Yamabe Equation and Geodesics

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Computation under Isothermal Coordinates

Isothermal Coordinates

Lemma (Isothermal Coordinates)

Let $(S, \mathbf{g})$ *be a metric surface, use isothermal coordinates*

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then we obtain

$$\boxed{\omega_1 = e^u dx \quad \omega_2 = e^u dy}$$

and the orthonormal frame is

$$\boxed{\mathbf{e_1} = e^{-u}\partial_x \quad \mathbf{e_2} = e^{-u}\partial_y}$$

and the connection

$$\boxed{\omega_{12} = -u_y dx + u_x dy}$$

Gaussian Curvature

Proof.

By direct computation, $ds^2 = \omega_1^2 + \omega_2^2$,

$$
\begin{aligned}
d\omega_1 &= de^u \wedge dx \\
&= e^u(u_x dx + u_y dy) \wedge dx \\
&= e^u u_y dy \wedge dx
\end{aligned}
\qquad
\begin{aligned}
d\omega_2 &= de^u \wedge dy \\
&= e^u(u_x dx + u_y dy) \wedge dy \\
&= e^u u_x dx \wedge dy.
\end{aligned}
$$

therefore

$$
\begin{aligned}
\omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2 \\
&= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\
\omega_{12} &= -u_y dx + u_x dy.
\end{aligned}
$$

□

Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvautre is given by

$$\boxed{K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u.}$$

Proof.

From

$$\omega_{12} = -u_y dx + u_x dy$$

we get

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = -\frac{(u_{xx} + u_{yy}) dx \wedge dy}{e^{2u} dx \wedge dy} = -\frac{1}{e^{2u}} \Delta u.$$

□

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{(1-x^2-y^2)^2}$, then $u = \log 2 - \log(1 - x^2 - y^2)$.

$$u_x = -\frac{-2x}{1 - x^2 - y^2} = \frac{2x}{1 - x^2 - y^2}.$$

□

Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1 - x^2 - y^2) - 2x(-2x)}{(1 - x^2 - y^2)^2} = \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}$$

similarly

$$u_{yy} = \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1 - x^2 - y^2)^2} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx} + u_{yy}) = -1.$$

□

Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, *then*

$$\boxed{\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda)}$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{-2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{-2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda)\\ &= \frac{1}{e^{2\lambda}}(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda)\\ &= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).\end{aligned}$$

Geodesics

Geodesic Equation

Lemma (Geodesic Equation on a Riemann Surface)

Suppose S is a Riemann surface with a metric, $\rho(z)dzd\bar{z} = e^{2u(z)}dzd\bar{z}$, then a geodesic γ with local representation $z(t)$ satisfies the equation:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.}$$

equivalently,

$$\ddot{\gamma} + 4u_\gamma\dot{\gamma}^2 \equiv 0.$$

Geodesic Equation

Proof.

Assume the velocity vector is $\dot{\gamma} = f_1\mathbf{e_1} + f_2\mathbf{e_2}$, which is parallel along γ, by parallel transport ODE,

$$\left\{\begin{array}{lcl} \frac{df_1}{dt} - f_2\frac{\omega_{12}}{dt} & = & 0 \\ \frac{df_2}{dt} + f_1\frac{\omega_{12}}{dt} & = & 0 \end{array}\right.$$

Suppose the geodesic has local representation $\gamma(t) = (x(t), y(t))$, then $d\gamma = \dot{x}\partial_x + \dot{y}\partial_y = e^u\dot{x}\mathbf{e_1} + e^u\dot{y}\mathbf{e_2}$, $\omega_{12}/dt = -u_y\dot{x} + u_x\dot{y}$, $\rho = e^u$,

$$\frac{d}{dt}(\rho\dot{x}) - (\rho\dot{y})(-u_y\dot{x} + u_x\dot{y}) = 0$$
$$\frac{d}{dt}(\rho\dot{y}) + (\rho\dot{x})(-u_y\dot{x} + u_x\dot{y}) = 0$$

□

Geodesic Equation

continued

in turn,

$$\rho\ddot{x} + \dot{\rho}\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{x} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{x} - \dot{y}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$
$$\rho\ddot{y} + \dot{\rho}\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = \rho\ddot{y} + (\rho_x\dot{x} + \rho_y\dot{y})\dot{y} + \dot{x}(-\rho_y\dot{x} + \rho_x\dot{y}) = 0$$

namely

$$\rho\ddot{x} + \rho_x(\dot{x}^2 - \dot{y}^2) + 2\rho_y\dot{x}\dot{y} = 0$$
$$\rho\ddot{y} - \rho_y(\dot{x}^2 - \dot{y}^2) + 2\rho_x\dot{x}\dot{y} = 0$$

The first row plus $\sqrt{-1}$ times the second row,

$$\rho(\ddot{x} + \sqrt{-1}\ddot{y}) + (\rho_x - \sqrt{-1}\rho_y)(\dot{x} + \sqrt{-1}\dot{y})^2 = 0.$$

Geodesic Equation

continued.

Represent $\gamma(t) = z(t)$, where $z = x + \sqrt{-1}y$, $\rho_z = \frac{1}{2}(\rho_x - \sqrt{-1}\rho_y)$, we obtain the equation for geodesic on complex domain,

$$\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.$$

□

Geodesic Curvature

Lemma

Given a curve γ on a surface $(S, \mathbf{g})$, with isothermal coordinates (x, y), the angle between ∂_x and $\dot{\gamma}$ is θ, then

$$\boxed{k_g(s) = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}}$$

Proof.

Construct an orthonormal frame $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$ by rotating $\{\mathbf{e}_1, \mathbf{e}_2\}$ by angle θ, hence $\bar{\mathbf{e}}_1$ is the tangent vector of γ.

$$\begin{cases} \bar{\mathbf{e}}_1 &= \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= -\sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_2 \end{cases}$$

$$\begin{aligned} d\bar{\mathbf{e}}_1 &= -\sin\theta d\theta \mathbf{e}_1 + \cos\theta d\mathbf{e}_1 + \cos\theta d\theta \mathbf{e}_2 + \sin\theta d\mathbf{e}_2 \\ &= (-\sin\theta d\theta - \sin\theta\omega_{12})\mathbf{e}_1 + (\cos\theta\omega_{12} + \cos\theta d\theta)\mathbf{e}_2 \\ &+ (\cos\theta\omega_{13} + \sin\theta\omega_{23})\mathbf{e}_3 \end{aligned}$$

continued

$$\begin{aligned}\bar{\omega}_{12} &= \langle d\bar{\mathbf{e}_1}, \bar{\mathbf{e}}_2 \rangle \\ &= (-sin\theta d\theta - \sin\theta\omega_{12})(-\sin\theta) + (\cos\theta\omega_{12} + \cos\theta d\theta)\cos\theta \\ &= d\theta + \omega_{12}.\end{aligned}$$

Therefore

$$k_g = \frac{\bar{\omega}_{12}}{ds} = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

□

Geodesic Curvature

Lemma (Geodesic Curvature)

Under the isothermal coordinates, the geodesic curvature is given by

$$\boxed{k_g = e^{-u}(k - \partial_{\mathbf{n}} u)}$$

where k is the curvature on the parameter plane, $\mathbf{n}$ *is the exterior normal to the cure on the parameter plane.*

Proof.

We have $\omega_{12} = -u_y dx + u_x dy$. On the parameter plane, the arc length is dt, then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} = \frac{dt}{ds}\left(\frac{d\theta}{dt} + \frac{-u_y dx + u_x dy}{dt}\right) \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_{\mathbf{n}} u) \end{aligned}$$

Geodesic Curvature

Lemma

Given a metric surface $(S, \mathbf{g})$*, under conformal deformation,* $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$*, the geodesic curvature satisfies*

$$\boxed{k_{\bar{\mathbf{g}}} = e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda)}$$

Proof.

$$\begin{aligned} k_{\bar{\mathbf{g}}} &= e^{-(u+\lambda)}(k - \partial_{\mathbf{n}}(u+\lambda)) \\ &= e^{-\lambda}(e^{-u}(k - \partial_{\mathbf{n}} u) - e^{-u}\partial_{\mathbf{n}}\lambda) \\ &= e^{-\lambda}(k_{\mathbf{g}} - \partial_{\mathbf{n},\mathbf{g}}\lambda) \end{aligned}$$

□

Geodesics

Definition (geodesic)

Given a metric surface $(S, \mathbf{g})$, a curve $\gamma : [0,1] \to S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

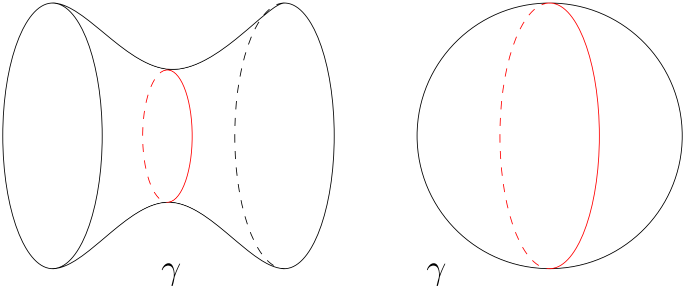


Figure: Stable and unstable geodesics.

Lemma (geodesic)

If γ is the shortest curve connecting p and q, then γ is a geodesic.

Proof.

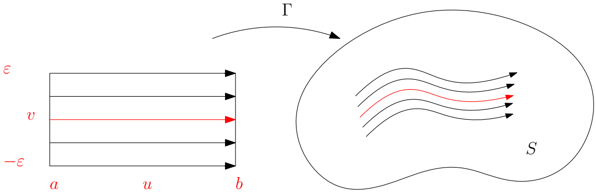
Consider a family of curves, $\Gamma : (-\varepsilon, \varepsilon) \to S$, such that $\Gamma(0,t) = \gamma(t)$, and

$$\Gamma(s,0) = p, \Gamma(s,1) = q, \frac{\partial \Gamma(s,t)}{\partial s} = \varphi(t)\mathbf{e_2}(t),$$

where $\varphi : [0,1] \to \mathbb{R}$, $\varphi(0) = \varphi(1) = 0$. Fix parameter s, curve $\gamma_s := \Gamma(s,\cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left|\frac{d\gamma_s(t)}{dt}\right| dt, \quad \frac{\partial L(s)}{\partial s} = -\int_0^1 \varphi k_{\mathbf{g}}(\tau) d\tau.$$

□



Let $\gamma_v : [a,b] \to M$, where $v \in (-\varepsilon, \varepsilon) \in \mathbb{R}$ be a 1-parameter family of paths. We define the map $\Gamma : [a,b] \times [0,1] \to M$ by

$$\Gamma(u,v) := \gamma_v(u).$$

Define the vector fields $\mathbf{u}$ and $\mathbf{v}$ along γ_v by

$$\mathbf{u} := \frac{\partial \Gamma}{\partial u} = \Gamma_*(\partial_u), \quad \text{and} \quad \mathbf{v} := \frac{\partial \Gamma}{\partial v} = \Gamma_*(\partial_v),$$

We call $\mathbf{u}$ the *tangent vector field* and $\mathbf{v}$ the *variation vector field*.

First Variation of arc length

Lemma (First variation of arc length)

If The length of γ_v is given by

$$L(\gamma_v) := \int_a^b |\mathbf{u}(\gamma_v(u))| du.$$

γ_0 is parameterized by arc length, that is, $|\mathbf{u}(\gamma_0(u))| \equiv 1$, then

$$\frac{d}{dv}\Big|_{v=0} L(\gamma_v) = -\int_a^b \langle D_{\mathbf{u}}\mathbf{u}, \mathbf{v}\rangle du + \langle \mathbf{u}, \mathbf{v}\rangle\Big|_a^b.$$

If we choose $\mathbf{u} = \mathbf{e_1}$, the tangent vector of γ, $\mathbf{v} = \mathbf{e_2}$ orthogonal to $\mathbf{e_1}$, and fix the starting and ending points of paths, then

$$\frac{d}{dv} L(\gamma_v) = -\int_a^b k_g ds.$$

First variation of arc length

Proof.

Fixing $u \in [a, b]$, we may consider $\mathbf{u}$ and $\mathbf{v}$ as vector fields along the path $v \mapsto \gamma_v(u)$. Then

$$\begin{aligned}
\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))| &= \frac{\partial}{\partial v}\sqrt{|\mathbf{u}(\gamma_v(u))|^2} \\
&= \frac{1}{2|\mathbf{u}(\gamma_v(u))|}\frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|^2 \\
&= \frac{1}{2|\mathbf{u}|}\mathbf{v}|\mathbf{u}|^2 = |\mathbf{u}|^{-1}\langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} = \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}}
\end{aligned}$$

□

First variation of arc length

Proof.

$$\frac{d}{dv}L(\gamma_v) = \int_a^b \frac{\partial}{\partial v}|\mathbf{u}(\gamma_v(u))|du = \int_a^b \langle D_{\mathbf{v}}\mathbf{u}, \mathbf{u}\rangle_{\mathbf{g}} du$$

Since $D_{\mathbf{v}}\mathbf{u} - D_{\mathbf{u}}\mathbf{v} = [\mathbf{v}, \mathbf{u}]$, and $[\mathbf{v}, \mathbf{u}] = \Gamma_*([\partial_v, \partial_u]) = 0$,

$$\begin{aligned} \frac{d}{dv}L(\gamma_v) &= \int_a^b \langle D_{\mathbf{u}}\mathbf{v}, \mathbf{u}\rangle_{\mathbf{g}} du \\ &= \int_a^b \left(\frac{d}{du}\langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}} - \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} \right) du \\ &= \langle \mathbf{u}, \mathbf{v}\rangle_{\mathbf{g}}\Big|_a^b - \int_a^b \langle \mathbf{v}, D_{\mathbf{u}}\mathbf{u}\rangle_{\mathbf{g}} du. \end{aligned}$$

□

Geodesics

The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.

Geodesics

Lemma (Uniqueness of geodesics)

Suppose $(S, \mathbf{g})$ *is a closed oriented metric surface,* $\mathbf{g}$ *induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.*

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ, by Gauss-Bonnet,

$$\int_\Sigma K dA + \int_{\partial\Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. □

Algorithm: Homotopy Detection

Input: A high genus closed mesh M, two loops γ_1 and γ_2;
Output: Whether $\gamma_1 \sim \gamma_2$;

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ_k to geodesics, $k = 1, 2$;
3. if two geodesics coincide, return true; otherwise, return false;

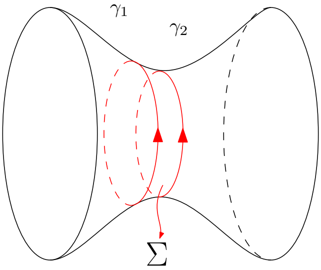


Figure: Geodesics uniqueness.

Algorithm: Shortest Word

Input: A high genus closed mesh M, one loop γ

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ to a geodesic;
3. Compute a set of canonical fundamental group basis;
4. Embed a finite portion of the universal covering space onto the Poincaré disk;
5. Lift γ to the universal covering space $\tilde{\gamma}$. If $\tilde{\gamma}$ crosses $b_i^{\pm}$, append $a_i^{\pm}$; crosses $a_i^{\pm}$, append $b_i^{\mp}$.

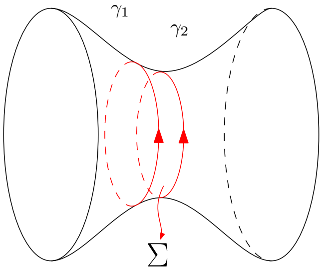


Figure: Geodesics uniqueness

Hyperbolic Geodesics

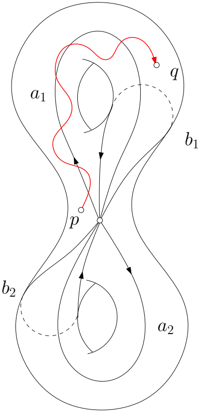
Lemma

Let Σ be a compact hyperbolic Riemann surface, $K \equiv -1$, $p, q \in \Sigma$, then there exists a unique geodesic in each homotopy class, the geodesic depends on p and q continuously.

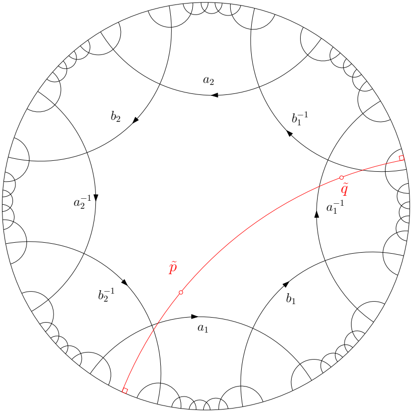
Proof.

Given a path $\gamma : [0,1] \to \Sigma$ connecting p and q. Let $\pi : \mathbb{H}^2 \to \Sigma$ be the universal covering space of Σ. Fix one point $\tilde{p} \in \pi^{-1}(p)$, then there exists a unique lifting of γ, $\tilde{\gamma} : [0,1] \to \mathbb{H}^2$, $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}(1) = \tilde{q}$. On the hyperbolic plane, the geodesic between $\tilde{p}$ and $\tilde{q}$ exists and is unique, $\tilde{\gamma}$ depends on $\tilde{p}$ and $\tilde{q}$ continuously. □

Hyperbolic Geodesic



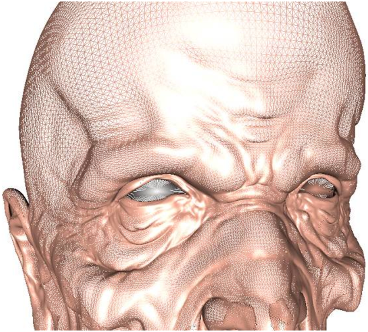
geodesic on surface



Poincaré's disk model

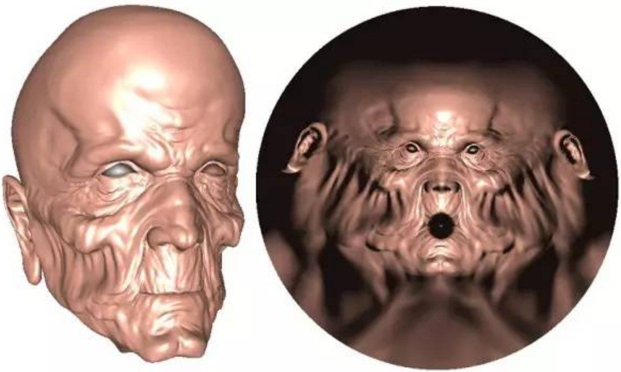
Finite Element Method

Finite Element Method



Given a smooth surface $(S, \mathbf{g})$, we can construct a sequence of triangle meshes $\varphi_n : S \to (M_n, \mathbf{d}_n)$, the pull back metric $\{\varphi_n^* \mathbf{d}_n\}$ converge to $\mathbf{g}$.

Finite Element Method



For each M_n, construct a harmonic map $f_n : M_n \to \mathbb{D}^2$. Then $\{f_n\}$ converge to the smooth harmonic map $f : S \to \mathbb{D}^2$.

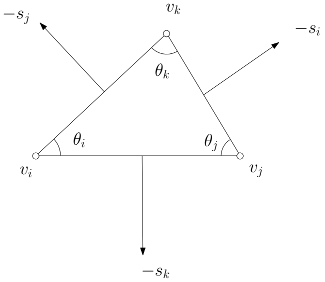
Finite Element Method

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \to \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij}(f(v_i) - f(v_j))^2.$$

$$w_{ij} = \cot \theta_{ij}^k + \cot \theta_{ji}^l.$$



Finite Element Method

Definition (Barry-centric Coordinates)

Given a Euclidean triangle with vertices $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$the bary-centric coordinates of a planar point $\mathbf{p} \in \mathbb{R}^2$ with respect to the triangle are $(\lambda_i, \lambda_j, \lambda_k)$, $\mathbf{p} = \lambda_i \mathbf{v_i} + \lambda_j \mathbf{v_j} + \lambda_k \mathbf{v_k}$, where

$$\lambda_i = \frac{(\mathbf{v}_j - \mathbf{p}) \times (\mathbf{v}_k - \mathbf{p}) \cdot \mathbf{n}}{(\mathbf{v}_j - \mathbf{v}_i) \times (\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}}$$

the ratio between the area of the triangle $\mathbf{p}, \mathbf{v}_j, \mathbf{v}_k$ and the area of $\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k$. λ_j and λ_k are defined similarly.

By direct computation, the sum of the bary-centric coordinates is 1

$$\lambda_i + \lambda_j + \lambda_k = 1.$$

If $\mathbf{p}$ is the interior point of the triangle, then all components of the bary-centric coordinates are positive.

Finite Element Method

Lemma

Suppose $f : \Delta \to \mathbb{R}$ is a linear function,

$$f(p) = \lambda_i f(v_i) + \lambda_j f(v_j) + \lambda_k f(v_k),$$

then the gradient of the function is

$$\nabla f(p) = \frac{1}{2A}(s_i f(v_i) + s_j f(v_j) + s_k f(v_k)).$$

its harmonic energy is

$$\int_\Delta \langle \nabla f, \nabla f \rangle dA = \frac{\cot \theta_i}{2}(f_j - f_k)^2 + \frac{\cot \theta_j}{2}(f_k - f_i)^2 + \frac{\cot \theta_k}{2}(f_i - f_j)^2. \quad (1)$$

Finite Element Method

Proof.

We have

$$\mathbf{s}_i + \mathbf{s}_j + \mathbf{s}_k = \mathbf{n} \times \{(\mathbf{v}_k - \mathbf{v}_j) + (\mathbf{v}_i - \mathbf{v}_k) + (\mathbf{v}_j - \mathbf{v}_i)\} = \mathbf{0}$$

therefore

$$\langle \mathbf{s}_i, \mathbf{s}_i \rangle = \langle \mathbf{s}_i, -\mathbf{s}_j - \mathbf{s}_k \rangle = -\langle \mathbf{s}_i, \mathbf{s}_j \rangle - \langle \mathbf{s}_i, \mathbf{s}_k \rangle.$$

pick a point $\mathbf{p} = \lambda_i \mathbf{v}_i + \lambda_j \mathbf{v}_j + \lambda_k \mathbf{v}_k$, bary-centric coordinates

$$\lambda_i = \frac{1}{2A}(\mathbf{v}_k - \mathbf{v}_j, \mathbf{p} - \mathbf{v}_j, \mathbf{n}) = \frac{1}{2A}\langle \mathbf{n} \times (\mathbf{v}_k - \mathbf{v}_j), \mathbf{p} - \mathbf{v}_j \rangle$$

hence

$$\lambda_i = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_j, \mathbf{s}_i \rangle, \lambda_j = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_k, \mathbf{s}_j \rangle, \lambda_k = \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_i, \mathbf{s}_k \rangle,$$

where A is the triangle area. □

Finite Element Method

continued

The linear function is

$$\begin{aligned} f(\mathbf{p}) &= \lambda_i f_i + \lambda_j f_j + \lambda_k f_k \\ &= \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_j, f_i\mathbf{s}_i\rangle + \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_k, f_j\mathbf{s}_j\rangle + \frac{1}{2A}\langle \mathbf{p} - \mathbf{v}_i, f_k\mathbf{s}_k\rangle \\ &= \langle \mathbf{p}, \frac{1}{2A}(f_i\mathbf{s}_i + f_j\mathbf{s}_j + f_k\mathbf{s}_k)\rangle - \frac{1}{2A}(\langle \mathbf{v}_j, f_i\mathbf{s}_i\rangle + \langle \mathbf{v}_k, f_j\mathbf{s}_j\rangle + \langle \mathbf{v}_i, f_k\mathbf{s}_k\rangle). \end{aligned}$$

Hence we obtain the gradient

$$\nabla f = \frac{1}{2A}(f_i\mathbf{s}_i + f_j\mathbf{s}_j + f_k\mathbf{s}_k).$$

Finite Element Method

continued

we compute the harmonic energy

$$
\begin{aligned}
&\int_{\Delta} \langle \nabla f, \nabla f \rangle dA \\
=&\frac{1}{4A} \langle f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k, f_i \mathbf{s}_i + f_j \mathbf{s}_j + f_k \mathbf{s}_k \rangle \\
=&\frac{1}{4A} \left(\sum_i \langle \mathbf{s}_i, \mathbf{s}_i \rangle f_i^2 + 2 \sum_{i<j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\
=&\frac{1}{4A} \left(-\sum_i \langle \mathbf{s}_i, \mathbf{s}_j + \mathbf{s}_k \rangle f_i^2 + 2 \sum_{i<j} \langle \mathbf{s}_i, \mathbf{s}_j \rangle f_i f_j \right) \\
=&-\frac{1}{4A} \left(\langle \mathbf{s}_i, \mathbf{s}_j \rangle (f_i - f_j)^2 + \langle \mathbf{s}_j, \mathbf{s}_k \rangle (f_j - f_k)^2 + \langle \mathbf{s}_k, \mathbf{s}_i \rangle (f_k - f_i)^2 \right)
\end{aligned}
$$

continued

Since

$$\frac{\langle \mathbf{s}_i, \mathbf{s}_j \rangle}{2A} = -\cot\theta_k, \frac{\langle \mathbf{s}_j, \mathbf{s}_k \rangle}{2A} = -\cot\theta_i, \frac{\langle \mathbf{s}_k, \mathbf{s}_i \rangle}{2A} = -\cot\theta_j.$$

Hence the harmonic energy is

$$\int_\Delta \langle \nabla f, \nabla f \rangle dA = \frac{\cot\theta_i}{2}(f_j - f_k)^2 + \frac{\cot\theta_j}{2}(f_k - f_i)^2 + \frac{\cot\theta_k}{2}(f_i - f_j)^2.$$

$\square$.

Finite Element Method

Lemma (Discrete Harmonic Energy)

Given a piecewise linear function $f : M \to \mathbb{R}$, then the harmonic energy of f is given by

$$E(f) = \frac{1}{2} \sum_{[v_i, v_j] \in M} w_{ij}(f(v_i) - f(v_j))^2.$$

$w_{ij} = \cot\theta_{ij}^k + \cot\theta_{ji}^l.$

Proof.

We add the harmonic energies on all faces together, and merge the items associated with the same edge, then each edge contributes $\frac{1}{2}w_{ij}(f_j - f_i)^2$. □