Discrete Euclidean Curvature Flow

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A surface $\Sigma$ with a Riemannian metric $g$, a local coordinate system $(u, v)$ is an isothermal coordinate system, if

$$g = e^{2\lambda(u,v)}(du^2 + dv^2).$$
Under the isothermal coordinates, the Riemannian metric is 
\[ g = e^{2\lambda(u,v)}(du^2 + dv^2) \], then the Gaussian curvature on interior points are 
\[ K = -\Delta_g \lambda = -\frac{1}{e^{2\lambda}} \Delta \lambda, \]
where 
\[ \Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \]
Conformal Metric Deformation

**Definition**

Suppose $\Sigma$ is a surface with a Riemannian metric, $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$.

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}g$ is also a Riemannian metric on $\Sigma$ and called a **conformal metric**. $\lambda$ is called the conformal factor.

$g \rightarrow e^{2\lambda}g$

Conformal metric deformation.

Angles are invariant measured by conformal metrics.
Suppose $\bar{g} = e^{2\lambda}g$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_g \lambda + K),$$

godesic curvature on the boundary

$$\bar{k}_g = e^{-\lambda}(-\partial_n \lambda + k_g).$$
Uniformization

Theorem (Poincaré Uniformization Theorem) Let $(\Sigma, g)$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{g} = e^{2\lambda} g$ conformal to $g$ which has constant Gauss curvature.
Figure: Closed surface uniformization.
Surface Uniformization

**Figure:** Open surface uniformization.
Surface Ricci Flow

Proposition

During the curvature flow \( \frac{d\lambda}{dt} = -K \), then

\[
\frac{d}{dt} K = 2K^2 + \Delta_g K.
\]

\[
\begin{align*}
\frac{d}{dt} K &= \frac{d}{dt} \left( -e^{-2\lambda} \Delta \lambda \right) \\
&= - \left( -2 \frac{d\lambda}{dt} \right) e^{-2\lambda} \Delta \lambda - e^{-2\lambda} \Delta \frac{d\lambda}{dt} \\
&= \left( -2 \frac{d\lambda}{dt} \right) \left[ -e^{-2\lambda} \Delta \lambda \right] - \left[ e^{-2\lambda} \Delta \right] \frac{d\lambda}{dt} \\
&= \left( -2 \frac{d\lambda}{dt} \right) K - \Delta_g \frac{d\lambda}{dt} \\
&= 2K^2 + \Delta_g K
\end{align*}
\]
Surface Ricci Flow

**Key Idea**

\[ K = -\Delta g \lambda, \]

Roughly speaking,

\[ \frac{dK}{dt} = \frac{d}{dt} \Delta g \lambda \]

Let \[ \frac{d\lambda}{dt} = -K, \]

\[ \frac{dK}{dt} = \Delta g K + 2K^2 \]

Diffusion and reaction equation!
Surface Ricci Flow

**Definition (Hamilton’s Surface Ricci Flow)**

A closed surface with a Riemannian metric $g$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$ 

The normalized surface Ricci flow,

$$\frac{dg_{ij}}{dt} = \frac{2\pi\chi(S)}{A(0)} - 2Kg_{ij},$$

where $A(0)$ is the initial surface area.

The normalized surface Ricci flow is area-preserving, the Ricci flow will converge to a metric such that the Gaussian curvature is constant \(\frac{2\pi\chi(S)}{A(0)}\) everywhere.
Ricci Flow

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) everywhere.
Surface Ricci Flow

- Conformal metric deformation

\[ g \rightarrow e^{2u} g \]

- Curvature Change - heat diffusion

\[ \frac{dK}{dt} = \Delta_g K + 2K^2 \]

- Ricci flow

\[ \frac{du}{dt} = \bar{K} - K. \]
- Surfaces are represented as polyhedron triangular meshes.
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- Isometric gluing of triangles in $\mathbb{F}^2$. 

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Surfaces are represented as polyhedron triangular meshes.

- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$. 
Discrete Generalization

Concepts

1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation
A Discrete Metric on a triangular mesh is a function defined on the vertices, \( l : E = \{ all \ edges \} \rightarrow \mathbb{R}^+ \), satisfies triangular inequality.

A mesh has infinite metrics.
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{ \text{vertices} \} \to \mathbb{R}$. 

$$K(v_i) = 2\pi - \sum_{jk} \theta_{jk}^i, \ v_i \notin \partial M; \ K(v_i) = \pi - \sum_{jk} \theta_{jk}, \ v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi \chi(M).$$
Discrete Metrics Determines the Curvatures

\[ \theta_i \theta_j \theta_k \theta_k \theta_k \theta_i \theta_j \theta_j \]

- \( E^2 \)
- \( H^2 \)
- \( S^2 \)

**cosine laws**

\[
\begin{align*}
\cos l_i &= \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} & S^2 \\
\cosh l_i &= \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} & H^2 \\
1 &= \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} & E^2
\end{align*}
\]
Discrete Conformal Metric Deformation

Conformal maps Properties

- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.
Discrete Conformal Metric Deformation vs CP
Circle Packing Metric

**CP Metric**

We associate each vertex $v_i$ with a circle with radius $\gamma_i$. On edge $e_{ij}$, the two circles intersect at the angle of $\Phi_{ij}$. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos \phi_{ij}$$

**CP Metric** $(\Sigma, \Gamma, \Phi)$, $\Sigma$ triangulation,

$$\Gamma = \{\gamma_i|\forall v_i\}, \Phi = \{\phi_{ij}|\forall e_{ij}\}$$

![Diagram of Circle Packing Metric](image)
Discrete Conformal Factor

Conformal Factor

Defined on each vertex $u : V \rightarrow \mathbb{R}$,

$$u_i = \begin{cases} 
\log \gamma_i & \mathbb{R}^2 \\
\log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\
\log \tan \frac{\gamma_i}{2} & \mathbb{S}^2
\end{cases}$$

Properties

- Symmetry
  $$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation
  $$dK = \Delta du,$$

$\Delta$ is a discrete Laplace-Beltrami operator.
Unified Framework of Discrete Curvature Flow

Analogy

- Curvature flow
  \[ \frac{du}{dt} = \bar{K} - K, \]

- Energy
  \[ E(u) = \int \sum_i (\bar{K}_i - K_i) du_i, \]

- Hessian of \( E \) denoted as \( \Delta \),
  \[ dK = \Delta du. \]
Criteria for Discretization

Key Points
- Convexity of the energy $E(u)$
- Convexity of the metric space ($u$-space)
- Admissible curvature space ($K$-space)
- Preserving or reflecting richer structures
- Conformality
Derivative Cosine law

\[ \theta_i(l_i, l_j, l_k) : \mathbb{R}^3_{>0} \rightarrow (0, \pi). \]

\[ \frac{\partial}{\partial l_i} \left(2l_j l_k \cos \theta_i \right) = \frac{\partial}{\partial l_i} \left(l_j^2 + l_k^2 - l_i^2 \right) \]

\[ -2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_i} = -2l_i \]

\[ \frac{d\theta_i}{dl_i} = \frac{l_i}{A} \]
Derivative Cosine law

\[ l_j = l_i \cos \theta_k + l_k \cos \theta_i \]

\[
\frac{\partial}{\partial l_j} \left( 2l_j l_k \cos \theta_i \right) = \frac{\partial}{\partial l_j} \left( l_j^2 + l_k^2 - l_i^2 \right)
\]

\[
2l_j = 2l_k \cos \theta_i - 2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_j}
\]

\[
\frac{d\theta_i}{dl_j} = \frac{l_k \cos \theta_i - l_j}{A}
\]

\[
= -\frac{l_i \cos \theta_k}{A}
\]

\[
= -\frac{d\theta_i}{dl_i} \cos \theta_k
\]
Derivative Cosine law

\[ l_k^2 = r_i^2 + r_j^2 + 2 \cos \tau_{ij} r_i r_j \]

\[
\frac{\partial}{\partial r_j} l_i^2 = \frac{\partial}{\partial r_j} \left( r_j^2 + r_k^2 + 2 r_j r_k \cos \tau_{jk} \right)
\]

\[
2l_i \frac{dl_i}{dr_j} = 2r_j + 2r_k \cos \tau_{jk}
\]

\[
\frac{dl_i}{dr_j} = \frac{2r_j^2 + 2r_j r_k \cos \tau_{jk}}{2l_i r_j}
\]

\[
= \frac{r_j^2 + r_k^2 + 2r_j r_k \cos \tau_{jk} + r_j^2 - r_k^2}{2l_i r_j}
\]

\[
= \frac{l_i^2 + r_j^2 - r_k^2}{2l_i r_j}
\]
Derivative Cosine law

Let $u_i = \log r_i$, then $\frac{d\theta}{du} = \frac{d\theta}{dl} \frac{dl}{dr} \frac{dr}{du}$, the Jacobian of $(\theta_i, \theta_j, \theta_k)(u_i, u_j, u_k)$ is

$$
\begin{pmatrix}
    d\theta_1 \\
    d\theta_2 \\
    d\theta_3
\end{pmatrix}
= -\frac{1}{A}
\begin{pmatrix}
    l_1 & 0 & 0 \\
    0 & l_2 & 0 \\
    0 & 0 & l_3
\end{pmatrix}
\begin{pmatrix}
    -1 & \cos \theta_3 & \cos \theta_2 \\
    \cos \theta_3 & -1 & \cos \theta_1 \\
    \cos \theta_2 & \cos \theta_1 & -1
\end{pmatrix}
\begin{pmatrix}
    \frac{l_1^2 + r_2^2 - r_3^2}{2l_1 r_2} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1 r_3} \\
    \frac{l_2^2 + r_1^2 - r_3^2}{2l_2 r_1} & \frac{l_2^2 + r_3^2 - r_1^2}{2l_2 r_3} \\
    \frac{l_3^2 + r_1^2 - r_2^2}{2l_3 r_1} & \frac{l_3^2 + r_2^2 - r_1^2}{2l_3 r_2}
\end{pmatrix}
\begin{pmatrix}
    r_1 & 0 & 0 \\
    0 & r_2 & 0 \\
    0 & 0 & r_3
\end{pmatrix}
\begin{pmatrix}
    du_1 \\
    du_2 \\
    du_3
\end{pmatrix}
$$
Derivative Cosine law

\[ l_k^2 = r_i^2 + r_j^2 + 2 \cos \tau_{ij} r_i r_j \]

\[
2 l_k \frac{dl_k}{dr_j} = 2 r_j + 2 r_i \cos \tau_{ij}
\]
\[
r_j \frac{dl_k}{dr_j} = \frac{2 r_j^2 + 2 r_i r_j \cos \tau_{ij}}{2 l_k}
\]
\[
= \frac{r_j^2 + r_i^2 + 2 r_i r_j \cos \tau_{ij} + r_j^2 - r_i^2}{2 l_k}
\]
\[
= \frac{l_k^2 + r_j^2 - r_i^2}{2 l_k}
\]

In triangle \([v_i, v_j, w_k],\)

\[
\frac{dl_k}{du_j} = 2 \frac{l_k r_j \cos \phi_{ji}}{2 l_k} = r_j \cos \phi_{ji} = d_{ji}
\]
There is a unique circle orthogonal to three circles \((v_i, r_i)\), the center is \(o\), the distance from \(o\) to edge \([v_i, v_j]\) is \(h_k\).

Theorem (Derivative Cosine Law)

\[
\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k} \\
\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i} \\
\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}
\]
Proof.

\[
\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_i}{\partial l_i} \frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k} \frac{\partial l_k}{\partial u_j} = \frac{\partial \theta_i}{\partial l_i} \left( \frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j} \cos \theta_j \right) = \frac{l_i}{A} (d_{jk} - d_{ji} \cos \theta_j) = \frac{dl_i}{l_i l_k \sin \theta_j} = \frac{h_k \sin \theta_j}{l_k \sin \theta_j} = \frac{h_k}{l_k}
\]
Derivative Cosine law

\[
\frac{\partial \langle v_j - v_i, v_j - v_i \rangle}{\partial u_j} = 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \rangle
\]

\[
\frac{\partial l_k^2}{\partial u_j} = 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \rangle
\]

\[
\frac{\partial l_k}{\partial u_j} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \rangle
\]

\[
d_{ji} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \rangle
\]

Similarly

\[
d_{jk} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_k}{l_i} \rangle
\]

So \( \frac{\partial v_j}{\partial u_j} = v_j - 0. \)
**Lemma**

For any three non-obtuse angles $\tau_{ij}, \tau_{jk}, \tau_{ki} \in [0, \frac{\pi}{2})$ and any three positive numbers $r_1, r_2$ and $r_3$, there is a configuration of 3 circles in Euclidean geometry, unique up to isometry, having radii $r_i$ and meeting in angles $\tau_{ij}$.

**Proof.**

\[
\max\{r_i^2, r_j^2\} < r_i^2 + r_j^2 + 2r_ir_j \cos \tau_{ij} \leq (r_i + r_j)^2
\]

\[
\max\{r_i^2, r_j^2\} < l_k \leq r_i + r_j
\]

so

\[
l_k \leq r_i + r_j < l_i + l_j.
\]
Lemma

ω is closed 1-form in Ω := \{(u_1, u_2, u_3) ∈ \mathbb{R}^3\}.

Because \( \frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} \), so

\[
d\omega = (\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_j}{\partial u_i}) \, du_j \wedge du_i + (\frac{\partial \theta_j}{\partial u_k} - \frac{\partial \theta_k}{\partial u_j}) \, du_k \wedge du_j + (\frac{\partial \theta_k}{\partial u_i} - \frac{\partial \theta_i}{\partial u_k}) \, du_i \wedge du_k
\]

= 0.
Lemma

The Ricci energy $E(u_1, u_2, u_3)$ is well defined.

Because $\Omega = \mathbb{R}^3$ is convex, closed 1-form is exact, therefore $E(u_1, u_2, u_3)$ is well defined,

$$E(u_1, u_2, u_3) = \int_{(0,0,0)}^{(u_1,u_2,u_3)} \omega.$$
Lemma

The Ricci energy \( E(u_1, u_2, u_3) \) is strictly concave on the subspace \( u_1 + u_2 + u_3 = 0 \).

The gradient \( \nabla E = (\theta_1, \theta_2, \theta_3) \), the Hessian matrix is

\[
H = \begin{pmatrix}
\frac{\partial \theta_1}{\partial u_1} & \frac{\partial \theta_1}{\partial u_2} & \frac{\partial \theta_1}{\partial u_3} \\
\frac{\partial \theta_2}{\partial u_1} & \frac{\partial \theta_2}{\partial u_2} & \frac{\partial \theta_2}{\partial u_3} \\
\frac{\partial \theta_3}{\partial u_1} & \frac{\partial \theta_3}{\partial u_2} & \frac{\partial \theta_3}{\partial u_3}
\end{pmatrix}
\]

because of \( \theta_1 + \theta_2 + \theta_3 = \pi \),

\[
\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_i}{\partial u_k} = -\frac{\partial \theta_j}{\partial u_i} - \frac{\partial \theta_k}{\partial u_i}
\]
Ricci energy

Proof.

\[ H = - \begin{pmatrix} \frac{h_3}{l_3} + \frac{h_2}{l_2} & -\frac{h_3}{l_3} & -\frac{h_2}{l_2} \\ -\frac{h_3}{l_3} & \frac{h_3}{l_3} + \frac{h_1}{l_1} & -\frac{h_1}{l_1} \\ -\frac{h_2}{l_2} & -\frac{h_1}{l_1} & h_2 + \frac{h_1}{l_1} \end{pmatrix} \]

\(-H\) is diagonal dominant, it has null space \((1, 1, 1)\), on the subspace \(u_1 + u_2 + u_3 = 0\), it is strictly negative definite. Therefore the discrete Ricci energy \(E(u_1, u_2, u_3)\) is strictly concave.
Lemma

The Ricci energy $E(u)$ is strictly convex on the subspace $\sum_{v_i \in M} u_i = 0$.

The gradient $\nabla E = (K_1, K_2, \cdots, K_n)$. The Ricci energy

$$E(u) = 2\pi \sum_{v_i \in M} u_i - \sum_{[v_i, v_j, v_k] \in M} E_{ijk}(u_i, u_j, u_k)$$

where $E_{ijk}$ is the Ricci energy defined on the face $[v_i, v_j, v_k]$. The linear term won’t affect the convexity of the energy. The null space of the Hessian is $(1, 1, \cdots, 1)$. In the subspace $\sum u_i = 0$, the energy is strictly convex.
Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \rightarrow \mathbb{R}$ is a strictly convex function, then the map

$$x \rightarrow \nabla f(x)$$

is one-to-one.

Proof.

Suppose $x_1 \neq x_2$, $\nabla f(x_1) = \nabla f(x_2)$. Because $\Omega$ is convex, the line segment $(1 - t)x_1 + tx_2$ is contained in $\Omega$. Construct a convex function $g(t) = f((1 - t)x_1 + tx_2)$, then $g'(t)$ is monotonous. But

$$g'(0) = \langle \nabla f(x_1), x_2 - x_1 \rangle = \langle \nabla f(x_2), x_2 - x_1 \rangle = g'(1),$$

contradiction.
Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \to \mathbb{R}$ is a strictly convex function, then the map

$$x \to \nabla f(x)$$

is one-to-one.
Uniqueness

**Theorem (Global Rigidity)**

Suppose $M$ is a mesh, with circle packing metric, all edge intersection angles are non-obtuse. Given the target curvature $(K_1, K_2, \cdots, K_n)$, $\sum_i K_i = 2\pi \chi(M)$. If the solution $(u_1, u_2, \cdots, u_n) \in \Omega(M), \sum_i u_i = 0$ exists, then it is unique.

**Proof.**

The discrete Ricci energy $E$ on $\Omega \cap \{ \sum_i u_i = 0 \}$ is convex,

$$\nabla E(u_1, u_2, \cdots, u_n) = (K_1, K_2, \cdots K_n).$$

Use previous lemma.
Theorem (Thurston)

Suppose \((T, \Phi)\) is a weighted generalized triangulation of a closed surface \(M\) and \(I\) is a proper subset of vertices of \(V\), here the weight is a map \(\Phi : E \to [0, \frac{\pi}{2})\). Then for any circle packing metric based on \((T, \Phi)\), we have

\[
\sum_{i \in I} K_i(u) > - \sum_{(e,v) \in Lk(I)} (\pi - \Phi(e)) + 2\pi \chi(F_I),
\]

where \(F_I\) is the CW-subcomplex of cells whose vertices are in \(I\) and

\[Lk(I) = \{(e, v)| v \in I, e \cap I = \emptyset, (e, v) \text{ form a triangle}\}\]