Assignment Seven: Verification of Abel-Jacobi Theorem

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Canonical Fundamental Group Generator

Step 1

Compute a set of canonical fundamental group generators of $S$,

$$\pi_1(S, p) = \langle a_1, b_1, \cdots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$ 

Based on Assignment 6 to compute handle loops and tunnel loops.
Holomorphic Differential

Step 2

Use Hodge decomposition algorithm to compute a holomorphic 1-form:

1. Compute $2g$ random 1-forms, $\{\tau_1, \tau_2, \cdots, \tau_{2g}\}$;
2. Decompose each $\tau_i$ to the harmonic 1-form $\omega_i$, $i = 1, 2, \cdots, 2g$;
3. Compute the Hodge star $\star \omega_i$ of $\omega_i = \sum_{j=1}^{2g} \lambda_{ij} \omega_j$;
4. Pair $\omega_i$ and $\star \omega_i$ to form a holomorphic 1-form $\varphi_i = \omega_i + \sqrt{-1} \star \omega_i$, $i = 1, 2, \cdots, 2g$
Cut the surface along the fundamental group generators to get a fundamental domain $\Omega$;

Select one holomorphic 1-form $\varphi$ and one base point $p_0 \in \Omega$;

Compute the Abel-Jacobi map

$$\mu(p) = \int_{p_0}^{p} \varphi.$$
Verify Abel Theorem

Step 4

1. Locate all the corner points of the polycube surfaces, the valence 3 corners are poles, denoted as \( q_i \)'s, the valence 5 corners are zeros, denoted as \( p_j \)'s;

2. Locate the zeros of the holomorphic 1-forms, denoted as \( c_k \)'s;

3. Compute

\[
\mu(D) := \sum_j \mu(p_j) - \sum_i \mu(q_i) - 4 \sum_k \mu(c_k)
\]

4. in theory, \( \mu(D) \) modulo the periods should be zero.
A genus one closed surface $S$, which is a polycube surface (union of canonical unit cubes). The holomorphic one form $\omega \in \Omega^1(S)$. 
The homology basis is \( \{a, b\} \), the surface is sliced along \( \{a, b\} \) to get a fundamental domain \( D \), \( \partial D = abab^{-1}b^{-1} \). The conformal mapping \( \mu : D \rightarrow \mathbb{C} \) is given by

\[
\mu(q) = \int_{p}^{q} \omega,
\]

where \( p \) is a base point and the integration path is arbitrarily chosen.
Suppose \( q_i \)'s are poles (degree 3), \( p_j \)'s are zeros (degree 5), then we have found that the number of poles equals to that of the zeros, furthermore,

\[
\sum_{j=1}^{22} \mu(p_j) - \sum_{i=1}^{22} \mu(q_i) = 0.
\]
Suppose $S$ is a genus two polycube surface, $\omega$ is a holomorphic one-form. The red circles show the poles (degree 3), the blue circles show the zeros (degree 5), the purple circles the zeros of $\omega$. 

(a). front view

(b). back view
The surface is sliced along $a_1, b_1, a_2, b_2, \tau$, and integrate $\omega$ to obtain $\mu : S \rightarrow \mathbb{C}$

$$\mu(q) = \int_p^q \omega,$$

it branch covers the plane, the branching points are zeros of $\omega, c_1, c_2$. 

(a). cuts  
(b). conformal fattening
Suppose $p_i$’s are zeros (degree 5), $q_j$’s are poles (degree 3), $c_k$’s are branch points, then we have

$$\sum_{i=1}^{16} \mu(p_i) - \sum_{j=1}^{8} \mu(q_j) = 4 \sum_{k=1}^{2} \mu(c_k).$$
Suppose $S$ is a genus two polycube surface, $\omega$ is a holomorphic one-form. The red circles show the poles (degree 3), the blue circles show the zeros (degree 5), the purple circles the zeros of $\omega$. 

(a). front view

(b). back view
Suppose $p_i$’s are **zeros** (degree 5), $q_j$’s are **poles** (degree 3), $c_k$’s are **branch points**, then we have

$$
\sum_{i=1}^{16} \mu(p_i) - \sum_{j=1}^{8} \mu(q_j) = 4 \sum_{k=1}^{2} \mu(c_k).
$$
You can also choose other topics, which are related to computational conformal geometry and can demonstrate your talent and skills. The project is due within one month. The solution is required to be written in generic C++ with a detailed technical report to describe your design of data structures, algorithms, potential applications and improvement direction.