Instruction for Assignment Four: Conformal Structure

David Gu

Computer Science Department
Stony Brook University

gu@cs.stonybrook.edu

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Conformal Structure: Theoretic Proofs
Problem (Torus)

Suppose $S$ is a topological torus, two Riemannian metrics $g_1$ and $g_2$ on $S$ are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:

$$g_1(p) = e^{2\lambda(p)}g_1(p),$$

the space of conformal equivalence classes of Riemannian metrics is called the Teichmüller space. What is the dimension of the Teichmüller space of a torus?
Teichmüller Space

Problem (Topological Quadrilateral)

Suppose $S$ is a topological disk with a single boundary, the boundary is piecewise analytic curve. There are four boundary points on $\partial S$, $p_0, p_1, p_2, p_3$.

- Show that there is a conformal map from $S$ to a planar rectangle, the four points $p_0, p_1, p_2, p_3$ are mapped to the corners of the rectangle.

- Two Riemannian metrics $g_1$ and $g_2$ on $S$ are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:

$$g_1(p) = e^{2\lambda(p)}g_1(p),$$

the space of conformal equivalence classes of Riemannian metrics is called the Teichmüller space. What is the dimension of the Teichmüller space of a topological quadrilateral?
Teichmüller Space

Problem (Poly-Annulus)

Suppose $S$ is a topological poly-annulus, two Riemannian metrics $g_1$ and $g_2$ on $S$ are called conformal equivalent, if they differ by a scalar function $\lambda : S \to \mathbb{R}$:

$$g_1(p) = e^{2\lambda(p)}g_1(p),$$

the space of conformal equivalence classes of Riemannian metrics is called the Teichmüller space. What is the dimension of the Teichmüller space of a poly-annulus?
Problem (Möbius Transformation)

The hyperbolic space is the unit disk $|z| < 1$ with a metric

$$ds^2 = \frac{dzd\bar{z}}{(1 - z\bar{z})^2}.$$ 

A Möbius transformation has the form

$$z \mapsto e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}.$$ 

Show that this type of Möbius transformation is hyperbolic isometric.
Suppose $S$ is a high genus surface with a hyperbolic metric, its universal covering space is the hyperbolic plane, the Deck transformations are Möbius transformations.

- Suppose $\{a_1, b_1, \cdots, a_g, b_g\}$ is a set of canonical fundamental group generators of $\pi_1(S, p)$, the corresponding Deck transformation group generators are $\{\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g\}$. What are the relations among the fundamental group generators? What are the relations among the Deck transformation generators?
- How many parameters do we need to describe the Deck transformation group generators?
- All possible hyperbolic metrics on $S$ form a space. What is the dimension of this space?