Fixed Point, Hopf-Poincarère Index Theorem, Characteristic Class

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$\omega(\rightarrow) = +1 \quad \omega(\leftarrow) = -1 \quad \omega(\Rightarrow) = -1$

**Figure:** $\gamma$ is the generator of $H_1(M, \mathbb{Z})$, $\omega$ is the generator of $H^1(M, \mathbb{R})$.

d$\omega = 0$ but $\int_\gamma \omega = 18$, so $\omega$ is closed but not exact.
Fixed Point
**Figure:** Brouwer fixed point.
Brouwer Fixed Point

**Theorem (Brouwer Fixed Point)**

Suppose $\Omega \subset \mathbb{R}^n$ is a compact convex set, $f : \Omega \to \Omega$ is a continuous map, then there exists a point $p \in \Omega$, such that $f(p) = p$.

**Proof.**

Assume $f : \Omega \to \Omega$ has no fixed point, namely $\forall p \in \Omega$, $f(p) \neq p$. We construct $g : \Omega \to \partial \Omega$, a ray starting from $f(p)$ through $p$ and intersect $\partial \Omega$ at $g(p)$, $g|_{\partial \Omega} = id$. $i$ is the inclusion map, $(g \circ i) : \partial \Omega \to \partial \Omega$ is the identity,

$$
\begin{align*}
\partial \Omega \xrightarrow{i} \Omega \xrightarrow{g} \partial \Omega
\end{align*}
$$

$(g \circ i)\# : H_{n-1}(\partial \Omega, \mathbb{Z}) \to H_{n-1}(\partial \Omega, \mathbb{Z})$ is $z \mapsto z$. But $H_{n-1}(\Omega, \mathbb{Z}) = 0$, then $g\# = 0$. Contradiction.
Definition (Index of Fixed Point)

Suppose $M$ is an $n$-dimensional topological space, $p$ is a fixed point of $f : M \to M$. Choose a neighborhood $p \in U \subset M$, $f_* : H_{n-1}(\partial U, \mathbb{Z}) \to H_{n-1}(\partial U, \mathbb{Z})$, $f_* : \mathbb{Z} \to \mathbb{Z}, z \mapsto \lambda z$,

where $\lambda$ is an integer, the algebraic index of $p$, $\text{Ind}(f, p) = \lambda$. 
Given a compact topological space $M$, and a continuous automorphism $f : M \rightarrow M$, it induces homomorphisms

$$f_* : H_k(M, \mathbb{Z}) \rightarrow H_k(M, \mathbb{Z}),$$

each $f_*k$ is represented as a matrix.

**Definition (Lefschetz Number)**

The Lefschetz number of the automorphism $f : M \rightarrow M$ is given by

$$\Lambda(f) := \sum_k (-1)^k \text{Tr}(f_*|_{H_k(M, \mathbb{Z})}).$$
Lefschetz Fixed Point

**Theorem (Lefschetz Fixed Point)**

Given a continuous automorphism of a compact topological space $f : M \to M$, if its Lefschetz number is non-zero, then there is a point $p \in M$, $f(p) = p$.

**Proof.**

Triangulate $M$, use a simplicial map to approximate $f$, then

$$
\sum_k (-1)^k Tr(f_k|C_k) = \sum_k (-1)^k Tr(f_k|H_k) = \Lambda(f).
$$

If $\Lambda(f) \neq 0$, $\exists \sigma \in C_k$, $f_k(\sigma) \subset \sigma$, from Brouwer fixed point theorem, there is a fixed point $p \in \sigma$.  

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Lefschetz Fixed Point

**Lemma**

\[
\sum_{k} (-1)^{k} Tr(f_k | C_k) = \sum_{k} (-1)^{k} Tr(f_k | H_k) = \Lambda(f).
\]

**Proof.**

\(C_k = C_k / Z_k \oplus Z_k\), \(Z_k\) is the closed chain space; \(Z_k = B_k \oplus H_k\), \(B_k\) is the exact chain space, \(H_k\) is the homology group. \(\partial_k : C_k / Z_k \to B_{k-1}\) is isomorphic.

\[
\begin{array}{ccc}
C_k / Z_k & \xrightarrow{f_k} & C_k / Z_k \\
\downarrow \partial_k & & \downarrow \partial_k \\
B_{k-1} & \xrightarrow{f_{k-1}} & B_{k-1}
\end{array}
\]
Lefschetz Fixed Point

Lemma

\[ \sum_k (-1)^k \text{Tr}(f_k|C_k) = \sum_k (-1)^k \text{Tr}(f_k|H_k) = \Lambda(f). \]

The left hand side depends on the triangulation, the right hand side is independent.

Proof.

\[ \partial_k \circ f_k \circ \partial_k^{-1} = f_{k-1}, \quad \text{Tr}(f_k|C_k/Z_k) = \text{Tr}(f_{k-1}|B_{k-1}), \]

\[ \text{Tr}(f_k|C_k) = \text{Tr}(f_k|C_k/Z_k) + \text{Tr}(f_k|Z_k) \]

\[ = \text{Tr}(f_{k-1}|B_{k-1}) + \text{Tr}(f_k|B_k) + \text{Tr}(f_k|H_k) \]
Lemma

Suppose $M$ is a compact oriented surface with genus $g$, $f : M \to M$ is a continuous automorphism of $M$, $f$ is homotopic to the identity map of $M$, then the Lefschetz number of $f$ equals to the Euler characteristic number of $M$,

$$\Gamma(f) = \chi(S).$$

Proof.

We construct a triangulation of $M$ and use a simplicial map to approximate the automorphism. Then

$$\Lambda(f) = \Lambda(Id) = |V| + |F| - |E| = \chi(S).$$
Poincaré-Hopf Theorem
Definition (Isolated Zero)

Given a smooth tangent vector field \( v : S \rightarrow TS \) on a smooth surface \( S \), \( p \in S \) is called a zero point, if \( v(p) = 0 \). If there is a neighborhood \( U(p) \), such that \( p \) is the unique zero in \( U(p) \), then \( p \) is an isolated zero point.

Figure: Isolated zero point.
Given a zero $p \in Z(\nu)$, choose a small disk $B(p, \varepsilon)$ define a map $\varphi : \partial B(p, \varepsilon) \to \mathbb{S}^1$, $q \mapsto \frac{\nu(q)}{|\nu(q)|}$. This map induces a homomorphism $\varphi_\# : \pi_1(\partial B) \to \pi_1(\mathbb{S}^1)$, $\varphi_\#(z) = kz$, where the integer $k$ is called the index of the zero.
**Zero Index**

*Figure*: Indices of zero points.

- **source** +1
- **saddle** −1
- **sink** +1
Theorem (Poincaré-Hopf Index)

Assume $S$ is a compact, oriented smooth surface, $v$ is a smooth tangent vector field with isolated zeros. If $S$ has boundaries, then $v$ point along the exterior normal direction, then we have

$$\sum_{p \in Z(v)} \text{Index}_p(v) = \chi(S),$$

where $Z(v)$ is the set of all zeros, $\chi(S)$ is the Euler characteristic number of $S$. 
Proof.

Given two vector fields $v_1$ and $v_2$ with different isolated zeros. We construct a triangulation $T$, such that each face contains at most one zero. Define two 2-forms, $\Omega_1$ and $\Omega_2$.

$$\Omega_k(\Delta) = \text{Index}_p(v_k), \quad p \in \Delta \cap Z(v_k), \quad k = 1, 2.$$  

Along $\gamma(t)$, $\theta(t)$ is the angle from $v_1 \circ \gamma(t)$ to $v_2 \circ \gamma(t)$. Define a one form,

$$\omega(\gamma) := \int_{\gamma} \dot{\theta}(\tau) d\tau.$$
Given a smooth tangent vector field $v$, we can define a one parameter family of automorphisms, $\varphi(p, t)$,

$$\frac{\partial \varphi(p, t)}{\partial t} = v \circ \varphi(p, t).$$

Then $f_t : p \mapsto \varphi(p, t)$ is an automorphism homotopic to the identity. According to lemma 7, the total index of fixed points of $f_t$ is $\chi(S)$. The fixed points of $f_t$ corresponds to the zeros of $v$ with the sample index.
continued.

Given a triangle $\Delta$, then the relative rotation of $v_2$ about $v_1$ is given by

$$\omega(\partial \Delta) = d\omega(\Delta)$$

then we get

$$\Omega_2 - \Omega_1 = d\omega.$$ 

Therefore $\Omega_1$ and $\Omega_2$ are cohomological. The total index of zeros of a vector field

$$\sum_{p \in v_k} \text{Index}_p(v_k) = \int_S \Omega_k$$
continued.

We construct a special vector field, then the total index of all the zeros is

\[ \sum_{p \in Z(v)} \text{Index}_p(v) = |V| + |F| - |E| = \chi(S). \]
Unit Tangent Bundle of the Sphere
Smooth Manifold

Figure: A manifold.
Definition (Manifold)

A manifold is a topological space $M$ covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps $U_\alpha$ to the Euclidean space $\mathbb{R}^n$. $(U_\alpha, \phi_\alpha)$ is called a coordinate chart of $M$. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of $M$. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.
Definition (Tangent Vector)

A tangent vector $\xi$ at the point $p$ is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at $p$ an $n$-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p) \xi^j.$$ 

A smooth vector field $\xi$ assigns a tangent vector for each point of $M$, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n) \frac{\partial}{\partial x^i}.$$ 

$\{\frac{\partial}{\partial x^i}\}$ represents the vector fields of the velocities of iso-parametric curves on $M$. They form a basis of all vector fields.
Definition (Push-forward)

Suppose $\phi : M \rightarrow N$ is a differential map from $M$ to $N$, $\gamma : (-\epsilon, \epsilon) \rightarrow M$ is a curve, $\gamma(0) = p$, $\gamma'(0) = v \in T_pM$, then $\phi \circ \gamma$ is a curve on $N$, $\phi \circ \gamma(0) = \phi(p)$, we define the tangent vector

$$\phi_*(v) = (\phi \circ \gamma)'(0) \in T_{\phi(p)}N,$$

as the push-forward tangent vector of $v$ induced by $\phi$. 
The unit tangent bundle of the unit sphere is the manifold

\[ UTM(S) := \{(p, v) | p \in S, v \in T_p(S), |v|_g = 1 \} . \]

The unit tangent bundle of a surface is a 3-dimensional manifold. We want to compute its triangulation and its fundamental group.
Sphere

Figure: Stereo-graphic projection

\[
(x, y) = \left( \frac{x_1}{1 - x_3}, \frac{x_2}{1 - x_3} \right)
\]

\[
r(x, y) = (x_1, x_2, x_3) = \left( \frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{-1 + x^2 + y^2}{1 + x^2 + y^2} \right)
\]
Sphere

\[ r_x = \partial_x = \frac{2}{(1 + x^2 + y^2)^2} (1 - x^2 + y^2, -2xy, 2x) \]

\[ r_y = \partial_y = \frac{2}{(1 + x^2 + y^2)^2} (-2xy, 1 + x^2 - y^2, 2y) \]

\[ \langle \partial_x, \partial_x \rangle = \frac{4}{(1 + x^2 + y^2)^2} \]

\[ \langle \partial_y, \partial_y \rangle = \frac{4}{(1 + x^2 + y^2)^2} \]

\[ \langle \partial_x, \partial_y \rangle = 0 \]
A tangent vector at $r(x, y)$ is given by: $dr(x, y) = r_x(x, y)dx + r_y(x, y)dy$. On the equator

$$((x, y), (dx, dy)) = ((\cos \theta, \sin \theta), (\cos \tau, \sin \tau)).$$
The unit tangent bundle of a hemisphere is a direct product $\mathbb{S}^1 \times \mathbb{D}^2$, where $\mathbb{S}^1$ is the fiber of each point, $\mathbb{D}^2$ is the hemisphere. The boundary of the UTM of the hemisphere is a torus $\mathbb{S}^1 \times \partial \mathbb{D}^2$. 

Figure: Unit tangent bundle.
Sphere

\[
(u, \nu) = \left( \frac{x_1}{1 + x_3}, \frac{-x_2}{1 + x_3} \right)
\]

\[
r(u, \nu) = (x_1, x_2, x_3) = \left( \frac{2u}{1 + u^2 + \nu^2}, \frac{-2\nu}{1 + u^2 + \nu^2}, \frac{1 - u^2 - \nu^2}{1 + u^2 + \nu^2} \right)
\]

\[
r_u = \partial_u = \frac{2}{(1 + u^2 + \nu^2)^2} (1 - u^2 + \nu^2, 2uv, -2u)
\]

\[
r_v = \partial_v = \frac{2}{(1 + u^2 + \nu^2)^2} (-2uv, -1 - u^2 + \nu^2, -2\nu)
\]

\[
\left\langle \partial_u, \partial_u \right\rangle = \frac{4}{(1 + u^2 + \nu^2)^2}
\]

\[
\left\langle \partial_v, \partial_v \right\rangle = \frac{4}{(1 + u^2 + \nu^2)^2}
\]

\[
\left\langle \partial_u, \partial_v \right\rangle = 0
\]
Let \( z = x + iy \) and \( w = u + iv \), Then

\[
\frac{1}{z} = \frac{x - iy}{x^2 + y^2} = \frac{x_1 - ix_2}{1 - x_3} \cdot \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{x_1 - ix_2}{1 + x_3} = w.
\]

Therefore \( dw = -\frac{1}{z^2} \, dz \),

\[
\begin{bmatrix}
du \\
dv
\end{bmatrix} =
\begin{bmatrix}
u_x & u_y \\
v_x & v_y
\end{bmatrix}
\begin{bmatrix}
dx \\
dy
\end{bmatrix}
\]

this gives the Jacobi matrix,

\[
\begin{bmatrix}
u_x & u_y \\
v_x & v_y
\end{bmatrix} = \frac{1}{(x^2 + y^2)^2}
\begin{bmatrix}
y^2 - x^2 & -2xy \\
2xy & y^2 - x^2
\end{bmatrix}
\]
Construct the unit tangent bundle of the sphere. The unit tangent bundle of the upper hemisphere is a solid torus, the unit tangent bundle of the lower hemisphere is also a solid torus. The unit tangent bundle of the equator is a torus, $\varphi: (z, dz) \mapsto (w, dw)$, $z = e^{i\theta}$, $dz = e^{i\tau}$,

$$\varphi: (z, dz) \mapsto \left(\frac{1}{z}, -\frac{1}{z^2} dz\right), (\theta, \tau) \mapsto (-\theta, \pi - 2\theta + \tau)$$
Automorphism of the Torus

\[ \varphi : (\tau, \theta) \mapsto (\tau - 2\theta + \pi, -\theta) \]

<table>
<thead>
<tr>
<th>(\varphi)</th>
<th>((\tau, \theta))</th>
<th>((\tau', \theta'))</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((0, 0))</td>
<td>((\pi, 0))</td>
</tr>
<tr>
<td>B</td>
<td>((2\pi, 0))</td>
<td>((3\pi, 0))</td>
</tr>
<tr>
<td>C</td>
<td>((2\pi, 2\pi))</td>
<td>((-\pi, -2\pi))</td>
</tr>
<tr>
<td>D</td>
<td>((0, 2\pi))</td>
<td>((-3\pi, -2\pi))</td>
</tr>
</tbody>
</table>

**Table:** Corresponding corner points.
This induces an automorphism of the fundamental group of the torus,
\[ \varphi^\#: \pi_1(T^2) \rightarrow \pi_1(T^2), \]
\[ \varphi^\#: a \mapsto a, \quad b \mapsto a^{-2}b^{-1}. \]
This induces an automorphism of the fundamental group of the torus,

$$\phi : \pi_1(T^2) \to \pi_1(T^2),$$

$$\phi : a \mapsto a, \quad b \mapsto a^{-2}b^{-1}.$$
Torus Automorphism on UCS

\[ \varphi(a) = a \]

\[ \varphi(b) = a^{-2}b^{-1} \]

\[ \pi_1(M_1) = \langle a_1 \rangle, \quad \pi_1(M_2) = \langle a_2 \rangle, \quad M_1 \cap M_2 = T^2, \quad \pi_1(T^2) = \langle a, b | [a, b] \rangle, \]

then the \( \pi_1 \) of the unit tangent bundle is

\[ \pi_1(M_1 \cup M_2) = \langle a_1, a_2 | a_1a_2, a_2^{-2}b_2^{-1} \rangle = \mathbb{Z}_2. \]
Obstruction Class

Figure: Local obstruction.
The topological obstruction for the existence of global section $\varphi : S^2 \to UTM(S^2)$ is constructed as follows:

1. Construct a triangulation $\mathcal{T}$, which is refined enough such that the fiber bundle of each face is trivial (direct product).
2. For each vertex $v_i$, choose a point on its fiber, $\varphi(v_i) \in F(v_i)$.
3. For each edge $[v_i, v_j]$, choose a curve connecting $\varphi(v_i)$ and $\varphi(v_j)$ in the restriction of the UTM on $[v_i, v_j]$, which is an annulus;
4. For each face $\Delta$, $\varphi(\partial \Delta)$ is a loop in the fiber bundle of $\Delta$, $[\varphi(\partial \Delta)]$ is an integer, an element in $\pi_1(UTM(\Delta))$, this gives a 2-form $\Omega$ on the original surface $M$,
   \[ \Omega(\Delta) = [\varphi(\partial \Delta)]. \]
5. If $\Omega$ is zero, then global section exists. Otherwise doesn’t exists.
6. Different constructions get different $\Omega$’s, but all of them are cohomological. Therefore $[\Omega] \in H^2(M, \mathbb{R})$ is the characteristic class of fiber bundle.
Lemma

Given two sections $\varphi, \bar{\varphi} : \mathbb{S} \to UTM(S)$, they induce two 2-forms $\Omega_2, \bar{\Omega}_2$. Then there exists a 1-form $h$, such that

$$\forall \sigma^2, \hspace{1em} \delta h(\sigma^2) = \Omega^2(\sigma^2) - \bar{\Omega}^2(\sigma^2).$$

Proof.

$\forall \sigma^0_a \in B^{(0)}$, construct a path in the fiber $p_a : [0, 1] \to F$, such that

$$p_a(0) = \bar{\varphi}(\sigma^0_a), \hspace{1em} p_a(1) = \varphi(\sigma^0_a)$$

Given a 1-simplex $\sigma^1_a$, with boundary $\partial \sigma^1_a = \sigma^0_j - \sigma^0_i$, construct a loop

$$l_a = p_i \varphi(\sigma^1_a) p_j^{-1} \bar{\varphi}(\sigma^1_a)^{-1}.$$
Figure: Denote $a = \varphi(\sigma_a^1)$, $b = \varphi(\sigma_b^1)$ and $c = \varphi(\sigma_c^1)$.

\[
\begin{align*}
    l_a &:= p_i \varphi(\sigma_a^1)p_j^{-1} \bar{\varphi}(\sigma_a^1)^{-1} = p_i a p_j^{-1} \bar{a}^{-1} \\
    l_b &:= p_j b p_k^{-1} \bar{b}^{-1} \sim \bar{a} p_j b p_k^{-1} \bar{b}^{-1} \bar{a}^{-1} \\
    l_c &:= p_k c p_i^{-1} \bar{c}^{-1} \sim \bar{a} b p_k c p_i^{-1} \bar{c}^{-1} \bar{b}^{-1} \bar{a}^{-1}
\end{align*}
\]
\[(l_a)[l_b][l_c] = (iaj^{-1}\bar{a}^{-1})(\bar{a}jbk^{-1}\bar{b}^{-1}\bar{a}^{-1})(\bar{a}\bar{b}kci^{-1}\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1})
\]
\[= iaj^{-1}jbk^{-1}kci^{-1}\bar{c}^{-1}\bar{b}^{-1}\bar{z}^{-1}
\]
\[= (iabc i^{-1})(\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1})
\]

Then

\[\delta h(\sigma^2) = [l_a][l_b][l_c]
\]
\[= [iabc i^{-1}][\bar{c}^{-1}\bar{b}^{-1}\bar{a}^{-1}]
\]
\[= [abc][(\bar{a}\bar{b}\bar{c})]^{-1}
\]
\[= C_2(\sigma^2)(\bar{C}(\sigma^2))^{-1}
\]