Chapter 1
Alexandrov Theory for Convex Polyhedra

1.1 Brunn-Minkowski Inequality

This section introduces the basic concepts and theorems in convex geometry, based on Andrejs Treiberg’s notes.

1.1.1 Minkowski Sum

Definition 1.1 (Minkowski Sum). Given two sets $P, Q \subset \mathbb{R}^d$, their Minkowski sum is defined to be

Fig. 1.1 Minkowski sum $P \oplus Q$. 

Fig. 1.1 Minkowski sum $P \oplus Q$. 

$P$ $Q$ $P \oplus Q$ 

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Fig. 1.2 A convex polyhedron with the heights $h_i, h_{ij}, h_{ijk}$.

$$P \oplus Q := \{p + q : p \in P \text{ and } q \in Q\}.$$  

Minkowski sum may be written as

$$P \oplus Q = \bigcup_{q \in Q} P \oplus \{q\}.$$  

It is the union of all translates of $P$ by points of $Q$. The $P$ set is smeared around by the $Q$ set.

**Definition 1.2 (Minkowski Dilation).** Given a set $P \subset \mathbb{R}^d$, for $s \geq 0$, the Minkowski dilation by factor $s$ is defined to be

$$sP := \{sp : p \in P\}.$$  

Minkowski proved the following theorem.

**Theorem 1.1 (Polynomial Nature of Sum).** Let $P$ and $Q$ be convex bodies in $\mathbb{R}^3$. Then the volume of the linear combination $sP \oplus tQ$ is a cubic form for nonnegative $s$ and $t$,

$$V(sP \oplus tQ) = V(P)s^3 + 3V(P,P,Q)s^2t + 3V(P,Q,Q)st^2 + V(Q)t^3,$$

where $V(P,P,Q)$ and $V(P,Q,Q)$ are called the mixed volumes.

**Proof.** We prove it for polyhedra. Suppose $X = sP_1 \oplus tP_2$, $P_1$ and $P_2$ have the similar combinatorial structures. We use $f_{\alpha}^i$ to represent the $i$-th face in $P_\alpha$, $\alpha = 1, 2$; $e_{ij}^\alpha$ the $j$-th edge in $f_{\alpha}^i$, $v_{ijk}^\alpha$ the $k$-th vertex on edge $e_{ij}^\alpha$. By translation, we move the origin to the interior of the polyhedra. The distance from $O$ to $f_{ij}^\alpha$ is $h_{ij}^\alpha$, the distance from the orthogonal foot in $f_{ij}^\alpha$ to $e_{ij}^\alpha$ is $h_{ij}^\alpha$, the orthogonal foot in $e_{ij}^\alpha$ to $v_{ijk}^\alpha$ is $h_{ijk}^\alpha$, as shown...
1.1 Brunn-Minkowski Inequality

in Fig. 1.2. The corresponding supporting distances of \( X \) are denoted as \( h_i, h_{ij} \) and \( h_{ijk} \) respectively.

The volume of \( X \) is given by

\[
V(X) = \frac{1}{3} \sum_{i=1}^{n} h_i A(F_i) = \frac{1}{3} \sum_{i=1}^{n} h_i \frac{1}{2} \sum_{j} h_{ij} L(e_{ij}) \\
= \frac{1}{3} \sum_{i=1}^{n} h_i \frac{1}{2} \sum_{j} h_{ij} = \frac{1}{6} \sum_{i,j,k} h_i h_{ij} h_{ijk} \\
= \frac{1}{6} \sum_{i,j,k} (sh_{ij}^1 + th_{ij}^2)(sh_{ij}^1 + th_{ij}^2)(sh_{ij}^1 + th_{ij}^2) \\
= \frac{1}{6} \sum_{i,j,k} (h_{ij}^1 h_{ij}^1 h_{ij}^1 s^3 + h_{ij}^1 h_{ij}^1 h_{ij}^1 s^3 t + h_{ij}^1 h_{ij}^1 h_{ij}^1 s^3 t^2 + h_{ij}^2 h_{ij}^2 h_{ij}^2 t^3)
\]

We denote the mixed volume as

\[
V(P_\alpha, P_\beta, P_\gamma) = \frac{1}{6} \sum_{i,j,k} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

Therefore the volume of \( sP_1 \oplus tP_2 \) is a cubic polynomial of \( s \) and \( t \) with coefficients as the mixed volumes of \( P_1 \) and \( P_2, V(P_\alpha, P_\beta, P_\gamma) \).

The mixed volume has the following properties.

**Proposition 1.1.** Suppose \( P, Q, R \) are convex sets and nonnegative \( s,t \), the following hold:

- \( V(P, Q, Q) \geq 0; \)
- \( V(P, Q, Q) = V(Q, Q, P); \)
- if \( \rho \) is a rigid motion, then \( V(\rho P, \rho Q, \rho Q) = V(P, Q, Q); \)
- \( V(P, P, P) = V(P); \)
- \( V(sP \oplus tQ, R, R) = sV(P, R, R) + tV(Q, R, R); \)
- If \( P \subset Q \) then \( V(P, R, R) \leq V(Q, R, R) \) and \( V(P, P, R) \leq V(Q, Q, R). \)
Lemma 1.1 (Derivative of Volume). Let $P$ be a polyhedron with facet directions $n_i$, support distances $h_i$,

$$P := \{ x \in \mathbb{R}^d : \langle n_i, x \rangle \leq h_i, \forall i = 1, \ldots, n \}$$

Suppose the volume of the $i$-th facet is $A_i$. Then

$$\frac{\partial V(h)}{\partial h_i} = A_i.$$  \hfill (1.1)

Proof. As shown in Fig. 1.3, if we increase the support distance from $h_i$ to $h_i + \Delta h_i$,

$$\Delta V = A_i \Delta h_i + o(\Delta h_i)$$

hence

$$\frac{\partial V}{\partial h_i} = A_i.$$  \hfill \Box

Lemma 1.2. For convex polyhedra $P_1$ and $P_2$ in $\mathbb{R}^3$, we have

$$V(P_1, P_1, P_2) = \frac{1}{3} \sum_{i=1}^{n} h_i^2 A_i.$$  \hfill (1.2)

Proof. We know the volume

$$V(sP_1 \oplus tP_2) = V(P_1, P_1, P_1)s^3 + 3V(P_1, P_1, P_2)s^2 t + 3V(P_1, P_2, P_2)st^2 + V(P_2, P_2, P_2)t^3.$$  

We differentiate volume in two ways,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} V(P_1 \oplus tP_2) = 3V(P_1, P_1, P_2).$$

Also, using the chain rule,

$$\left. \frac{\partial}{\partial t} \right|_{t=0} V(P_1 \oplus tP_2) = \left. \frac{\partial}{\partial t} \right|_{t=0} V(P(h_1 + th_2))$$

$$= \sum_{i=1}^{n} h_i^2 \left. \frac{\partial}{\partial h_i} V(P(h_1 + th_2)) \right|_{t=0}$$

$$= \sum_{i=1}^{n} h_i^2 A_i(h_1 + th_2) \bigg|_{t=0}$$

$$= \sum_{i=1}^{n} h_i^2 A_i.$$  \hfill \Box
1.1 Brunn-Minkowski Inequality

The theorem of Hermann Karl Brunn says that since the Minkowski sum tends to “round out” the shapes being added, the volume of the added shape exceeds the volume of the summands.

Definition 1.3 (Homotheticity). Two polyhedra $A$ and $B$ are homothetic, i.e., they are similar and are similarly situated, which means there is a translation and dilation so that

$$A = rB \oplus \{x\}.$$

Now we focus on the proof of classical Brunn-Minkowski inequality.

Theorem 1.2 (Brunn-Minkowski Inequality). Let $A, B \subset \mathbb{R}^d$ be proper convex sets and $0 \leq \lambda \leq 1$. Then

$$\forall \xi, \eta \in \mathbb{R}, \quad \frac{V((1-\lambda)A \oplus \lambda B)}{\lambda} \geq (1-\lambda)V(A) + \lambda V(B).$$

Equality holds if and only if $A$ and $B$ are homothetic.

Proof. We use H. Kneser and W. Süss’s method for the proof. The idea is to prove the inequality inductively on dimension. First, we assume $A, B \subset \mathbb{R}^d$ are convex bodies with $V(A) = V(B) = 1$. Choose a direction, say $x_1$ axis. The projection of $A$ is $[\alpha_1, \alpha_2]$ on the axis. Cut the body along $x_1 = \xi$ plane. Define the left portion and the face (see Fig. 1.5) by

$$A[\xi] = \{x \in A : x_1 \leq \xi\}, \quad a[\xi] = \{x \in A : x_1 = \xi\}.$$

Similarly for $B$. Let $\tau \in [0, 1]$ denote the volume of the portion and $\rho(\tau)$ and $\sigma(\tau)$ denote the corresponding $x_1$ coordinates,
\[ \tau = V(A[\rho(\tau)]) = V(B[\sigma(\tau)]). \]

Rewriting the volumes,
\[ \tau = \int_{\alpha_1}^{\alpha_2} v(a[\xi]) d\xi = \int_{\beta_1}^{\beta_2} v(b[\eta]) d\eta, \]
where \( v(\cdot) \) is the \((d-1)\) dimensional volume. Differentiating, we find
\[ 1 = v(a[\rho(\tau)]) \frac{d\rho}{d\tau} = v(b[\sigma(\tau)]) \frac{d\sigma}{d\tau}. \quad (1.4) \]

**Base Case** For \( d = 1 \), if \( A = [\alpha_1, \alpha_2] \) and \( B = [\beta_1, \beta_2] \) then
\[ (1 - \lambda)A \oplus \lambda B = [(1 - \lambda)\alpha_1 + \lambda\beta_1, (1 - \lambda)\alpha_2 + \lambda\beta_2], \]
so that thus the volume of the Minkowski sum
\[ V((1 - \lambda)A + \lambda B) = (1 - \lambda)\alpha_2 + \lambda\beta_2 - [(1 - \lambda)\alpha_1 + \lambda\beta_1] \]
\[ = (1 - \lambda)(\alpha_2 - \alpha_1) + \lambda(\beta_2 - \beta_1) \]
\[ = (1 - \lambda)V(A) + \lambda V(B). \]

**Induction Case** For \( d > 1 \) we assume that Brunn-Minkowski holds for \( d - 1 \). Let \( \gamma(\tau) = (1 - \lambda)\rho(\tau) + \lambda\sigma(\tau) \). The Minkowski sum
\[ S_\lambda = (1 - \lambda)A \oplus \lambda B \]
is defined for \( x_1 \in [(1 - \lambda)\alpha_1 + \lambda\beta_1, (1 - \lambda)\alpha_2 + \lambda\beta_2] \). Its \( \gamma(\tau) \) slice contains the Minkowski sum of the sections
\[ s_\lambda[\gamma(\tau)] \supset (1 - \lambda)a[\rho(\tau)] \oplus \lambda b[\sigma(\tau)]. \quad (1.5) \]
1.1 Brunn-Minkowski Inequality

The volume of $S_{\lambda}$ is given by the integral

$$V(S_{\lambda}) = \int_{(1-\lambda)\alpha_1 + \lambda \beta_1}^{(1-\lambda)\alpha_2 + \lambda \beta_2} v(s_{\lambda}[\xi])d\xi.$$  

change variables to $\xi = \gamma(\tau)$ using (1.4). By (1.5) and the induction hypothesis,

$$V(S_{\lambda}) \geq \int_0^1 v((1-\lambda)a[p(\tau)] + \lambda b[\sigma(\tau)]) \left( (1-\lambda) \frac{d\rho}{d\tau} + \lambda \frac{d\sigma}{d\tau} \right) d\tau \geq 1 = (1-\lambda)V(A) + \lambda V(B),$$

using Jensen’s inequality and unity of volumes.

As shown in Fig. 1.6, Jensen’s inequality for $\varphi(u) = u^{-\frac{1}{\alpha-1}}$ says

$$(1-\lambda)\varphi(u_1) + \lambda \varphi(u_2) \geq \varphi((1-\lambda)u_1 + \lambda u_2).$$

Let $u_1 = \frac{1}{v(a[p(\tau)]})$ and $u_2 = \frac{1}{v(b[\sigma(\tau)])}$. Hence

$$(1-\lambda) \left( \frac{1}{v(a[p(\tau)]}) \right)^{-\frac{1}{\alpha-1}} + \lambda \left( \frac{1}{v(b[\sigma(\tau)])} \right)^{-\frac{1}{\alpha-1}} \geq \left( (1-\lambda) \frac{1}{v(a[p(\tau)]}) + \lambda \frac{1}{v(b[\sigma(\tau)])} \right)^{-\frac{1}{\alpha-1}}$$

Equivalently

$$(1-\lambda) v(a[p(\tau)])^{-\frac{1}{\alpha-1}} + \lambda v(b[\sigma(\tau)])^{-\frac{1}{\alpha-1}} \geq \left( (1-\lambda) \frac{1}{v(a[p(\tau)]}) + \lambda \frac{1}{v(b[\sigma(\tau)])} \right)^{-\frac{1}{\alpha-1}}.$$
and the desired inequality follows.

Equality implies \( u_1 = u_2 \). To see that equality in (1.3) implies \( A \) and \( B \) are homothetic, it suffices to show in case \( V(A) = V(B) \) that \( A \) and \( B \) are translates.

To this end we translate \( A \) and \( B \) so that they have a common center of mass since (1.3) is independent of translation. Expressing the \( x_1 \) coordinate of the center of mass, by 

\[
\alpha_2 v(a[\xi]) d\xi = \int_0^{\beta_1} \sigma(\tau) v(b[\sigma(\tau)]) d\sigma \, d\tau = \int_0^{\beta_1} \sigma(\tau) d\tau
\]

(1.6)

Equality in (1.3) implies equality in Jensen’s Inequality. Thus for every \( \tau \in (0, 1) \) we have

\[ v(a[\rho(\tau)]) = v(b[\sigma(\tau)]) \]

It follows from (1.4) that

\[ \rho(\tau) - \sigma(\tau) = \text{const}, \]

and from (1.6) that the constant is 0.

Finally, since \( \rho(\tau) = \sigma(\tau) \) for \( 0 < \tau < 1 \), it follows that

\[ \alpha_2 = \lim_{\tau \to 1} \rho(\tau) = \lim_{\tau \to 1} \sigma(\tau) = \beta_2. \]

Thus in the direction of the \( x_1 \)-axis, the support planes of \( A \) and \( B \) coincide, and thus the support distance in this direction are equal. Since we could have chosen any direction for the \( x_1 \)-axis, the support distances of \( A \) and \( B \) are identical in all directions, hence \( A \) and \( B \) are identical, as claimed.

Non-unit volume case. In general cases, we change \( A \) and \( B \) to \( V(A)^{-\frac{1}{d}}A \) and \( V(B)^{-\frac{1}{d}}B \), which have unit volumes, and set

\[ \tilde{\lambda} = \frac{V(B)^{\frac{1}{d}}}{(1 - \tilde{\lambda})V(A)^{\frac{1}{d}} + \tilde{\lambda} V(B)^{\frac{1}{d}}}. \]

Then apply (1.3) for the unit volume case, we obtain:

\[ V \left( (1 - \tilde{\lambda}) \frac{A}{V(A)^{1/d}} \oplus \tilde{\lambda} B \frac{B}{V(B)^{1/d}} \right)^{\frac{1}{d}} \geq \frac{(1 - \tilde{\lambda})}{V(A)^{\frac{1}{d}}} V(A)^{\frac{1}{d}} + \frac{\tilde{\lambda}}{V(B)^{\frac{1}{d}}} V(B)^{\frac{1}{d}} = 1. \]

But the left side is

\[
V \left( (1 - \tilde{\lambda})V(A)^{\frac{1}{d}} \oplus \frac{\lambda \tilde{\lambda}}{V(A)^{\frac{1}{d}}} \frac{A}{V(A)^{\frac{1}{d}}} \oplus \frac{\tilde{\lambda}}{V(B)^{\frac{1}{d}}} \frac{B}{V(B)^{\frac{1}{d}}} \right) \frac{1}{d} = \frac{V((1 - \tilde{\lambda})A \oplus \tilde{\lambda} B)^{\frac{1}{d}}}{(1 - \tilde{\lambda})V(A)^{\frac{1}{d}} + \tilde{\lambda} V(B)^{\frac{1}{d}}}
\]
Therefore, we obtain
\[ V((1 - \lambda)A \oplus \lambda B)^{\frac{1}{d}} \geq (1 - \lambda)V(A)^{\frac{1}{d}} + \lambda V(B)^{\frac{1}{d}}. \]

This finishes the whole proof. \( \square \)

**Theorem 1.3 (Minkowski’s Inequality).** Suppose we have two bounded convex polyhedra \( P \) and \( P' \) in \( \mathbb{R}^3 \) with the same normals \( \{v_i\}_{i=1}^n \). Let \( A_i, h_i \) and \( A'_i \) denote areas and support distances in \( v_i \) directions. Some areas may be zero. Then the mixed volume \( V(P, P', P') \) satisfies
\[
V(P, P', P') = \frac{1}{3} \sum_{i=1}^n h_i A'_i \geq V(P)^{1/3} V(P')^{2/3}. \tag{1.7}
\]

If equality holds, then \( P \) and \( P' = cP + a \) are homothetic translates.

**Proof.** To show for all proper polyhedra \( A, B \),
\[
V(A, A, B) \geq V(A)^{2/3} V(B)^{1/3} \tag{1.8}
\]
and equality implies \( A, B \) are homothetic. Using
\[
V((1 - t)A \oplus tB) = V(A)(1 - t)^3 + 3V(A, A, B)(1 - t)^2 t + 3V(A, B, B)(1 - t)^2 + V(B)t^3,
\]
By Brunn-Minkowski, \( f : [0, 1] \rightarrow \mathbb{R} \) is non-negative and concave down,
\[
f(t) = V((1 - t)A \oplus tB)^{\frac{1}{d}} - (1 - t)V(A)^{\frac{1}{d}} - tV(B)^{\frac{1}{d}}. \]
Differentiating at \( t = 0 \), the result follows from
\[
0 \leq \frac{df}{dt} \bigg|_{t=0} = V(A)^{-\frac{2}{d}} [-V(A) + V(A, A, B)] + V(A)^{\frac{1}{d}} - V(B)^{\frac{1}{d}}.
\]
As shown in Fig. 1.4, equality in (1.8) implies \( f(t) \equiv 0 \) so equality holds in the Brunn-Minkowski theorem, hence \( A \) and \( B \) are homothetic. \( \square \)

**Theorem 1.4 (Brunn-Minkowski).** Let \( A, B \in \mathbb{R}^d \) be proper convex polyhedra. Then
\[
V(A \oplus B)^{\frac{1}{d}} \geq V(A)^{\frac{1}{d}} + V(B)^{\frac{1}{d}} \tag{1.9}
\]
Equality holds if and only if \( A \) and \( B \) are homothetic.

**Proof.** Here we prove \( d = 3 \) case. We expand \( V(sA \oplus tB) \) as
\[
V(sA \oplus tB) = V(A, A, A)s^3 + 3V(A, A, B)s^2 t + 3V(A, B, B)s t^2 + V(B, B, B)t^3
\]
Because
\[
V(A, A, B) \geq V(A)^{\frac{3}{d}} V(B)^{\frac{1}{d}}, V(A, B, B) \geq V(A)^{\frac{1}{d}} V(B)^{\frac{2}{d}},
\]
therefore
\[ V(A \oplus B) \geq V(A) + 3V(A)^{\frac{1}{3}}V(B)^{\frac{1}{3}} + 3V(A)^{\frac{1}{3}}V(B)^{\frac{1}{3}} + V(B) = \left( V(A)^{\frac{1}{3}} + V(B)^{\frac{1}{3}} \right)^3. \]

\[ \square \]

**Theorem 1.5 (Uniqueness for Minkowski’s Problem for Polyhedra).** Suppose we have two bounded convex polyhedra \( P \) and \( P' \) in \( \mathbb{R}^3 \) whose outer normals \( v_i = v'_i \) and corresponding areas \( A_i = A'_i > 0 \) coincide. Then \( P' \) is a translation of \( P \).

**Proof.** Since \( A_i = A'_i \), then the mixed volume \( V(P, P', P') = \frac{1}{3} \sum_{i=1}^{n} h_i A'_i = \frac{1}{3} \sum_{i=1}^{n} h_i A_i = V(P) \)

From Minkowski’s inequality,
\[ V(P) = V(P, P', P') \geq V(P)^{1/3}V(P')^{2/3} \]

follows
\[ V(P) \geq V(P'). \]

By reversing the roles of \( P \) and \( P' \), we find by the same argument that \( V(P') \geq V(P) \).

It follows that \( V(P) = V(P') \) and equality holds in (1.7). By the uniqueness statement in Minkowski’s theorem, \( P \) and \( P' \) are homothetic translate. But since they have the same volume, \( c = 1 \), so \( P \) and \( P' \) are translates and correspond to the same point in \( \mathcal{M} \), the realization manifold of polyhedra.

\[ \square \]

### 1.2 Isoperimetric Inequality

Isoperimetric Inequality says that the largest the volume convex set can have for a given surface area is the volume attained by the ball.

**Theorem 1.6 (Isoperimetric Inequality).** Suppose \( E \subset \mathbb{R}^d \) is a measurable set, \( B \) is the unit ball in \( \mathbb{R}^d \), then isoperimetric inequality holds
\[ V(E) \geq \frac{1}{d} \frac{V(\partial E)}{dV(B)^{\frac{1}{d}}} \]  

(1.10)

where \( V(E) = \mathcal{L}^d(E), V(B) = \mathcal{L}^d(B) \) and \( V(\partial E) = \mathcal{H}^{d-1}(\partial E) \).

**Proof.** We will prove this inequality via Brenier’s theorem, neglecting the smoothness issues. Let
\[ \mu := \frac{1}{V(E)} \mathcal{L}^d|_E \quad \nu := \frac{1}{V(E)} \mathcal{L}^d|_E \]

and \( T : E \to B \) be the optimal transport map with respect to the cost given by the distance squared. The change of variable formula gives
\[ \frac{1}{V(E)} = \det(\nabla T(x)) \frac{1}{V(B)}, \quad \forall x \in E. \]

Since we know that \( T(x) = \nabla u(x) \) is the gradient of a convex function, we have that \( DT(x) = D^2 u(x) \) is a symmetric matrix with non-negative eigenvalues for every \( x \in E \). Hence the arithmetic-geometric mean inequality ensures that

\[ (\det D^2 u(x))^{\frac{1}{d}} \leq \frac{\nabla \cdot \nabla u(x)}{d}, \quad x \in E. \]

Coupling the last two equations we get

\[ \frac{1}{V(E)^{\frac{1}{d}}} \leq \frac{\nabla \cdot T(x)}{d} \frac{1}{V(B)^{\frac{1}{d}}} \quad \forall x \in E. \]

Integrating over \( E \) and applying the divergence theorem we get

\[ L(E)^{1 - \frac{1}{d}} \leq \frac{1}{dL(B)^{1/d}} \int_{\partial E} \langle T(x), v(x) \rangle d\mathcal{H}^{d-1}(x), \]

where \( v : \partial E \to \mathbb{R}^d \) is the outer unit normal vector. Since \( T(x) \in B \) for every \( x \in E \), we have \( |T(x)| \leq 1 \) for \( x \in \partial E \) and thus \( \langle T(x), v(x) \rangle \leq 1 \). We conclude with

\[ L(E)^{1 - \frac{1}{d}} \leq \frac{1}{dL(B)^{1/d}} \int_{\partial E} \langle T(x), v(x) \rangle d\mathcal{H}^{d-1}(x) \leq \frac{V(\partial E)}{dL(B)^{1/d}}. \]

\[ \Box \]

Minkowski’s inequality also gives us the isoperimetric Inequality.

**Theorem 1.7 (Isoperimetric Inequality).** Suppose \( K \subset \mathbb{R}^3 \) is a compact convex set in \( \mathbb{R}^3 \), then

\[ A(\partial K)^{\frac{2}{3}} \geq 6\sqrt{\pi}V(K). \]

If equality holds then \( K \) is a ball.

**Proof.** Without loss of generality, we first assume \( K \) is a convex polyhedron. Let \( B \) be the unit ball, then

\[ 3V(B,K,K) = \sum_{i=1}^{n} h_i(B)\lambda_i(K) = A(\partial K). \]

By Minkowski’s inequality

\[ \frac{1}{3}A(\partial K) = V(B,K,K) \geq V(B)^{\frac{1}{d}}V(K)^{\frac{d}{d}} = \left( \frac{4}{3\pi} \right)^{\frac{1}{d}} V(K)^{\frac{d}{d}}. \]

In other words,

\[ A(\partial K)^{\frac{2}{3}} \geq 6\sqrt{\pi}V(K). \]  \( (1.11) \)
Equality holds if and only if $K$ is a ball. For the ball $B$ of radius $R$,

$$A(\partial B)^2 = (4\pi R^2)^{\frac{3}{2}} = 6\sqrt{\pi} \left(\frac{4}{3} \pi R^3\right)^{\frac{1}{2}} = 6\sqrt{\pi} V(B).$$

If $K$ is a general compact convex set in $\mathbb{R}^3$, it can be approximated by a sequence of convex polyhedra, by taking the limit, the inequality holds. \qed