is bounded on the compact set \( \mathcal{X} \).

There is a Lipschitz constant \( M \), such that
\[ |g_i(\xi_1) - g_i(\xi_2)| \leq M|\xi_1 - \xi_2| \]
For our construction, at the \( t \)-th step, we set \( \xi = u_t, \quad \bar{\xi} = u_{t+1} \),
\[ g_i(\xi_0, \ldots, \xi_{i-1}, \bar{\xi}, \xi_{i+1}, \ldots, \xi_n) = v_i \]
Then we can estimate
1.3 Oliker-Prussner Method

Fig. 1.9 A human facial surface captured by a 3D camera.

\[
|g_i(u_{i+1}) - v_i| = |g_i(\bar{\xi}) - g_i(\xi_0, \ldots, \xi_{i-1}, \bar{\xi}, \xi_{i+1}, \ldots, \xi_n)| \\
\leq M|\bar{\xi} - \xi| = M|u_{i+1} - u_i|.
\]

Taking the limit \( t \to \infty \), since \( \{u_i\} \) is monotonously decreasingly convergent, the right hand side goes to zero; the left hand side \( g_i(u_{i+1}) \to g(u^*) \), hence \( g_i(u^*) = v_i \). \( \square \)

Recently, the approximation error of Oliker-Brussner’s algorithm has been proven for regular grid tessellation by Wang. The work consider the integer standard grid only, scaled by a small positive constant \( h \). The boundary of \( \Omega \) intersecting the grid lines, the convex hull of intersection points is denoted as \( \Omega_h \). For convex set \( \Omega \), \( \Omega_h \subseteq \Omega \).

**Theorem 1.3 (Wang).** Let \( u \) and \( v_n \) be the solutions to (1.16) and (1.18), respectively. We assume that \( \Omega \) is bounded, uniformly convex domain in the Euclidean space \( \mathbb{R}^d \) with \( C^3 \) smooth boundary \( \partial \Omega \), \( f \) is a positive function in \( \Omega \) satisfying \( \lambda \leq f \leq \Lambda \) for two positive constants \( \Lambda \geq \lambda > 0 \) in (1.16), and \( \phi \in C^3(\Omega) \) is a convex function in (1.17).

1. Assume that \( f \in C^\alpha(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \). Then we have the estimate

\[
\|u - v_n\|_{L^\infty(\Omega_h)} \leq Ch^\alpha.
\]

2. If \( \partial \Omega \in C^{3, \alpha} \), \( \phi \in C^{3, \alpha}(\overline{\Omega}) \), and \( f \in C^{1, \alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \), then we have the estimate

\[
\|u - v_n\|_{L^\infty(\Omega_h)} \leq Ch^{1+\alpha}.
\]

3. If \( \partial \Omega \in C^{4, \alpha} \), \( \phi \in C^{4, \alpha}(\overline{\Omega}) \), and \( f \in C^{2, \alpha}(\overline{\Omega}) \) for some \( \alpha \in (0, 1) \) then we have the estimate

\[
\|u - v_n\|_{L^\infty(\Omega_h)} \leq Ch^2.
\]

In the above estimates, the constant \( C \) depends on \( n, \alpha, \Omega, \phi \) and \( f \).
Fig. 1.10 The human facial surface is conformally mapped onto the planar unit disk. The small circles on the plane are mapped to small circles on the surface.

Fig. 1.11 The facial surface is mapped onto the planar unit disk, the mapping is measure-preserving. The circles on the plane are mapped to ellipses on the surface with the same area.

Fig. 1.12 The triangulation of the facial surface.
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The facial surface is conformally mapped onto the unit disk. The surface area measure is mapped to the disk. An optimal transport map is computed between the measure induced by the conformal mapping to the Lebesgue measure on the disk.

Fig. 1.13 The facial surface is conformally mapped onto the unit disk. The surface area measure is mapped to the disk. An optimal transport map is computed between the measure induced by the conformal mapping to the Lebesgue measure on the disk.