Fundamental Group and Covering Space

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July 3, 2022
Algebraic Topology: Fundamental Group
Figure: Escher. Ants
Surface Genus

Figure: How to differentiate the above two surfaces.

Topological Sphere  Topological Torus
Key Idea

Figure: Check whether all loops on the surface can shrink to a point.

All oriented compact surfaces can be classified by their genus $g$ and number of boundaries $b$. Therefore, we use $(g, b)$ to represent the topological type of an oriented surface $S$. 
Figure: Handle detection by finding the handle loops and the tunnel loops.
**Figure:** Topological Denoise in medical imaging.
Philosophy

Associate groups with manifolds, study the topology by analyzing the group structures.

\[ \mathcal{C}_1 = \{\text{Topological Spaces, Homeomorphisms}\} \]
\[ \mathcal{C}_2 = \{\text{Groups, Homomorphisms}\} \]
\[ \mathcal{C}_1 \rightarrow \mathcal{C}_2 \]

Functor between categories.
Suppose $q$ is a base point, all the oriented closed curves (loops) through $q$ can be classified by homotopy. All the homotopy classes form the so-called fundamental group of $S$, or the first homotopy group, denoted as $\pi_1(S, q)$. The group structure of $\pi_1(S, q)$ determines the topology of $S$. 
Figure: Path homotopy.
Let $S$ be a two manifold with a base point $p \in S$,

**Definition (Curve)**

A curve is a continuous mapping $\gamma : [0, 1] \to S$.

**Definition (Loop)**

A closed curve through $p$ is a curve, such that $\gamma(0) = \gamma(1) = p$.

**Definition (Homotopy)**

Let $\gamma_1, \gamma_2 : [0, 1] \to S$ be two curves. A homotopy connecting $\gamma_1$ and $\gamma_2$ is a continuous mapping $F : [0, 1] \times [0, 1] \to S$, such that

$$f(0, t) = \gamma_1(t), f(1, t) = \gamma_2(t).$$

We say $\gamma_1$ is homotopic to $\gamma_2$ if there exists a homotopy between them.
Lemma

Homotopy relation is an equivalence relation.

Proof.

$\gamma \sim \gamma, F(s, t) = \gamma(t)$. If $\gamma_1 \sim \gamma_2, F(s, t)$ is the homotopy, then $F(1 - s, t)$ is the homotopy from $\gamma_2$ to $\gamma_1$.

Corollary

All the loops through the base point can be classified by homotopy relation. The homotopy class of a loops $\gamma$ is denoted as $[\gamma]$. 
Fundamental Group

Definition (Loop product)

Suppose $\gamma_1, \gamma_2$ are two loops through the base point $p$, the product of the two loops is defined as

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} 
\gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\
\gamma_2(2t - 1) & \frac{1}{2} \leq t \leq t
\end{cases}$$

Definition (Loop inverse)

$$\gamma^{-1}(t) = \gamma(1 - t).$$
Loop Inversion

Figure: Loop inversion
Figure: Loop product
Definition (Fundamental Group)

Given a topological space $S$, fix a base point $p \in S$, the set of all the loops through $p$ is $\Gamma$, the set of all the homotopy classes is $\Gamma/\sim$. The product is defined as:

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2],$$

the unit element is defined as $[e]$, the inverse element is defined as

$$[\gamma]^{-1} := [\gamma^{-1}],$$

then $\Gamma/\sim$ forms a group, the fundamental group of $S$, and is denoted as $\pi_1(S, p)$. 
Let $G = \{g_1, g_2, \cdots, g_n\}$ be $n$ symbols, a word generated by $G$ is a sequence

$$w = g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_k}^{e_k}, g_{i_j} \in G, e_j \in \mathbb{Z}.$$ 

- The empty word $\emptyset$ is also treated as the unit element.
- Given two words $w_1 = \alpha_1 \cdots \alpha_{n_1}$ and $w_2 = \beta_1 \cdots \beta_{n_2}$, the product is defined as concatenation:

$$w_1 \cdot w_2 = \alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2}.$$ 

- The inverse of a work is defined as

$$(g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_k}^{e_k})^{-1} = g_{i_k}^{-e_k} g_{i_{k-1}}^{-e_{k-1}} \cdots g_{i_1}^{-e_1}.$$ 

All words form a group, freely generated by $G$,

$$\langle g_1, g_2, \cdots, g_n \rangle.$$
The relations \( R = \{R_1, R_2, \cdots, R_m\} \) are \( m \) words, such that we can replace \( R_k \) by the empty word.

**Definition (word equivalence relation)**

Two words are equivalent if we can transform one to the other by finite many steps of the following two elementary transformations:

1. Insert a relation word anywhere.

\[
\alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_l \mapsto \alpha_1 \cdots \alpha_i R_k \alpha_{i+1} \cdots \alpha_l
\]

2. If a subword is a relation word, remove it from the word.

\[
\alpha_1 \cdots \alpha_i R_k \alpha_{i+1} \cdots \alpha_l \mapsto \alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_l.
\]
Definition (Word Group)

Given a set of generators $G$ and a set of relations $R$, all the equivalence classes of the words generated by $G$ form a group under the concatenation, denoted as

$$\langle g_1, g_2, \cdots, g_n | R_1, R_2, \cdots, R_m \rangle.$$ 

If there is no relations, then the word group is called a free group.
Definition (Intersection Index)

Suppose $\gamma_1(t), \gamma_2(\tau) \subset S$ intersect at $q \in S$, the tangent vectors satisfy

$$\frac{d\gamma_1(t)}{dt} \times \frac{d\gamma_2(\tau)}{d\tau} \cdot n(q) > 0,$$

then the index of the intersection point $q$ of $\gamma_1$ and $\gamma_2$ is +1, denoted as $\text{Ind}(\gamma_1, \gamma_2, q) = +1$. If the mixed product is zero or negative, then the index is 0 or −1.
Definition (Algebraic Intersection Number)

The algebraic intersection number of $\gamma_1(t), \gamma_2(\tau) \subset S$ is defined as

$$\gamma_1 \cdot \gamma_2 := \sum_{q_i \in \gamma_1 \cap \gamma_2} \text{Ind}(\gamma_1, \gamma_2, q_i).$$
Algebraic Intersection Number Homotopy Invariance

Suppose $\gamma_1$ is homotopic to $\tilde{\gamma}_1$, then the algebraic intersection number

$$\gamma_1 \cdot \gamma_2 = \tilde{\gamma}_1 \cdot \gamma_2.$$
Figure: Canonical fundamental group representation.
Canonical Representation of $\pi_1(S, p)$

### Definition (Canonical Basis)

Suppose $S$ is a compact, oriented surface, there exists a set of generators of the fundamental group $\pi_1(S, p)$,

$$G = \{ [a_1], [b_1], [a_2], [b_2], \cdots, [a_g], [b_g] \}$$

such that

$$a_i \cdot b_j = \delta_{ij}, \quad a_i \cdot a_j = 0, \quad b_i \cdot b_j = 0,$$

where $a_i \cdot b_j$ represents the algebraic intersection number of loops $a_i$ and $b_j$, $\delta_{ij}$ is the Kronecker symbol, then $G$ is called a set of canonical basis of $\pi_1(S, p)$. 

Theorem (Surface Fundamental Group Canonical Representation)

Suppose $S$ is a compact, oriented surface, $p \in S$ is a fixed point, the fundamental group has a canonical representation,

$$\pi_1(S, p) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] \rangle,$$

where

$$[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1},$$

$g$ is the genus of the surface.
Non-uniqueness

The canonical representation of the fundamental group of the surface is not unique. It is NP hard to verify if two given representations are isomorphic.
Theorem
Suppose $\pi_1(S_1, p_1)$ is isomorphic to $\pi_2(S_2, p_2)$, then $S_1$ is homeomorphic to $S_2$, and vice versa.

Proof.
For each surface, find a canonical basis, slice the surface along the basis to get a $4g$ polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition.
Theorem (Seifert-Van Kampen)

Topological space $M$ is decomposed into the union of $U$ and $V$, the intersection of $U$ and $V$ is $W$, $M = U \cup V$, $W = U \cap V$, where $U$, $V$ and $W$ are path connected. $i : W \to U$, $j : W \to V$ are the inclusions. Pick a base point $p \in W$, the fundamental groups

$$
\pi_1(U, p) = \langle u_1, \cdots, u_k | \alpha_1, \cdots, \alpha_l \rangle \\
\pi_1(V, p) = \langle v_1, \cdots, v_m | \beta_1, \cdots, \beta_n \rangle \\
\pi_1(W, p) = \langle w_1, \cdots, w_p | \gamma_1, \cdots, \gamma_q \rangle
$$

then the $\pi_1(M, p)$ is given by

$$
\pi_1(M, p) = \langle u_1, \ldots, u_k, v_1, \ldots, v_m | \alpha_i, \beta_j, i(w_1)j(w_1)^{-1}, \ldots, i(w_p)j(w_p)^{-1} \rangle
$$
Definition (Connected Sum)

Let $S_1$ and $S_2$ be two surfaces, $D_1 \subset S_1$ and $D_2 \subset S_2$ are two topological disks. $f : \partial D_1 \to \partial D_2$ is a homeomorphism between the boundaries of the disks. The connected sum is $S_1 \oplus S_2 := S_1 \cup S_2 / \{p \sim f(p)\}$. 
Theorem (Surface Topological Classification)

All the compact closed surfaces can be represented as

\[ S \cong T^2 \oplus T^2 \oplus \cdots \oplus T^2 \]

for oriented surfaces, or

\[ S \cong RP^2 \oplus RP^2 \oplus \cdots \oplus RP^2. \]

\(RP^2\) is gluing a Möbius band with a disk along its single boundary.
Canonical Representation of $\pi_1(S, p)$

Figure: $\pi_1(T, p) = \langle a, b | aba^{-1}b^{-1} \rangle$.

Lemma

The fundamental group of a torus is $\pi_1(T, p) = \langle a, b | aba^{-1}b^{-1} \rangle$.

Proof.

Homotopic deform a loop $\gamma$, such that $\gamma$ intersects $a$ and $b$ only at $p$; decompose $\gamma$ to $\gamma_1\gamma_2 \ldots \gamma_k$, such that $\gamma_i$ starts and ends at $p$, the interior doesn’t intersect $a$ and $b$; each $\gamma_i$ is generated by $a, b$. $\square$
Canonical Representation of $\pi_1(S, p)$

Figure: Punctured torus, fundamental group $\pi_1(T \setminus \{q\}, p) = \langle a, b \rangle$. 
Canonical Representation of $\pi_1(S, p)$

Figure: Divide conquer method.

**Fundamental Groups**

\[
\pi_1(T_1, p) = \langle a_1, b_1 \rangle, \quad \pi_1(T_2, p) = \langle a_2, b_2 \rangle, \quad \pi_1(T_1 \cap T_2, p) = \langle \gamma \rangle
\]
Theorem

Show that $\pi_1(S)$ is $\langle a_1, b_1, \cdots, a_g, b_g | \prod_{i=1}^{g} [a_i, b_i] \rangle$ for a surface $S = \bigoplus_{i=1}^{g} T^2$.

Proof.

By induction. If $g = 1$, obvious. Let $g = 2$,

\[
\begin{align*}
\pi_1(T_1) &= \langle a_1, b_1 \rangle \\
\pi_1(T_2) &= \langle a_2, b_2 \rangle \\
\pi_1(T_1 \cap T_2) &= \langle \gamma \rangle
\end{align*}
\]

$[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1}$ in $\pi_1(T_1)$, $[\gamma] = (a_2 b_2 a_2^{-1} b_2^{-1})^{-1}$ in $\pi_1(T_2)$, so

$\pi_1(T_1 \cup T_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2] \rangle$.

where $[a_k, b_k] = a_k b_k a_k^{-1} b_k^{-1}$. 
Suppose it is true for $g - 1$ case. Then for $g$ case, the intersection is an annulus,

$$\pi_1(T_1 \cup T_2 \ldots T_{g-1}) = \langle a_1, b_1, \ldots a_{g-1}, b_{g-1} \mid \prod_{k=1}^{g-1} [a_k, b_k] \rangle$$

$$\pi_1(T_g) = \langle a_g, b_g \mid [a_g, b_g] \rangle$$

$$\pi_1(S \cap T_g) = \langle \gamma \rangle$$

$$[\gamma] = \pi_{k=1}^{g-1} [a_k, b_k] \text{ in } \pi_1(T_1 \cup T_2 \ldots T_{g-1}) \text{ and } [a_g, b_g] \in \pi_1(T_g).$$
Computational Topology: Fundamental Group
**Definition (Cut Graph)**

\( \Gamma \) is a graph on the surface \( S \), such that \( S \setminus \Gamma \) is a topological disk, then \( \Gamma \) is a cut graph of \( S \).

**Figure**: Cut graph of a genus two surface.
Cut Graph Algorithm

Input: A closed triangle mesh $M$;
Output: A cut graph $\Gamma$ of $M$.

1. Compute the dual mesh $\bar{M}$ of the input mesh $M$;
2. Compute a spanning tree $\bar{T}$ of $\bar{M}$;
3. The cut graph is given by

$$\Gamma := \{ e \in M | \bar{e} \notin \bar{T} \}.$$
Figure: Fundamental group generators of a genus two surface.
Algorithm for Fundamental Group Generators

Fundamental Group Generators Algorithm

Input: A closed triangle mesh $M$;
Output: A set of generators of $\pi_1(M, p)$.

1. Compute a cut graph $\Gamma$ of the input mesh $M$;
2. Compute a spanning tree $T$ of $\Gamma$;
3. Select an edge $e_i \in \Gamma \setminus T$, $e_i \cup T$ has a unique loop $\gamma_i$;
4. $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ is a set of generators of the fundamental group of $M$. 
Algorithm for Fundamental Group Relations

Input: A closed triangle mesh \( M \);
Output: The relations in \( \pi_1(M, p) \).

1. Compute a cut graph \( \Gamma \) of the input mesh \( M \);
2. Compute a spanning tree \( T \) of \( \Gamma \), \( \Gamma \setminus T = \{e_1, e_2, \ldots, e_k\} \);
3. For each oriented edge, \( e_i \cup T \) has an oriented loop \( \gamma_i \), \( \{\gamma_1, \gamma_2, \ldots, \gamma_k\} \);
4. Cut the mesh \( M \) along \( \Gamma \) to obtain \( \bar{M} \);
5. Set \( \gamma = \partial \bar{M} \), traverse \( \gamma \). Set \( w = \emptyset \), once \( e_i^{\pm 1} \) is encountered, append \( \gamma_i^{\pm 1} \) to \( w \), \( w \leftarrow w \gamma_i^{\pm 1} \).
Algebraic Topology: Universal Covering Space
Figure: Universal Covering Space
Definition (Covering Space)

Given topological spaces $\tilde{S}$ and $S$, a continuous map $p: \tilde{S} \rightarrow S$ is surjective, such that for each point $q \in S$, there is a neighborhood $U$ of $q$, its preimage $p^{-1}(U) = \bigcup \tilde{U}_i$ is a disjoint union of open sets $\tilde{U}_i$, and the restriction of $p$ on each $\tilde{U}_i$ is a local homeomorphism, then $(\tilde{S}, p)$ is a covering space of $S$, $p$ is called a projection map.

Definition (Deck Transformation)

The automorphisms of $\tilde{S}$, $\tau: \tilde{S} \rightarrow \tilde{S}$, are called deck transformations, if they satisfy $p \circ \tau = p$. All the deck transformations form a group, the covering group, and denoted as $Deck(\tilde{S})$. 
Suppose $\tilde{q} \in \tilde{S}$, $p(\tilde{q}) = q$. The projection map $p : \tilde{S} \to S$ induces a homomorphism between their fundamental groups, $p_* : \pi_1(\tilde{S}, \tilde{q}) \to \pi_1(S, q)$, if $p_*\pi_1(\tilde{S}, \tilde{q})$ is a normal subgroup of $\pi_1(S, q)$ then

**Theorem (Covering Group Structure)**

The quotient group of \[ \frac{\pi_1(S)}{p_*\pi_1(\tilde{S}, \tilde{q})} \] is isomorphic to the deck transformation group of $\tilde{S}$.

\[ \frac{\pi_1(S, q)}{p_*\pi_1(\tilde{S}, \tilde{q})} \cong \text{Deck}(\tilde{S}). \]
Definition (Universal Covering Space)
If a covering space $\tilde{S}$ is simply connected (i.e. $\pi_1(\tilde{S}) = \{e\}$), then $\tilde{S}$ is called a universal covering space of $S$.

For universal covering space

$$\pi_1(S) \cong Deck(\tilde{S}).$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space.
Figure: Universal Covering Space of a genus two surface.
Path homotopy classes form the universal covering space.
Theorem

Suppose the topological manifold is path connected, then there is a universal covering space \( p : \tilde{S} \to S \).

Proof.

Fix a base point \( q \in S \), consider all the paths starting from \( q \),
\( \Gamma := \{ \gamma : [0, 1] \to S | \gamma(0) = q \} \). Define \( \tilde{\Gamma} := \Gamma / \sim \), the homotopy classes of paths in \( \Gamma \). Pick a path \( \gamma \in \Gamma \), \( \gamma(1) = q_0 \), let \( U \subset S \) be an open set of \( q_1 \). For each point \( q' \in U \), there is a path \( \alpha(q') \subset U \) connecting \( q' \) to \( q_0 \). Then we define an open set \( \tilde{U} \subset \tilde{\Gamma} \) of \([\gamma] \) as

\[
\tilde{U} := \{ [\tau] | \tau(1) \in U, \tau \cdot \alpha(\tau(1)) \sim \gamma \}.
\]

The \( \{ \tilde{U} \} \) define a topology of \( \tilde{\Gamma} \). \( p : \tilde{\Gamma} \to S, [\gamma] \mapsto \gamma(1) \) is a universal covering space of \( S \).
Figure: Universal Covering Space
Figure: Universal Covering Space
Lifting to Universal Covering Space

Figure: Lifting to the Universal Covering Space
Lifting to Universal Covering Space

Let \((\tilde{S}, p)\) be the universal covering space of \(S\), \(q\) be the base point. The orbit of base is \(p^{-1}(q) = \{\tilde{q}_k\}\). Given a loop through \(q\), there exists a unique lift of \(\gamma\), \(\tilde{\gamma} \subset \tilde{S}\), starting from \(\tilde{q}_0\).

**Lemma**

\(\gamma_1\) and \(\gamma_2\) are two loops through the base point, their lifts are \(\tilde{\gamma}_1\) and \(\tilde{\gamma}_2\). \(\gamma_1 \sim \gamma_2\) if and only if the end points of \(\tilde{\gamma}_1\) and \(\tilde{\gamma}_2\) coincide.

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{\tilde{\gamma}} & \tilde{M} \\
\downarrow{id} & & \downarrow{p} \\
[0, 1] & \xrightarrow{\gamma} & M
\end{array}
\]
Let $G$ be an unoriented graph, $T$ is a spanning tree of $G$, $G - T = \{e_1, e_2, \cdots, e_n\}$, where $e_k$ is an edge not in the tree. Then $T \cup e_k$ has a unique loop $\gamma_k$. Choose one orientation of $\gamma_k$.

**Lemma**

*The fundamental group of $G$ is $\pi_1(G) = \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$, which is a free group.*
Definition (CW-cell decomposition)

A $k$ dimensional cell $D_k$ is a $k$ dimensional topological disk. Suppose $M$ is a $n$-dimensional manifold.

1. 0-skeleton $S_0$ is the union of a set of 0-cells.
2. $k$-skeleton $S_k$
   
   $$S_k = S_{k-1} \cup D_k^1 \cup D_k^2 \cdots \cup D_k^{n_k},$$

   such that
   
   $$\partial D_k^i \subset S_{k-1}.$$

   The $k$-skeleton is constructed by gluing $k$-cells to the $k-1$ skeleton, all the boundaries of the cells are in the $k-1$ skeleton.

3. $S_n = M.$
Theorem (CW-cell decomposition)

\[ \pi_1(S_2) = \pi_1(S_3) \cdots \pi_1(S_n) = \pi_1(M) \]

Proof.

using induction. \( S_2 \cap D^1_3 \) is \( \partial D^1_3 \), which is a topological sphere. \( \pi_1(D^1_3) = \langle e \rangle \), \( \pi_1(S^2) \) is \( \langle e \rangle \).
Computational Topology: Universal Covering Space
Algorithm for Universal Covering Space

Universal Covering Space Algorithm

Input: A closed triangle mesh $M$;
Output: A finite portion of the universal covering space $\tilde{M}$.

1. Compute a cut graph $\Gamma$ of $M$, divide $\Gamma$ into nodes and oriented segments, $\{s_1, s_2, \ldots, s_k\}$;
2. Slice $M$ along $\Gamma$ to obtain one fundamental domain $\bar{M}$;
3. Initialize $\tilde{M} \leftarrow \bar{M}$
4. Choose an oriented segment $s_i$ on the boundary of $\tilde{M}$, glue a copy of $\bar{M}$ with $\tilde{M}$ along $s_i$,
   \[ \tilde{M} \leftarrow \tilde{M} \cup \partial \bar{M} \supset s_i \sim s_i^{-1} \subset \partial \bar{M} \bar{M} \]
5. Trace the boundary of $\tilde{M}$, if there are two adjacent segments $s_i, s_{i+1} \subset \partial \tilde{M}$, such that $s_i^{-1} = s_{i+1}$, then glue them together;
6. Repeat step 4 and step 5, until $\tilde{M}$ is large enough.
Algorithm for Homotopy Detection

Homotopy Detection Algorithm

Input: A closed triangle mesh $M$, two loops $\gamma_1$ and $\gamma_2$ through a base point $p$;
Output: Verify whether $\gamma_1 \sim \gamma_2$.

1. Compute a finite portion of the universal covering space $\tilde{M}$ of $M$;
2. Lift $\gamma_1 \cdot \gamma_2^{-1}$ to $\tilde{M}$, the lifted path is $\tilde{\gamma}$;
3. If $\tilde{\gamma}$ is a closed loop, then return Yes; otherwise, return No.