Harmonic Maps and Conformal Maps

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Harmonic Maps

Harmonic Map

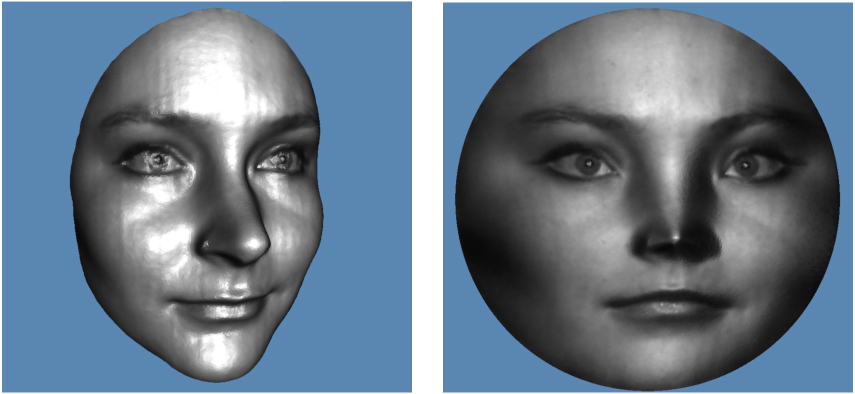


Figure: Harmonic map between topological disks.

Harmonic Map

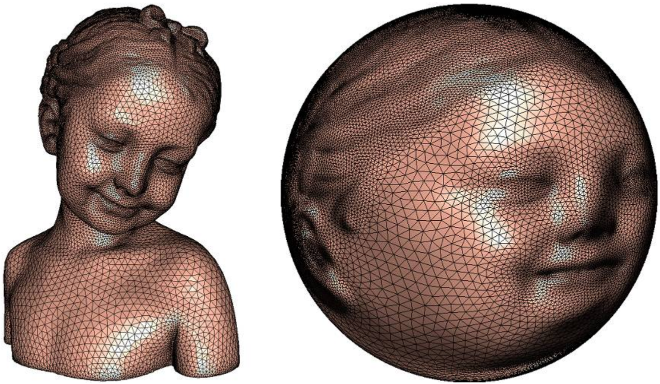


Figure: Harmonic map between topological spheres.

Surface Double Covering Algorithm

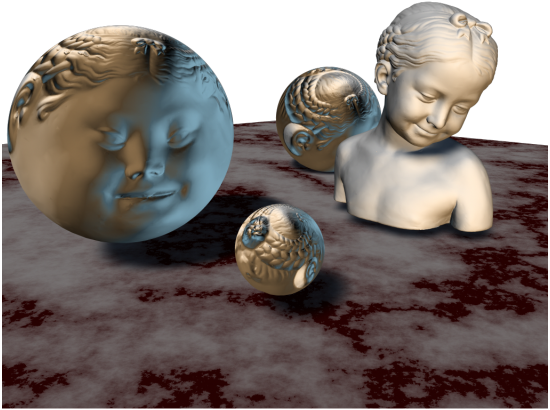


Figure: Spherical harmonic map.

Surface Double Covering Algorithm

Figure: Spherical harmonic map.

Harmonic Map

Figure: Harmonic map induced foliations.

Harmonic Function

Given a planar domain $\Omega \subset \mathbb{R}^2$, consider the electric potential $u : \Omega \to \mathbb{R}$. The gradient of the potential induces electric currents, and produces heat. The heat power is represented as *harmonic energy*

$$E(u) := \int_\Omega \langle \nabla u, \nabla u \rangle dxdy.$$

In nature, the distribution of u minimizes the heat power, and is called a *harmonic function*. Assume $h \in C_0^\infty(\Omega)$, then $E(u + \varepsilon h) \geq E(u)$,

$$\frac{d}{d\varepsilon} \int_\Omega \langle \nabla u + \varepsilon \nabla h, \nabla u + \varepsilon \nabla h \rangle dxdy \Big|_{\varepsilon=0} = 2 \int_\Omega \langle \nabla u, \nabla h \rangle dxdy = 0.$$

Harmonic Function

By relation

$$\nabla \cdot (h\nabla u) = \langle \nabla h, \nabla u\rangle + h\nabla \cdot \nabla u,$$

we obtain

$$\begin{aligned}\int_\Omega \langle \nabla u, \nabla h\rangle &= \int_\Omega h\Delta u dxdy - \int_\Omega \nabla \cdot (h\nabla u) dxdy \\ &= \int_\omega h\Delta u dxdy - \int_{\partial\Omega} h\nabla u ds = \int_\omega h\Delta u dxdy.\end{aligned}$$

We obtain the classical *Laplace equation*

$$\boxed{\begin{cases} \Delta u & \equiv & 0 \\ u|_{\partial\Omega} & = & g \end{cases}}$$

Steady temperature field, static electric field, elastic deformation, diffusion field, all are governed by the Laplace equation.

Harmonic Function

Theorem (Mean Value)

Assume $\Omega \subset \mathbb{R}^2$ is a planar open set, $u : \Omega \to \mathbb{R}$ is a harmonic function, then for any $p \in \Omega$

$$u(p) = \frac{1}{2\pi\varepsilon} \oint_\gamma u(q) ds, \tag{1}$$

where γ is a circle centered at p, with radius ε.

Proof.

u is harmonic, du is a harmonic 1-form, its Hodge star $^*du$ is also harmonic. Define the conjugate function v, $dv = {}^*du$, then $\varphi(z) := u + \sqrt{-1}v$ is holomorphic. By Cauchy integration formula,

$$\varphi(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\varphi(\zeta)}{\zeta - z} dz \tag{2}$$

Hence, we obtain the mean value property of harmonic function. □

Harmonic Function

Corollary (Maximal value principle)

Assume $\Omega \subset \mathbb{R}^2$ is a planar domain, and $u : \overline{\Omega} \to \mathbb{R}$ is a non-constant harmonic function, then u can't reach extremal values in the interior of Ω.

Proof.

Assume p is an interior point in Ω, p is a maximal point of u, $u(p) = C$. By mean value property, we obtain for any point q on the circle $B(p, \varepsilon)$, $u(q) = C$, where ε is arbitrary, therefore u is constant in a neighborhood of p. Therefore $u^{-1}(C)$ is open. On the other hand, u is continuous, $u^{-1}(C)$ is closed, hence $u^{-1}(C) = \Omega$. Contradiction. □

Uniqueness of Harmonic Functions

Corollary

Suppose $\Omega \subset \mathbb{R}^2$ is a planar domain, $u_1, u_2 : \Omega \to \mathbb{R}$ are harmonic functions with the same boundary value, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 = u_2$ on Ω.

Proof.

$u_1 - u_2$ is also harmonic, with 0 boundary value, therefore the maximal and minimal values of $u_1 - u_2$ must be on the boundary, namely they are 0, hence u_1, u_2 are equal in Ω. □

Disk Harmonic Maps

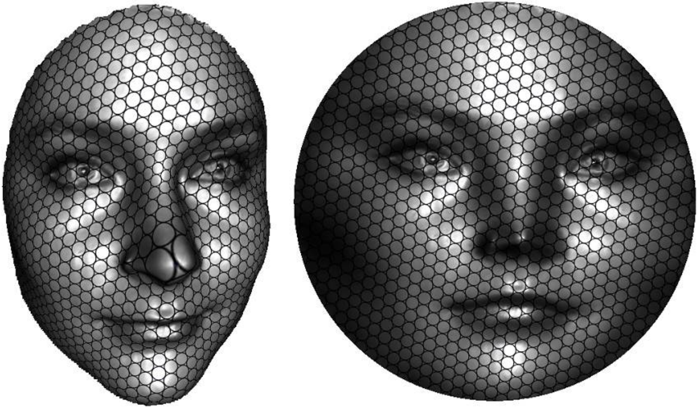


Figure: Harmonic map between topological disks.

Diffeomorphic Property of Disk Harmonic Maps

Theorem (Rado)

Suppose a harmonic map $\varphi : (S, \mathbf{g}) \to (\Omega, dx^2 + dy^2)$ *satisfies:*

1. *planar domain* Ω *is convex*
2. *the restriction of* $\varphi : \partial S \to \partial \Omega$ *on the boundary is homoemorphic,*

then u *is diffeomorphic in the interior of* S*.*

Proof.

By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi : (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated as p, there are two constants $a, b \in \mathbb{R}$, not being zeros simultaneously, such that

$$a\nabla u(p) + b\nabla v(p) = 0.$$

By $\Delta u = 0$,$\Delta v = 0$, the auxiliary function $f(q) = au(q) + bv(q)$ is also harmonic. □

Diffeomorphic Property of Disk Harmonic Maps

continued

By $\nabla f(p) = 0$, p is an saddle point of f. Consider the level set of f near p

$$\Gamma = \{q \in \Omega | f(q) = f(p) - \varepsilon\}$$

Γ has two connected components, intersecting ∂S at 4 points.
But Ω is a planar convex domain, $\partial\Omega$ and the line $au + bv = const$ have two intersection points. By assumption, the mapping φ restricted on the boundary $\varphi : \partial S \to \partial\Omega$ is homeomorphic. Contradiction.

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

General Harmonic Map

Definition (Harmonic Energy)

Let (Σ_1, z) and (Σ_2, u) be two Riemann surfaces, with Riemannian metrics $\sigma(z)dzd\bar{z}$ and $\rho(u)dud\bar{u}$. Given a C^1 map $u : \Sigma_1 \to \Sigma_2$, then the harmonic energy of u is defined as

$$E(z, \rho, u) := \int_{\Sigma_1} \rho^2(u)(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \frac{i}{2} dz d\bar{z}$$

where $u_z := \frac{1}{2}(u_x - iu_y)$, $u_{\bar{z}} := \frac{1}{2}(u_x + iu_y)$ and $dz \wedge d\bar{z} = -2idx \wedge dy$.

Definition (Harmonic Map)

If the C^1 map $u : \Sigma_1 \to \Sigma_2$ minimizes the harmonic energy, then u is called a harmonic map.

General Harmonic Map

Theorem (Euerl-Larange Equation for Harmonic Maps)

Suppose $u:\Sigma_1 \to \Sigma_2$ is a C^2 harmonic map between Riemannian surfaces, then

$$\boxed{u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0}$$

Geodesics are special harmonic maps, harmonic maps are generalized geodesics:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0 \quad u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} \equiv 0}$$

General Harmonic Map

Proof.

Suppose u is harmonic, u_t is a variation in a local coordinates system,

$$u + t\varphi, \quad \varphi \in C^0 \cap W_0^{1,2}(\Sigma_1, \Sigma_2)$$

we obtain

$$\frac{d}{dt} E(u + t\varphi)\Big|_{t=0} = 0,$$

□

General Harmonic Map

continued

$$
\begin{aligned}
0 =& \frac{d}{dt}\left\{ \int \rho^2(u+t\varphi)((u+t\varphi)_z(\bar{u}+t\bar{\varphi})_{\bar{z}} \right. \\
& \left. + (\bar{u}+t\bar{\varphi})_z(u+t\varphi)_{\bar{z}}) i dz d\bar{z} \right\}\Big|_{t=0} \\
=& \int \left\{ \rho^2(u)(u_z\bar{\varphi}_{\bar{z}} + \bar{u}_{\bar{z}}\varphi_z + \bar{u}_z\varphi_{\bar{z}} + u_{\bar{z}}\bar{\varphi}_z) \right. \\
& \left. + 2\rho(\rho_u\varphi + \rho_{\bar{u}}\bar{\varphi})(u_z\bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \right\} i dz d\bar{z}.
\end{aligned}
$$

General Harmonic Map

continued

We set $\varphi = \frac{\psi}{\rho^2(u)}$,

$$\rho^2 \varphi_z = \psi_z - \frac{2\psi}{\rho}(\rho_u u_z + \rho_{\bar{u}} \bar{u}_z)$$

$$\rho^2 \varphi_{\bar{z}} = \psi_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}} \bar{u}_{\bar{z}})$$

$$\rho^2 \bar{\varphi}_z = \bar{\psi}_z - \frac{2\bar{\psi}}{\rho}(\rho_u u_z + \rho_{\bar{u}} \bar{u}_z)$$

$$\rho^2 \bar{\varphi}_{\bar{z}} = \bar{\psi}_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_u u_{\bar{z}} + \rho_{\bar{u}} \bar{u}_{\bar{z}})$$

General Harmonic Map

continued

$$\bar{u}_{\bar{z}}\rho^2\varphi_z = \psi_z\bar{u}_{\bar{z}} - \frac{2\psi}{\rho}(\rho_u u_z\bar{u}_{\bar{z}} + \rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}})$$

$$\bar{u}_z\rho^2\varphi_{\bar{z}} = \psi_{\bar{z}}\bar{u}_z - \frac{2\psi}{\rho}(\rho_u u_{\bar{z}}\bar{u}_z + \rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z)$$

$$u_{\bar{z}}\rho^2\bar{\varphi}_z = \bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_z u_{\bar{z}} + \rho_u u_z u_{\bar{z}})$$

$$u_z\rho^2\bar{\varphi}_{\bar{z}} = \bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho}(\rho_{\bar{u}}\bar{u}_{\bar{z}} u_z + \rho_u u_{\bar{z}} u_z)$$

General Harmonic Map

continued

$$\frac{2}{\rho}(\rho_u \psi + \rho_{\bar{u}} \bar{\psi})(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}})$$
$$= \frac{2\psi}{\rho}\rho_u(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) + \frac{2\bar{\psi}}{\rho}\rho_{\bar{u}}(\bar{u}_z u_{\bar{z}} + u_z \bar{u}_{\bar{z}})$$

Take summation,

$$\bar{u}_{\bar{z}}\rho^2\varphi_z + u_z\rho^2\bar{\varphi}_{\bar{z}} = \left(\psi_z \bar{u}_{\bar{z}} - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_z\bar{u}_{\bar{z}}\right) + \left(\bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho}\rho_u u_{\bar{z}} u_z\right)$$
$$\bar{u}_z\rho^2\varphi_{\bar{z}} + u_{\bar{z}}\rho^2\bar{\varphi}_z = \left(\psi_{\bar{z}} \bar{u}_z - \frac{2\psi}{\rho}\rho_{\bar{u}}\bar{u}_{\bar{z}}\bar{u}_z\right) + \left(\bar{\psi}_z u_{\bar{z}} - \frac{2\bar{\psi}}{\rho}\rho_u u_z u_{\bar{z}}\right)$$

General Harmonic Map

continued

The above equation becomes

$$
\begin{aligned}
0 =& 2\Re \int \left(\bar{\psi}_{\bar{z}} u_z - \frac{2\bar{\psi}}{\rho} \rho_u u_{\bar{z}} u_z \right) idzd\bar{z} \\
&+ 2\Re \int \left(\psi_{\bar{z}} \bar{u}_z - \frac{2\psi}{\rho} \rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z \right) idzd\bar{z}
\end{aligned}
$$

If $u \in C^2$, we can integrate by parts, $(u_z\bar{\psi})_{\bar{z}} = u_{z\bar{z}}\bar{\psi} + u_z\bar{\psi}_{\bar{z}}$,

$$
\begin{aligned}
0 =& 2\Re \int \left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_{\bar{z}} u_z \right) \bar{\psi} idzd\bar{z} \\
&+ 2\Re \int \left(\bar{u}_{z\bar{z}} + \frac{2\rho_{\bar{u}}}{\rho} \bar{u}_{\bar{z}} \bar{u}_z \right) \psi idzd\bar{z}
\end{aligned}
$$

General Harmonic Map

continued

Therefore

$$0 = 2\Re \int \left(u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_{\bar{z}} u_z \right) \bar{\psi} i dz d\bar{z}$$

□

Hopf Differential of Harmonic Maps

Theorem (Hopf Diffential of Harmonic Maps)

Let $u : (\Sigma_1, \lambda^2(z)dzd\bar{z}) \to (\Sigma_2, \rho^2(u)dud\bar{u})$ *is harmonic, then the Hopf differential of the map*

$$\boxed{\Phi(u) := \rho^2 u_z \bar{u}_z dz^2}$$

is holomorphic quadratic differential on Σ_1*. Furthermore* $\Phi(u) \equiv 0$*, if and only if* u *is holomorphic or anti-holomorprhic.*

Proof.

If u is harmonic, then

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(\rho^2 u_z \bar{u}_z) &= \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z \\ &= (\rho^2 u_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z)\bar{u}_z + (\rho^2 \bar{u}_{z\bar{z}} + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z) u_z = 0.\end{aligned}$$

Therefore $\Phi(u)$ is holomorphic.

Hopf Differential of Harmonic Maps

Proof.

If $\Phi(u) = \rho^2 u_z \bar{u}_z \equiv 0$, then either $u_z = 0$ or $\bar{u}_z = 0$. Since the Jacobian determinant equals to

$$|u_z|^2 - |u_{\bar{z}}|^2 > 0,$$

therefore $\bar{u}_z = 0$, namely $u_{\bar{z}} = 0$, u is holomorphic or anti-holomorphic. u is holomorphic, equivalent to $L \equiv 0$; u is anti-holomorphic, equivalent to $H \equiv 0$. We know H and L have isolated zeros, unless they are zero everywhere. Hence u is entirely holomorphic or anti-holomorphic. □

Spherical Harmonic Map

Lemma

A holomorphic quadratic differential ω is on the unit sphere, then ω is zero.

Proof.

Choose two charts z and $w = \frac{1}{z}$. Let $\omega = \varphi(z)dz^2$, then

$$\varphi(z)dz^2 = \varphi\left(\frac{1}{w}\right)\left(\frac{dz}{dw}\right)^2 dw^2 = \varphi\left(\frac{1}{w}\right)\frac{1}{w^4}dw^2.$$

since ω is globally holomorphic, when $w \to 0$,

$$\varphi\left(\frac{1}{w}\right)\frac{1}{w^4} < \infty,$$

hence $z \to \infty$, $\varphi(z) \to 0$. By Liouville theorem, $\varphi \equiv 0$. □

Spherical Harmonic Map

Theorem (Spherical Harmonic Maps)

Harmonic maps between genus zero closed metric surfaces must be conformal.

Proof.

Suppose $u : \Sigma_1 \to \Sigma_2$ is a harmonic map, then $\Phi(u)$ must be a holomorphic quadratic differential. Since Σ_1 is of genus zero, therefore $\Phi(u) \equiv 0$. Hence u is holomorphic. □

Spherical Harmonic Map

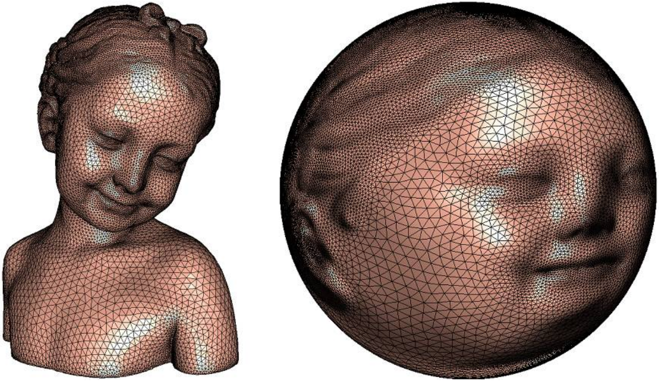


Figure: Spherical Harmonic Map

Uniqueness Spherical Harmonic Map

Definition (Möbius Transformation)

A Möbius transformation $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc=1.$$

Given $\{z_0, z_1, z_2\}$, there is a unique Möbius transformation, that maps them to $\{0, 1, \infty\}$,

$$z \mapsto \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Theorem (Uniquess of Spherical Conformal Automorphisms)

Suppose $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a biholomorphic automorphism, then f must be a Möbius transformation.

Uniqueness of Spherical Harmonic Map

Proof.

By stereo-graphic projection, we map the sphere to the extened complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. First, the poles of f must be finite. Suppose there are infinite poles of f, because $\mathbb{S}^2$ is compact, there must be accumulation points, then f must be a constant value function.

Let $z_1, z_2, \dots, z_n$ be the finite poles of f, with degrees $e_1, e_2, \dots, e_n$. Let $g = \Pi_i (z - z_i)^{e_i}$, then fg is a holomorphic function on $\mathbb{C}$, therefore fg is entire, namely, fg is a polynomial. Therefore

$$f = \frac{\sum_{i=1}^{n} a_i z^i}{\sum_j b_j z^j},$$

if $n > 1$ then f has multiple zeros, contradict to the condition that f is an automorphism. Therefore $n = 1$. Similarly $m = 1$. □

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;
Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian
$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$
3. project the Laplacian to the tangent plane,
$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i)\rangle \varphi(v_i)$$
4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;
5. compute the mass center $c = \sum A_i \varphi(v_i) / \sum_j A_j$; normalize $\varphi(v_i) = \varphi(v_i) - c / |\varphi(v_i) - c|$;
6. Repeat step 2 through 5, until the Laplacian norm is less than ε.

General theory for Surface Harmonic Maps

Existence of Harmonic Map

Theorem (Existence of Harmonic Maps)

Assume Σ is a Riemann surface, $(N, \rho(u)du d\bar{u})$ is a metric surface, then for any smooth mapping $\varphi : \Sigma \to N$, there is a harmonic map $f : \Sigma \to N$ homotopic to φ.

This can be proven using Courant-Lebesgue lemma, which controls the geodesic distance between image points by harmonic energy.

Regularity of Harmonic Map

Theorem (Regularity of Harmonic Maps)

Let $u : \Sigma_1 \to \Sigma_2$ be a (weak) harmonic map between Riemann surfaces, Σ_2 is with hyperbolic metric, the harmonic energy of u is finite, then u is a smooth map.

This is based on the regularity theory of ellptic PDEs.

Theorem (Diffeomorphic Properties of Harmonic Maps)

Let Σ_1 and Σ_2 are compact Riemann surfaces with the same genus, $K_2 \leq 0$. If $u : \Sigma_1 \to \Sigma_2$ is a degree one harmonic map, then u is a diffeomorphism.

Uniqueness of Harmonic Map

Theorem (Uniqueness of Harmonic Map)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

Theorem

Suppose Σ_1 and Σ_2 are Riemann surfaces, the Riemannian metric on Σ_2 induces non-positive curvature K. Let $u \in C^2(\Sigma_1, \Sigma_2)$, $\varphi(z,t)$ is the variation of u, $\dot{\varphi} \neq 0$. If u is harmonic, or for any point $z \in \Sigma_1$, $\varphi(z_1, \cdot)$ is geodesic, then

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} \geq 0. \tag{3}$$

If $K < 0$, then either

$$\frac{d}{dt^2} E(u + \varphi(t))\Big|_{t=0} > 0. \tag{4}$$

or

$$u_z \bar{u}_{\bar{z}} - u_{\bar{z}} \bar{u}_z \equiv 0, \tag{5}$$

Namely the rank of u is ≤ 1 everywhere.

Uniqueness of Harmonic Map

Consider the variation of the mapping u, $u(z)+\varphi(z,t)$, where $\varphi(z,0)\equiv 0$. Let $\dot{\varphi}=\frac{\partial}{\partial t}\varphi$, $\ddot{\varphi}:=\frac{\partial^2}{\partial t^2}\varphi$. $K=-\Delta\log\rho=-\frac{4}{\rho^4}(\rho\rho_{u\bar{u}}-\rho_u\rho_{\bar{u}})$

$$
\begin{aligned}
\frac{d^2}{dt^2}E(u+\varphi(t))\Big|_{t=0} = 2\int\Big\{&\rho^2\left(\dot{\varphi}_z+2\frac{\rho_u}{\rho}u_z\dot{\varphi}\right)\left(\dot{\bar{\varphi}}_{\bar{z}}+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_{\bar{z}}\dot{\bar{\varphi}}\right)\\
&+\rho^2\left(\dot{\bar{\varphi}}_z+2\frac{\rho_{\bar{u}}}{\rho}\bar{u}_z\dot{\bar{\varphi}}\right)\left(\dot{\varphi}_{\bar{z}}+2\frac{\rho_u}{\rho}u_{\bar{z}}\dot{\varphi}\right)\\
&-\rho^4\frac{K}{2}(u_z\dot{\bar{\varphi}}-\bar{u}_z\dot{\varphi})(\bar{u}_{\bar{z}}\dot{\varphi}-u_{\bar{z}}\dot{\bar{\varphi}})\\
&-(\rho^2\ddot{\varphi}+2\rho\rho_u\dot{\varphi}^2)\left(\bar{u}_{z\bar{z}}+\frac{2\rho_{\bar{u}}}{\rho}\bar{u}_z\bar{u}_{\bar{z}}\right)\\
&-(\rho^2\ddot{\bar{\varphi}}+2\rho\rho_{\bar{u}}\dot{\bar{\varphi}}^2)\left(u_{z\bar{z}}+\frac{2\rho_u}{\rho}u_z u_{\bar{z}}\right)\Big\}\,idzd\bar{z}
\end{aligned}
\tag{6}
$$

Uniqueness of Harmonic Map

If u is harmonnic, then

$$u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0,$$

or if $\varphi(z,\cdot)$ is geodesic, then

$$\rho^2 \ddot{\varphi} + 2\rho\rho_u \dot{\varphi}^2 = 0.$$

Then, the last two items vanish. Since $K \leq 0$, there first three items are non-negative.
If $K < 0$, then $\frac{d^2}{dt^2} E(u+\varphi(t))|_{t=0}$ is either positive or zero. If it is 0, then the integrands must be 0 everywhere, therefore

$$u_z \dot{\bar{\varphi}} - \bar{u}_z \dot{\varphi} \equiv \bar{u}_{\bar{z}} \dot{\varphi} - u_{\bar{z}} \dot{\bar{\varphi}} \equiv 0. \tag{7}$$

Uniqueness of Harmonic Map

Furthermore

$$\frac{\partial}{\partial z}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = (\rho^2\dot{\varphi}_z + 2\rho\rho_u u_z\dot{\varphi})\dot{\bar{\varphi}} + (\rho^2\dot{\bar{\varphi}} + 2\rho\rho_u \bar{u}_z\dot{\bar{\varphi}})\dot{\varphi} = 0. \quad (8)$$

Similarly

$$\frac{\partial}{\partial \bar{z}}(\rho^2\dot{\varphi}\dot{\bar{\varphi}}) = 0. \quad (9)$$

We obtain

$$\rho^2\dot{\varphi}\dot{\bar{\varphi}} \equiv \text{const}. \quad (10)$$

By assumption $\dot{\varphi} \not\equiv 0$, the constant is non-zero, hence $\dot{\varphi}$ and $\dot{\bar{\varphi}}$ are non-zero everywhere, by (7) we get

$$|u_z||\dot{\varphi}| = |\bar{u}_z||\dot{\bar{\varphi}}|$$

hence

$$|u_z| = |\bar{u}_z| = |u_{\bar{z}}|$$

we get (5). □

Uniqueness of Harmonic Map

Theorem (Uniqueness)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

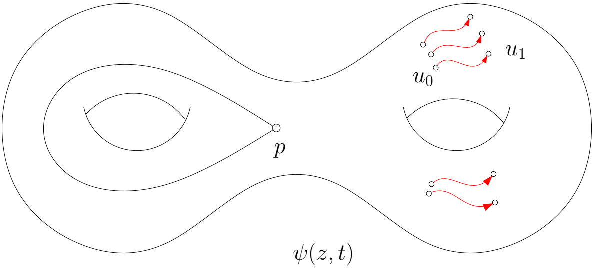
Proof.

Given a homotopy connecting u_0 and u_1, $h(z,t) : \Sigma_1 \times [0,t] \to \Sigma_2$, such that $h(z,0) = u_0(z)$, $h(z,1) = u_1(z)$. Let $\psi(z,t)$ is a geodesic from $u_0(z)$ to $u_1(z)$ and homotopic to $h(z,t)$, with parameter

$$\rho(\psi(z,t))|\dot{\psi}(z,t)| \equiv \text{const}$$

then $u_t(z) := \psi(z,t)$ is also a homotopy connecting u_0 and u_1. □

Uniqueness of Harmonic Map

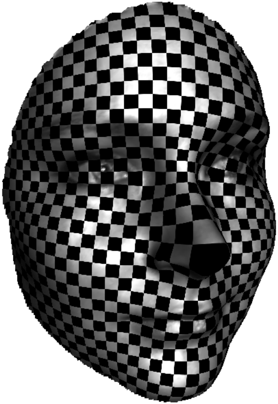


continue

We define function $f(t) := E(u_t)$. By above theorem, $\forall t \in [0,1]$, $\ddot{f}(t) \geq 0$, hence $f(t)$ is convex. Since u_0 and u_1 are harmonic, $\dot{f}(0) = \dot{f}(1) = 0$. By the assumption of the Jacobian matrix, either $\ddot{f}(0) > 0$ or $\ddot{f}(1) > 0$, hence we must have $\dot{\psi}(t) \equiv 0$, namely $u_0 \equiv u_1$. $\square$

Riemann Mapping

Riemann Mapping



Riemann Mapping Proof I

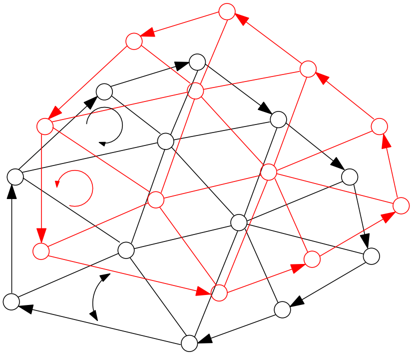


Figure: Surface double covering.

Surface Double Covering Algorithm

Input: A oriented surface with boundaries M;
Output: The double covering $\bar{M}$;

1. Make a copy of M, denoted as M';
2. Reverse the order of the vertices of each face of M';
3. Glue M and M' along their corresponding boundary edges to obtain $\bar{M}$.

Surface Double Covering Algorithm

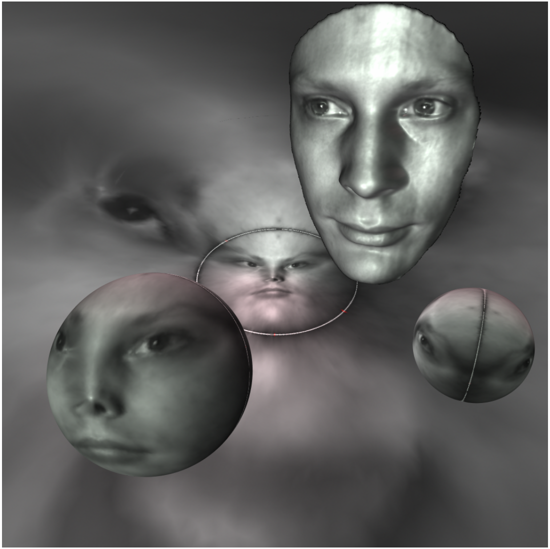


Figure: Spherical harmonic map for a double covering of a facial surface.

Surface Riemann Mapping Theorem

Theorem (Riemann Mapping)

Suppose $(S, \mathbf{g})$ *is a topological disk with a Riemannian metric* $\mathbf{g}$*, then there exists a conformal map* $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$*. Furthermore, such kind of mappings differ by a Möbius transformation with the form*

$$z \to e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad |z_0| < 1, \theta \in [0, 2\pi). \tag{11}$$

Surface Riemann Mapping Theorem

Proof.

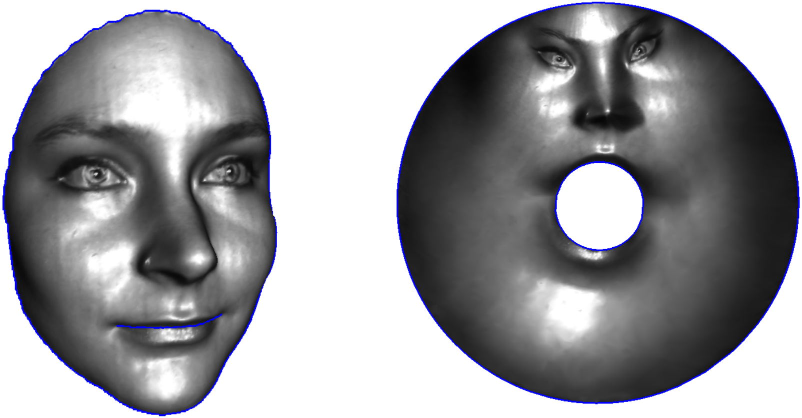
First, we double cover $(S, \mathbf{g})$ to obtain $\bar{S}$ with a Riemannian metric $\bar{\mathbf{g}}$. According to the symmetry, $\bar{\mathbf{g}}$ is well defined. Then there is a harmonic map $\varphi : (\bar{S}, \bar{\mathbf{g}}) \to \mathbb{S}^2$. We use a Möbius transformation to adjust the mapping, such that the boundary ∂S is mapped to the equator. Due to the symmetry of $\bar{S}$, such kind of Möbius transformation exists. Then by the stereo-graphic projection, we map one hemi-sphere onto the unit planar disk. Then the composition of the stereo-graphic projection and φ gives the desired conformal mapping.
All the Mobius transformations maps the unit disk onto itself has the form of Eqn. 11. $\square$

Riemann Mapping Algorithm I

Input: A topological disk surface M;
Output: A Riemann mapping $\varphi : M \to \mathbb{D}^2$.

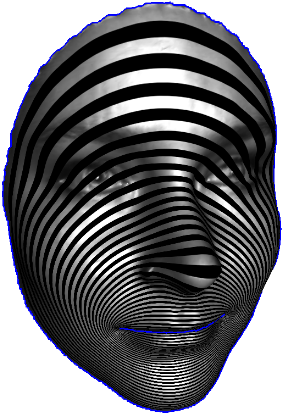
1. Compute a double covering $\bar{M}$ of M;
2. Compute a harmonic map $\varphi : \bar{M} \to \mathbb{S}^2$;
3. Use a stereo-graphics projection to $\tau : \mathbb{S}^2 \hat{\mathbb{C}}$;
4. Use a Möbius transformation, to maps the hemisphere to the unit disk.

Topological Annulus



Conformal mapping for topological annulus.

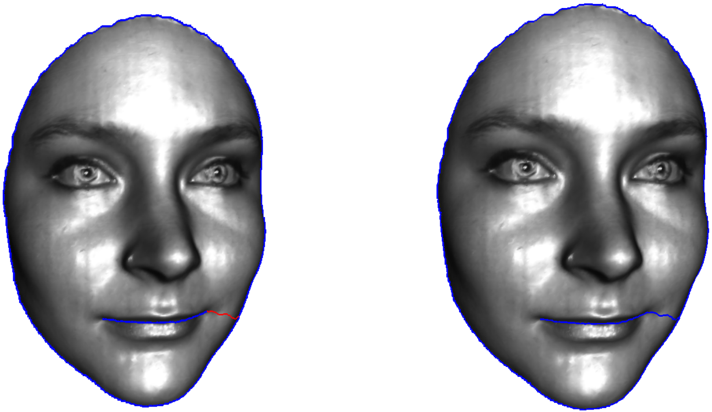
Topological Annulus



exact harmonic form

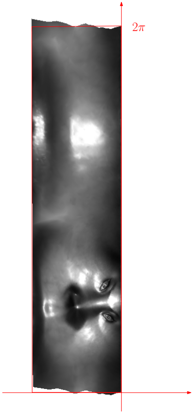
closed harmonic 1-form

Topological Fundamental Domain



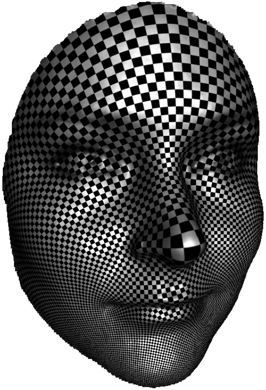
Find the shortest path τ connecting γ_0 and γ_1, slice the mesh along τ to get a topological disk $\bar{M}$.

Integration



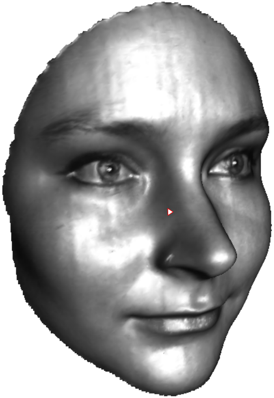
Integrating $\omega + \sqrt{-1}^*\omega$ on $\bar{M}$, normalize the rectangular image $\varphi(\bar{M})$, such that $\varphi(\gamma_0)$ is along the imaginary axis, the height is 2π, $\varphi(\gamma_1)$ is $x = -c$, $c > 0$ is a real number.

Integration



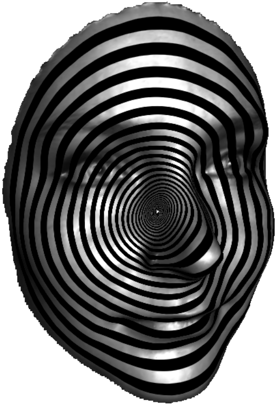
Compute the polar map e^{φ}, which maps $\varphi(\bar{M})$ to an annulus.

Riemann Mapping

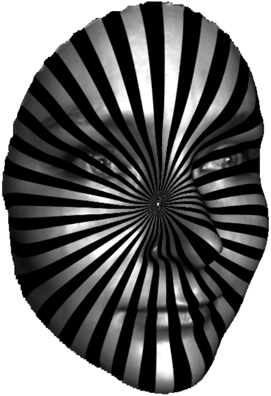


Riemann mapping can be obtained by puncturing a small hole on the surface, then use topological annulus conformal mapping algorithm.

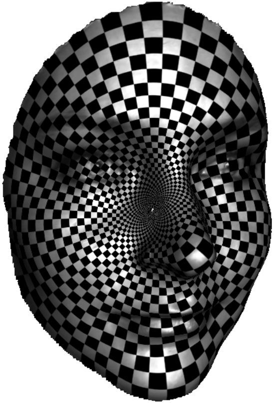
Riemann Mapping



ω



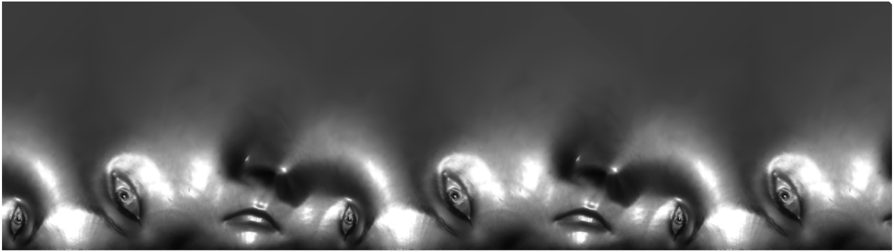
$^*\omega$



$\omega + \sqrt{-1}^*\omega$

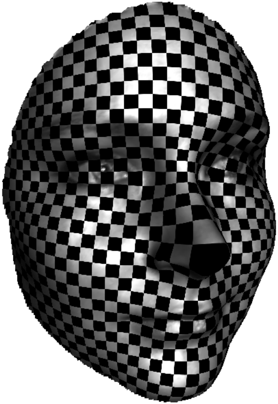
Exact harmonic 1-form and closed, non-exact harmonic 1-form.

Riemann Mapping



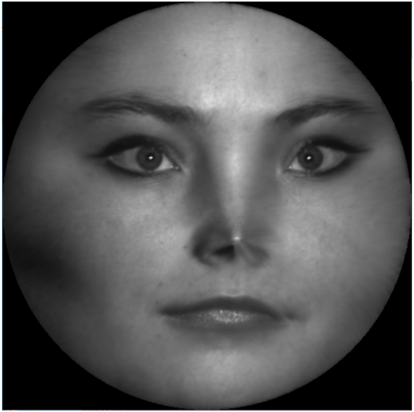
Periodic conformal mapping image $\varphi(M)$.

Riemann Mapping



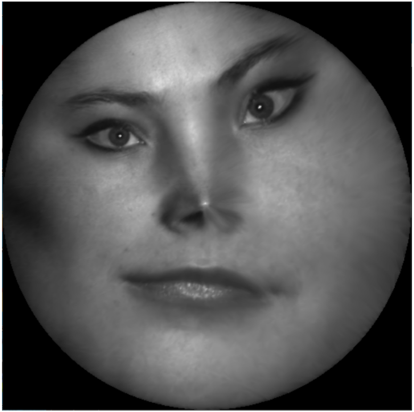
Polar map $e^{\varphi(p)}$ induces the Riemann mapping.

Riemann Mapping



The choice of the central puncture, and the rotation determine a Möbius transformation.

Riemann Mapping

The conformal automorphism of the unit disk is the Möbius transformation group.

Riemann Mapping Proof II

Proof.

We punch a small hole in the center of S, to get an annulus $\bar{S}$, with two boundary components $\partial\bar{S} = \gamma_0 - \gamma_1$. By Hodge theory, there exists a harmonic 1-form ω, such that $\int_{\gamma_0} \omega = 2\pi$.
The Laplace equation

$$\Delta_{\mathbf{g}} f \equiv 0, \quad f|_{\partial\gamma_0} = 0, f|_{\partial\gamma_1} = -1,$$

has unique solution. Then there is a constant λ, such that

$$\lambda^* df = \omega,$$

then $\lambda df + \sqrt{-1}\omega$ is a holomorphic 1-form. Choose a point $p \in \bar{S}$, then mapping

$$\varphi(q) := \exp\left(\int_p^q \lambda df + \sqrt{-1}\omega\right)$$

maps $\bar{S}$ to a planar canonical annulus.

Riemann Mapping Proof II

Proof.

When the punched hole shrinks to a point, φ converges to a global conformal map, which is the desired Riemann mapping.
The choice of the puncture, and the rotation of the unit disk gives all possible conformal automorphisms of the unit planar disk, which is the Möbius transformation. □

Riemann Mapping Algorithm II

Input: A topological surface $(M, \mathbf{g})$;
Output: A Riemann mapping $\varphi : M \to \mathbb{D}^2$;

1. Punch a small hole on M, to get $\bar{M}$, $\partial \bar{M} = \gamma_0 - \gamma_1$;
2. Solve Laplace equation $\Delta_{\mathbf{g}} f \equiv 0$ with Dirichlet boundary condition, $f|_{\gamma_0} = 0$ and $f|_{\gamma_1} = -1$;
3. Compute a harmonic 1-form ω, such that $\int_{\gamma_0} \omega = 2\pi$;
4. Find a constant , such that ${}^*\omega = -\lambda df$;
5. Find the shortest path τ between γ_0 and γ_1, slice $\bar{M}$ along τ to get an open surface $\hat{M}$;
6. Choose a base point $p \in \hat{M}$, compute the mapping

$$\varphi(q) := \exp\left(\int_p^q \lambda df + \sqrt{-1}\omega\right),$$

7. The mapping φ is the desired conformal mapping.