Circle Domain Mapping: Koebe’s Theorem

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Motivation
Figure: Conformal mapping from a poly-annulus to a circle domain.
Circle Domain

**Definition (Circle Domain)**
Suppose \( \Omega \subset \hat{\mathbb{C}} \) is a planar domain, if \( \partial \Omega \) has finite number of connected components, each of them is either a circle or a point, then \( \Omega \) is called a circle domain.

**Theorem (Koebe)**
Suppose \( S \) is of genus zero, \( \partial S \) has finite number of connected components, then \( S \) is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.
**Schwartz Reflection Principle**

**Definition (Mirror Reflection)**

Given a circle $\Gamma : |z - z_0| = \rho$, the reflection with respect to $\Gamma$ is defined as:

$$\varphi_\Gamma : re^{i\theta} + z_0 \mapsto \frac{\rho^2}{r} e^{i\theta} + z_0.$$  \hfill (1)

Two planar domains $S$ and $S'$ are symmetric about $\Gamma$, if $\varphi_\Gamma(S) = S'$.

**Figure**: Reflection about a circle.
Definition (Reflection)

Suppose $\Gamma$ is an analytic curve, domain $S, S'$ and $\Gamma$ are included in a planar domain $\Omega$. There is a conformal map $f : \Omega \rightarrow \hat{\mathbb{C}}$, such that $f(\Gamma)$ is a canonical circle, $f(S)$ and $f(S')$ are symmetric about $f(\Gamma)$, then we say $S$ and $S'$ are symmetric about $\Gamma$, and denoted as

$$S|S' \ (\Gamma).$$

Figure: General symmetry.
Theorem (Schwartz Reflection Principle)

Assume $f$ is an analytic function, defined on the upper half disk \( \{ |z| < 1, \Im(z) > 0 \} \). If $f$ can be extended to a real continuous function on the real axis, then $f$ can be extended to an analytic function $F$ defined on the whole disk, satisfying

\[
F(z) = \begin{cases} 
  f(z), & \Im(z) \geq 0 \\
  \frac{f(\bar{z})}{f(z)}, & \Im(z) < 0
\end{cases}
\]

Figure: Schwartz reflection principle.
Multiple Reflection

\[ \Gamma_1 \quad C^0 \quad \Gamma_2 \]

\[ \Gamma_3 \]
Multiple Reflection
Multiple Reflection

1. Initial circle domain $C^0$: complex plane remove three disks, its boundary is $\{\Gamma_1, \Gamma_2, \Gamma_3\}$;

2. First level reflection: $C^0$ is reflected about $\Gamma_{i_1}$ to $C^{i_1}$, $i_1 = 1, 2, 3$;

$$\partial C^{i_1} = \Gamma^{i_1}_{i_1} - \sum_{j \neq i_1} \Gamma^{i_1}_j,$$

where $\Gamma^{i_1}_{i_1} = \Gamma_{i_1}$.

3. Second level reflection: $C^{i_1}$ is reflected about $\Gamma_{i_2}$ to $C^{i_1i_2}$, $i_1 \neq i_2$; the boundary of $C^{i_1i_2}$ are $\Gamma^{i_1i_2}_j$, when $j \neq i_1$, $\Gamma^{i_1i_2}_j$ is an interior boundary; when $j = i_1$, $\Gamma^{i_1i_2}_j$ is the exterior boundary, $\Gamma^{i_1i_2}_{i_1} = \Gamma^{i_2}_{i_1}$.

$$\partial C^{i_1i_2} = \Gamma^{i_2}_{i_1} - \sum_{j \neq i_1} \Gamma^{i_1i_2}_j$$

when $j = i_1$, $\Gamma^{i_1i_2}_{i_1} = \Gamma^{i_2}_{i_1}$. 
Multiple Reflection

Third level reflection: $C^{i_1i_2}$ is reflected about $\Gamma_{i_3}$ to $C^{i_1i_2i_3}$, $i_1 \neq i_2$, $i_2 \neq i_3$; the boundary of $C^{i_1i_2i_3}$ are $\Gamma^j_{i_1i_2i_3}$, when $j \neq i_1$, $\Gamma^j_{i_1i_2i_3}$ is an interior boundary; when $j = i_1$, $\Gamma^j_{i_1i_2i_3}$ is the exterior boundary, $\Gamma^j_{i_1i_2i_3} = \Gamma_{i_1}^{i_2i_3}$.

$$\partial C^{i_1i_2i_3} = \Gamma_{i_1}^{i_2i_3} - \sum_{j \neq i_1} \Gamma^j_{i_1i_2i_3}.$$

5. The $m$-level reflection: $C^{i_1i_2...i_{m-1}}$ is reflected about $\Gamma_m$ to $C^{i_1i_2...i_{m-1}i_m}$, $i_k \neq i_{k+1}$; the boundary of $C^{i_1i_2...i_{m-1}i_m}$, $i_k \neq i_{k+1}$ are $\Gamma^j_{i_1i_2...i_{m-1}i_m}$, when $j \neq i_1$, $\Gamma^j_{i_1i_2...i_{m-1}i_m}$ is an interior boundary; when $j = i_1$, $\Gamma^j_{i_1i_2...i_{m-1}i_m}$ is the exterior boundary, $\Gamma^j_{i_1i_2...i_{m-1}i_m} = \Gamma_{i_1}^{i_2...i_{m-1}i_m}$ is an interior boundary,

$$\partial C^{i_1i_2...i_m} = \Gamma_{i_1}^{i_2i_3...i_m} - \sum_{j \neq i_1} \Gamma^j_{i_1i_2...i_m}.$$
Multiple Reflection

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]

\[ C^0 \]

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]

\[ C^1 \]

\[ C^{21} \]

\[ C^{31} \]

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]

\[ C^{12} \]

\[ C^{32} \]

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]

\[ C^2 \]

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]

\[ C^3 \]

\[ C^{13} \]

\[ C^{23} \]

\[ \Gamma_1 \quad \Gamma_2 \quad \Gamma_3 \]
Multiple Reflection

- Each node represents a domain $C^{i_1i_2\ldots i_m}$;
- Each edge represents a circle $\Gamma_k$, $k = 1, \ldots, n$;
- Father and Son share an edge $i_1$
  \[ \Gamma_{i_1}^{i_1i_2\ldots i_m} = \Gamma_{i_1}^{i_2\ldots i_m}. \]
- Each node $C^{(i)}$, $(i) = i_1i_2\ldots i_m$ is the path from the root to $C^{(i)}$,
  \[ C^{(i)} = \varphi_{\Gamma_{i_m}} \circ \varphi_{\Gamma_{i_{m-1}}} \cdots \varphi_{\Gamma_{i_1}}(C^0). \]
Multiple Reflection

- Father node $C^{i_2 \cdots i_m}$ and child node $C^{i_1 i_2 \cdots i_m}$ is connected by edge $i_1$, the exterior boundary of child equals to an interior boundary of the father

$$\Gamma_{i_1}^{i_2 \cdots i_m} = \Gamma_{i_1}^{i_2 \cdots i_m}.$$

- From the root $C^0$ to $C^{i_1 \cdots i_m}$, the path is inverse to the index

$$(i)^{-1} = i_m i_{m-1} \cdots i_2 i_1,$$

starting from $C^0$ crosses $\Gamma^{i_m}$ to $C^{i_m}$, crosses $\Gamma^{i_m}_{i_{m-1}}$ to $C^{i_{m-1} i_m}$; when arrives at $C^{i_{k-1} \cdots i_1}$, crosses $\Gamma^{i_{k-1} \cdots i_1}_{i_k}$ to $C^{i_k i_{k-1} \cdots i_1}$; and eventually reach $C(i)$. 

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Figure: Embedding tree.
Lemma

Suppose \( C^{(i)} \) is an interior node in the reflection tree, \( (i) = i_1 i_2 \cdots i_m \), its exterior boundary is \( \Gamma^{(i)}_{i_1} \), interior boundaries are \( \Gamma^{(i)}_{j} \), \( j \neq i_1 \), we have the estimate:

\[
\sum_{j \neq i_1} \alpha(\Gamma^{(i)}_{j}) \leq \mu^4 \alpha(\Gamma^{(i)}_{i_1}).
\]
Hole Area Estimation

Figure: Hole area estimation.

Enlarge all $\Gamma_k$'s by factor $\mu^{-1}$ to $\tilde{\Gamma}_k$, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_3$ touch each other; reflect $C^0$ about $\Gamma_2$

- $\Gamma_k|\Gamma_k^2$ ($\Gamma_2$).
- $\tilde{\Gamma}_k|\tilde{\Gamma}_k^2$ ($\Gamma_2$).

\[
\alpha(\tilde{\Gamma}_1^2) = \mu^{-2}\alpha(\Gamma_1^2) \\
\alpha(\tilde{\Gamma}_3^2) = \mu^{-2}\alpha(\Gamma_3^2) \\
\alpha(\tilde{\Gamma}_2^2) = \mu^2\alpha(\Gamma_2^2)
\]

\[
\alpha(\Gamma_1^2) + \alpha(\Gamma_3^2) = \mu^2(\alpha(\tilde{\Gamma}_1^2) + \alpha(\tilde{\Gamma}_3^2)) \leq \mu^2\alpha(\tilde{\Gamma}_2^2) = \mu^4\alpha(\Gamma^2).
\]
Lemma

Suppose the boundaries of the initial circle domain $C^0$ are $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$, consider the reflection tree with $m$ layers, then the total area of the holes bounded by the interior boundaries of leaf nodes is no greater than $\mu^{4m}$ times the area bounded by $\Gamma_k$’s,

$$\sum_{(i)=i_1i_2\ldots i_m \ k \neq i_1} \sum \alpha(\Gamma_k^{(i)}) \leq \mu^{4m} \sum_{i=1}^n \alpha(\Gamma_i).$$  \hspace{1cm} (2)

Proof.

By induction on $m$. The area bounded by the exterior boundaries of the nodes in the $k+1$-layer is no greater than $\mu^4$ times that of the $k$-layer. The total area of the interior boundaries of leaf nodes is no greater than the area bounded by the exterior boundaries of leaf nodes.
Theorem (Uniqueness)

Given two circle domains \( C_1, C_2 \subset \hat{\mathbb{C}} \), \( f : C_1 \to C_2 \) is a univalent holomorphic function, then \( f \) is a linear rational, namely a Möbius transformation.

Proof.

Assume both \( C_1 \) and \( C_2 \) include \( \infty \), and \( f(\infty) = \infty \). Since \( f \) is holomorphic, it maps the boundary circles of \( C_1 \) to those of \( C_2 \). By Schwartz reflection principle, \( f \) can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. 2, the multiple reflected domains cover the whole \( \hat{\mathbb{C}} \), hence \( f \) can be extended to the whole \( \hat{\mathbb{C}} \), since \( f(\infty) = \infty \), \( f \) is a linear function. If \( f(\infty) \neq \infty \), we can use a Möbius map to transform \( f(\infty) \) to \( \infty \).
Existence

Definition (Kernel)
Suppose \( \{B_n\} \) is a family of domains on the complex plane, \( \infty \in B_k \) for all \( k \). Suppose \( B \) is the maximal set: \( \infty \in B \), and for any closed set \( K \subset B \), there is an \( N \), such that for any \( n > N \), \( K \subset B_n \). Then \( B \) is called the kernel of \( \{B_n\} \).

Definition (Domain Convergence)
We say a sequence \( \{B_n\} \) converges to its kernel \( B \), if any sub-sequence \( \{B_{n_k}\} \) of \( \{B_n\} \) has the same kernel \( B \). We denote \( B_n \rightarrow B \).
Theorem (Goluzin)

Let \( \{A_n\} \) be a sequence of domains on the complex domain. Any domain \( A_n \) includes \( \infty \), \( n = 1, 2, \cdots \). Assume \( \{A_n\} \) converges to its kernel \( A \). Let \( \{f_n(z)\} \) be a family of analytic function, for all \( n \), \( f_n(z) \) maps \( A_n \) to \( B_n \) surjectively, such that \( f_n(\infty) = \infty \), \( f_n'(\infty) = 1 \). Then \( \{f_n(z)\} \) uniformly converges to a univalent analytic function \( f(z) \) in the interior of \( A \), if and only if \( \{B_n\} \) converges to its kernel \( B \), then the univalent analytic function \( f(z) \) maps \( A \) to \( B \) surjectively.
Existence

Theorem (Existence)

On the $z$-plane, every $n$-connected domain $\Omega$ can be mapped to a circle domain on the $\zeta$-plane by a univalent holomorphic function. Choose a point $a \in \Omega$, there is a unique map which maps $a$ to $\zeta = \infty$, and in a neighborhood of $z = a$, the map has the power series

$$\frac{1}{z - a} + a_1(z - a) + \cdots \text{ if } a \neq \infty$$

$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$
Proof.

According to Hilbert theorem, all $n$-connected domains are conformally equivalent to slit domains. We can assume $\Omega$ is a slit domain. We use $S$ represent all the $n$-connected slit domains with horizontal slits, and $C$ the $n$-connected circle domains. We label all the boundaries of the domains, $\partial \Omega = \bigcup_{k=1}^{n} \gamma_k$. For each slit $\gamma_k$, we represent it by the starting point $p_k$ and the length $l_k$, then we get the coordinates of the slit domain $\Omega$

$$(p_1, l_1, p_2, l_2, \cdots, p_n, l_n).$$

Hence $S$ is a connected open set in $\mathbb{R}^{3n}$. Similarly, consider a circle domain $D \in C$, we use the center and the radius to represent each circle $(q_k, r_k)$, and the coordinates of $D$ are given by,

$$(q_1, r_1, q_2, r_2, \cdots, q_n, r_n).$$

$C$ is also a connected open set in $\mathbb{R}^{3n}$. \qed
Existence

continued

Consider a normalized univalent holomorphic function \( f : \Omega \to \mathcal{D}, \Omega \in S \) and \( \mathcal{D} \in \mathcal{C} \), \( f \) maps the \( k \)-th boundary curve \( \gamma_k \) to the \( k \)-th circular boundary of \( \mathcal{D} \). By the existence of slit mapping and the uniqueness of circle domain mapping, we have

1. Every circle domain \( \mathcal{D} \in \mathcal{C} \) corresponds to a unique slit domain \( \Omega \in S \);
2. Every slit domain \( \Omega \in S \) corresponds to at most one circle domain \( \mathcal{D} \in \mathcal{C} \).

Then we establish a mapping from circle domains to slit domains \( \varphi : \mathcal{C} \to S \).
Assume \( \{D_n\} \) is a family of circle domains, converge to the kernel \( D^* \). The domain convergence definition is consistent with the convergence of coordinates, namely, the boundary circles of \( D_n \) converge to the corresponding boundary circles of \( D^* \), denoted as \( \lim_{n \to \infty} D_n = D^* \). The convergence of slit domains can be similarly defined. By Goluzin’s theorem, we obtain the mapping \( \varphi : C \to S \) is continuous:

\[
\varphi\left( \lim_{n \to \infty} D_n \right) = \lim_{n \to \infty} \varphi(D_n).
\]

By the uniqueness of circle domain mapping, we obtain \( \varphi \) is injective. We will prove the mapping \( \varphi \) is surjective.
Existence

continued

\( C \) is an open set in Euclidean space \( \varphi : C \rightarrow S \) is injective continuous map. According to invariance of domain theorem, \( \varphi(C) \) is an open set, \( \varphi : C \rightarrow \varphi(C) \) is a homeomorphism.

Choose a circle domain \( D_0 \in C \), its corresponding slit domain is \( \varphi(D_0) = \Omega_0 \in S \), then \( \Omega_0 \in \varphi(C) \). Choose another slit map \( \Omega_1 \in S \), we don’t know if \( \Omega_1 \) is in \( \varphi(C) \) or not. We draw a path \( \Gamma : [0, 1] \rightarrow S \), \( \Gamma(0) = \Omega_0 \) and \( \Gamma(1) = \Omega_1 \). Let

\[
t^* = \sup\{ t \in [0, 1] | \forall 0 \leq \tau \leq t, \Gamma(\tau) \in \varphi(C) \},
\]

namely \( \Gamma \) from starting point to \( t^* \) belongs to \( \varphi(C) \).
By the definition of domain convergence,

$$\lim_{n \to \infty} \Gamma(t_n) \to \Gamma(t^*).$$

By \( \{\Gamma(t_n)\} \subset \varphi(C) \), there is a family of circle domains \( \{D_n\} \subset C \), \( \varphi(D_n) = \Gamma(t_n) \). Let \( \lim_{n \to \infty} D_n = D^* \), by domain limit theorem, we have

$$\varphi(D^*) = \varphi(\lim_{n \to \infty} D_n) = \lim_{n \to \infty} \varphi(D_n) = \lim_{n \to \infty} \Gamma(t_n) = \Gamma(t^*),$$

namely \( \varphi(D^*) = \Gamma(t^*) \), hence \( \Gamma(t^*) \in \varphi(C) \). But \( \varphi(C) \) is an open set, hence if \( t^* < 1 \), \( t^* \) can be further extended. This contradict to the choice of \( t^* \), hence \( t^* = 1 \). Therefore \( \Omega_1 \in \varphi(C) \). Since \( \Omega_1 \) is arbitrarily chosen, hence \( \varphi : C \to S \) is surjective. This proves the existence of the circle domain mapping.