Summary of Computational Conformal Geometry

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Algebraic Topology

Surface Topology

Philosophy

Associate groups with manifolds, study the topology by analyzing the group structures.

$$\begin{aligned} \mathfrak{C}_1 &= \{\textit{Topological Spaces}, \textit{Homeomorphisms}\} \\ \mathfrak{C}_2 &= \{\textit{Groups}, \textit{Homomorphisms}\} \\ \mathfrak{C}_1 &\to \mathfrak{C}_2 \end{aligned}$$

Functor between categories.

Fundamental Group

Definition (Fundamental Group)

Given a topological space S, fix a base point $p \in S$, the set of all the loops through p is Γ, the set of all the homotopy classes is $\Gamma/\sim$. The product is defined as:

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2],$$

the unit element is defined as $[e]$, the inverse element is defined as

$$[\gamma]^{-1} := [\gamma^{-1}],$$

then $\Gamma/\sim$ forms a group, the fundamental group of S, and is denoted as $\pi_1(S, p)$.

Canonical Representation of $\pi_1(S, p)$

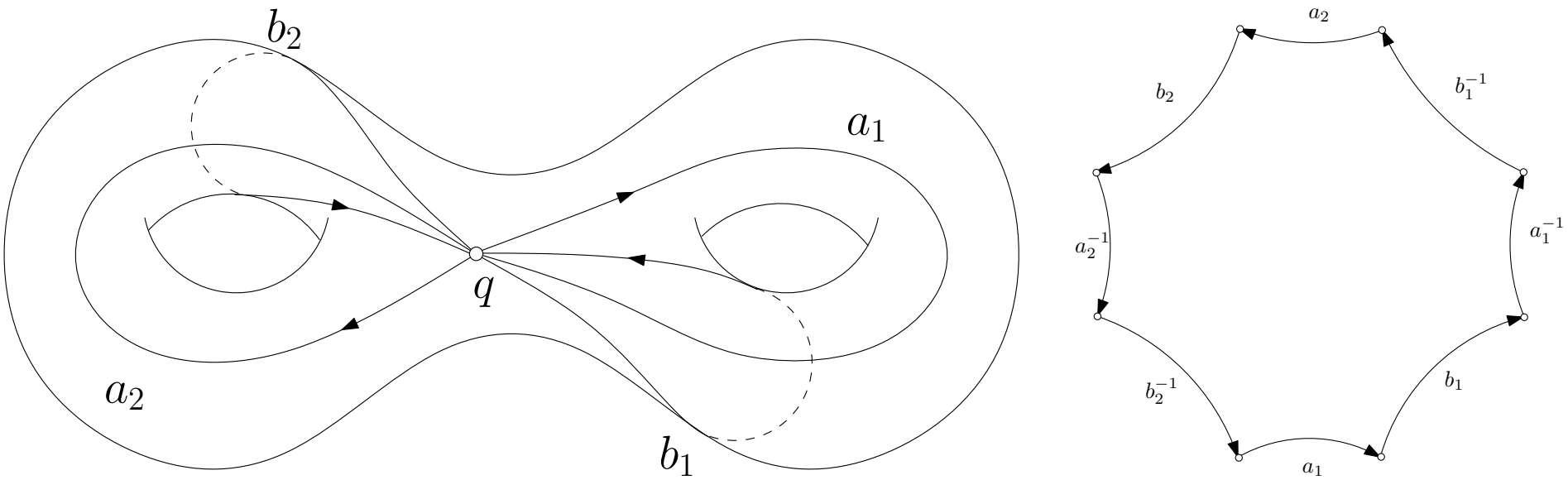


Figure: Canonical fundamental group representation.

Canonical Representation of $\pi_1(S, p)$

Theorem (Surface Fundamental Group Canonical Representation)

Suppose S is a compact, oriented surface, $p \in S$ is a fixed point, the fundamental group has a canonical representation,

$$\pi_1(S, p) = \langle a_1, b_1, a_2, b_2, \cdots, a_g, b_g | \Pi_{i=1}^{g} [a_i, b_i] \rangle,$$

where

$$[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1},$$

g is the genus of the surface.

Fundamental Group Representation

Theorem

Suppose $\pi_1(S_1, p_1)$ is isomorphic to $\pi_2(S_2, p_2)$, then S_1 is homeomorphic to S_2, and vice versa.

Proof.

For each surface, find a canonical basis, slice the surface along the basis to get a $4g$ polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition. □

Seifert-Van Kampen Theorem

Theorem (Seifert-Van Kampen)

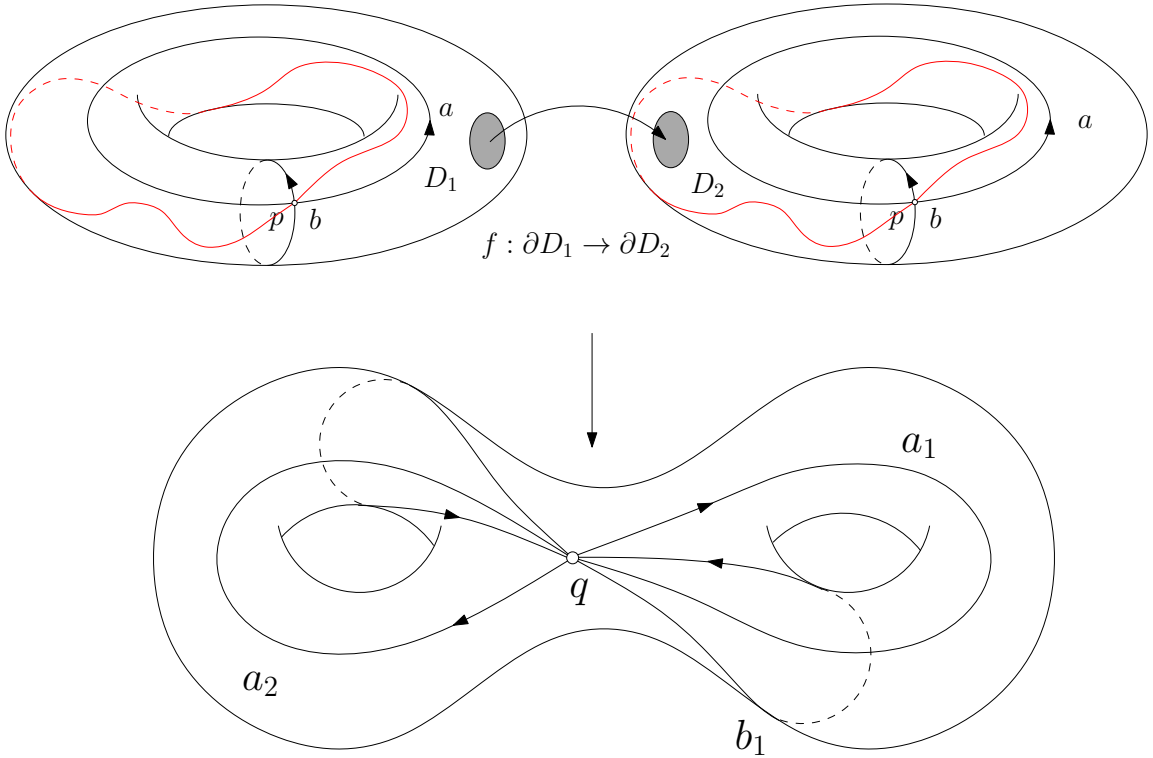
Topological space M is decomposed into the union of U and V, the intersection of U and V is W, $M = U \cup V$, $W = U \cap V$, where U, V and W are path connected. $i : W \to U$, $j : W \to V$ are the inclusions. Pick a base point $p \in W$, the fundamental groups

$$\pi_1(U, p) = \langle u_1, \cdots, u_k | \alpha_1, \cdots, \alpha_l \rangle$$
$$\pi_1(V, p) = \langle v_1, \cdots, v_m | \beta_1, \cdots, \beta_n \rangle$$
$$\pi_1(W, p) = \langle w_1, \cdots, w_p | \gamma_1, \cdots, \gamma_q \rangle$$

then the $\pi_1(M, p)$ is given by

$$\pi_1(M, p) = \langle u_1, \ldots, u_k, v_1, \ldots, v_m | \alpha_i, \beta_j, i(w_1)j(w_1)^{-1}, \ldots, i(w_p)j(w_p)^{-1} \rangle$$

Canonical Representation of $\pi_1(S, p)$



Definition (Connected Sum)

Let S_1 and S_2 be two surfaces, $D_1 \subset S_1$ and $D_2 \subset S_2$ are two topological disks. $f : \partial D_1 \to \partial D_2$ is a homeomorphism between the boundaries of the disks. The connected sum is $S_1 \oplus S_2 := S_1 \cup S_2 / \{p \sim f(p)$.

Surface Topology

Theorem (Surface Topological Classification)

All the compact closed surfaces can be represented as

$$S \cong T^2 \oplus T^2 \oplus \cdots \oplus T^2$$

for oriented surfaces, or

$$S \cong RP^2 \oplus RP^2 \oplus \cdots \oplus RP^2.$$

RP^2 is gluing a Möbius band with a disk along its single boundary.

Universal Covering Space

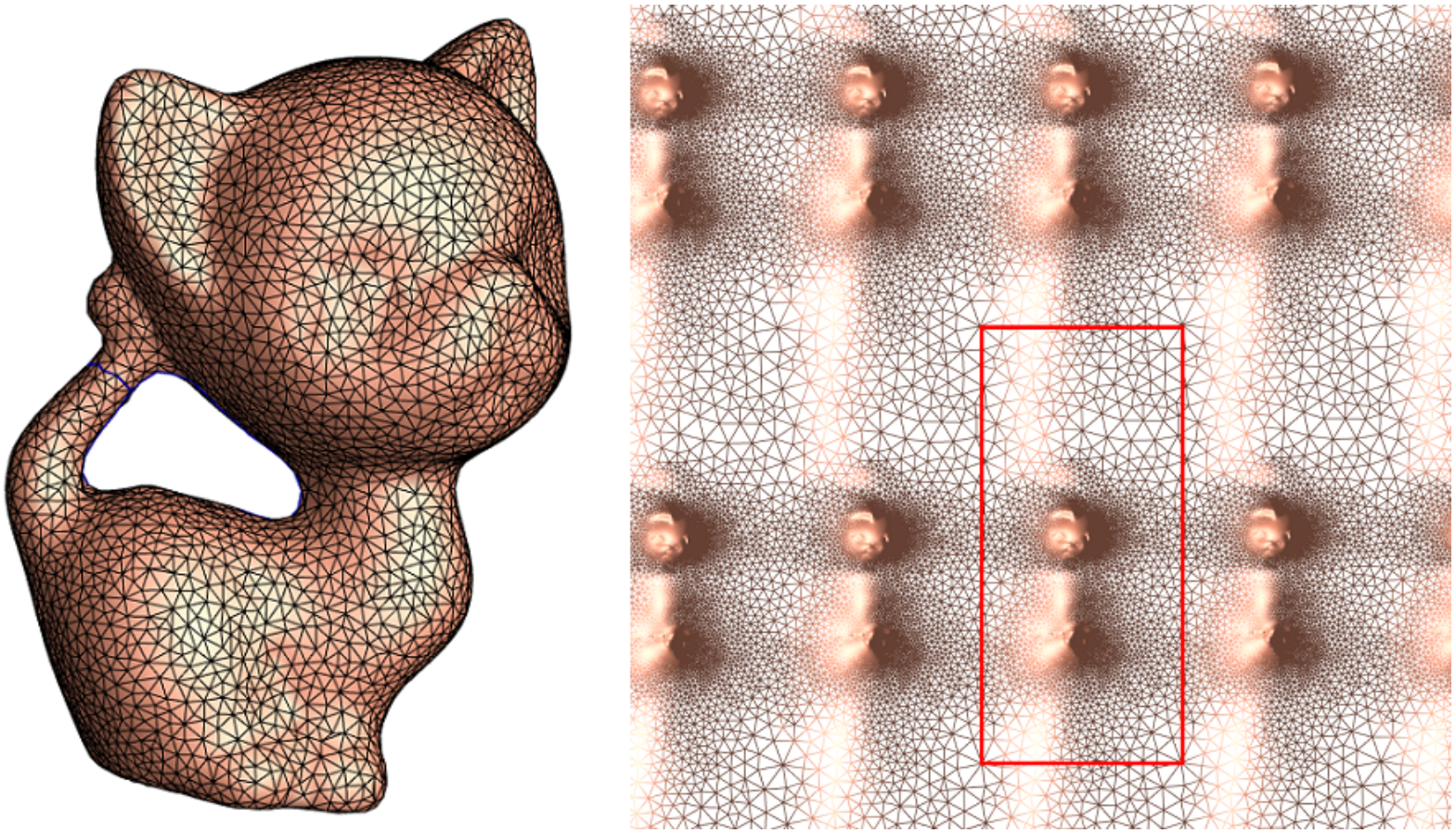


Figure: Universal Covering Space

Covering Space

Definition (Covering Space)

Given topological spaces $\tilde{S}$ and S, a continuous map $p : \tilde{S} \to S$ is surjective, such that for each point $q \in S$, there is a neighborhood U of q, its preimage $p^{-1}(U) = \cup_i \tilde{U}_i$ is a disjoint union of open sets $\tilde{U}_i$, and the restriction of p on each $\tilde{U}_i$ is a local homeomorphism, then $(\tilde{S}, p)$ is a *covering space* of S, p is called a *projection map*.

Definition (Deck Transformation)

The automorphisms of $\tilde{S}$, $\tau : \tilde{S} \to \tilde{S}$, are called *deck transformations*, if they satisfy $p \circ \tau = p$. All the deck transformations form a group, the *covering group*, and denoted as $Deck(\tilde{S})$.

Covering Group

Suppose $\tilde{q} \in \tilde{S}$, $p(\tilde{q}) = q$. The projection map $p : \tilde{S} \to S$ induces a homomorphism between their fundamental groups, $p_* : \pi_1(\tilde{S}, \tilde{q}) \to \pi_1(S, q)$, if $p_*\pi_1(\tilde{S}, \tilde{q})$ is a normal subgroup of $\pi_1(S, q)$ then

Theorem (Covering Group Structure)

The quotient group of $\frac{\pi_1(S)}{p_*\pi_1(\tilde{S},\tilde{q})}$ *is isomorphic to the deck transformation group of* $\tilde{S}$.

$$\frac{\pi_1(S, q)}{p_*\pi_1(\tilde{S}, \tilde{q})} \cong Deck(\tilde{S}).$$

universal covering space

Definition (Universal Covering Space)

If a covering space $\tilde{S}$ is simply connected (i.e. $\pi_1(\tilde{S}) = \{e\}$), then $\tilde{S}$ is called a *universal covering space* of S.

For universal covering space

$$\pi_1(S) \cong Deck(\tilde{S}).$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space.

Universal Covering Space

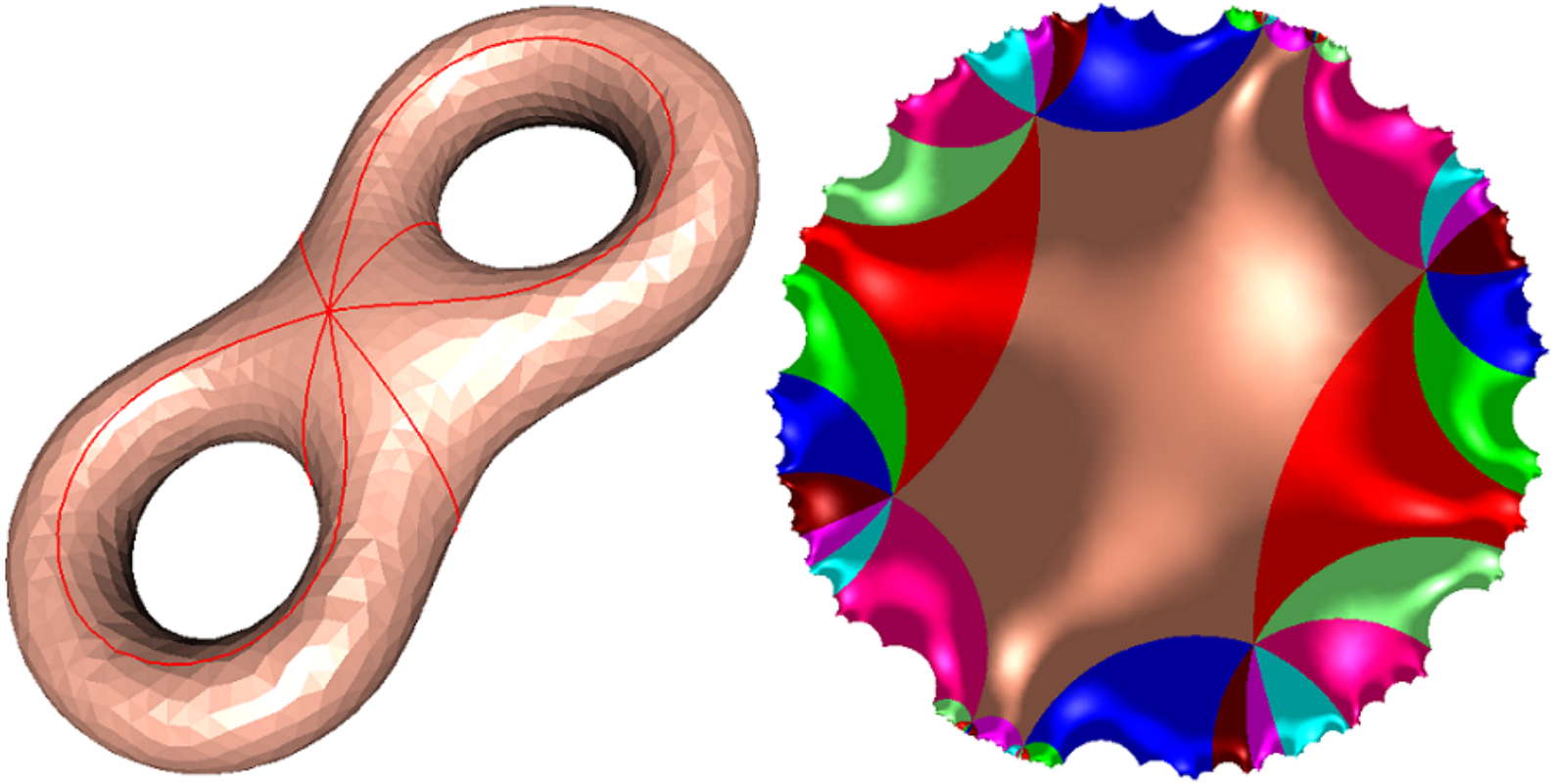


Figure: Universal Covering Space of a genus two surface.

Universal Covering Space

Theorem

Suppose the topological manifold is path connected, then there is a universal covering space $p : \tilde{S} \to S$.

Proof.

Fix a base point $q \in S$, consider all the paths starting from q, $\Gamma := \{\gamma : [0,1] \to S | \gamma(0) = q\}$. Define $\tilde{S} := \Gamma / \sim$, the homotopy classes of paths in Γ. Pick a path $\gamma \in \Gamma$, $\gamma(1) = q_0$, let $U \subset S$ be an open set of q_1. For each point $q' \in U$, there is a path $\alpha(q') \subset U$ connecting q' to q_0. Then we define an open set $\tilde{U} \subset \tilde{S}$ of $[\gamma]$ as

$$\tilde{U} := \{[\tau] | \tau(1) \in U, \tau \cdot \alpha(\tau(1)) \sim \gamma\}.$$

The $\{\tilde{U}\}$ define a topology of $\tilde{\Gamma}$. $p : \tilde{\Gamma} \to S$, $[\gamma] \mapsto \gamma(1)$ is a universal covering space of S. $\square$

Lifting to Universal Covering Space

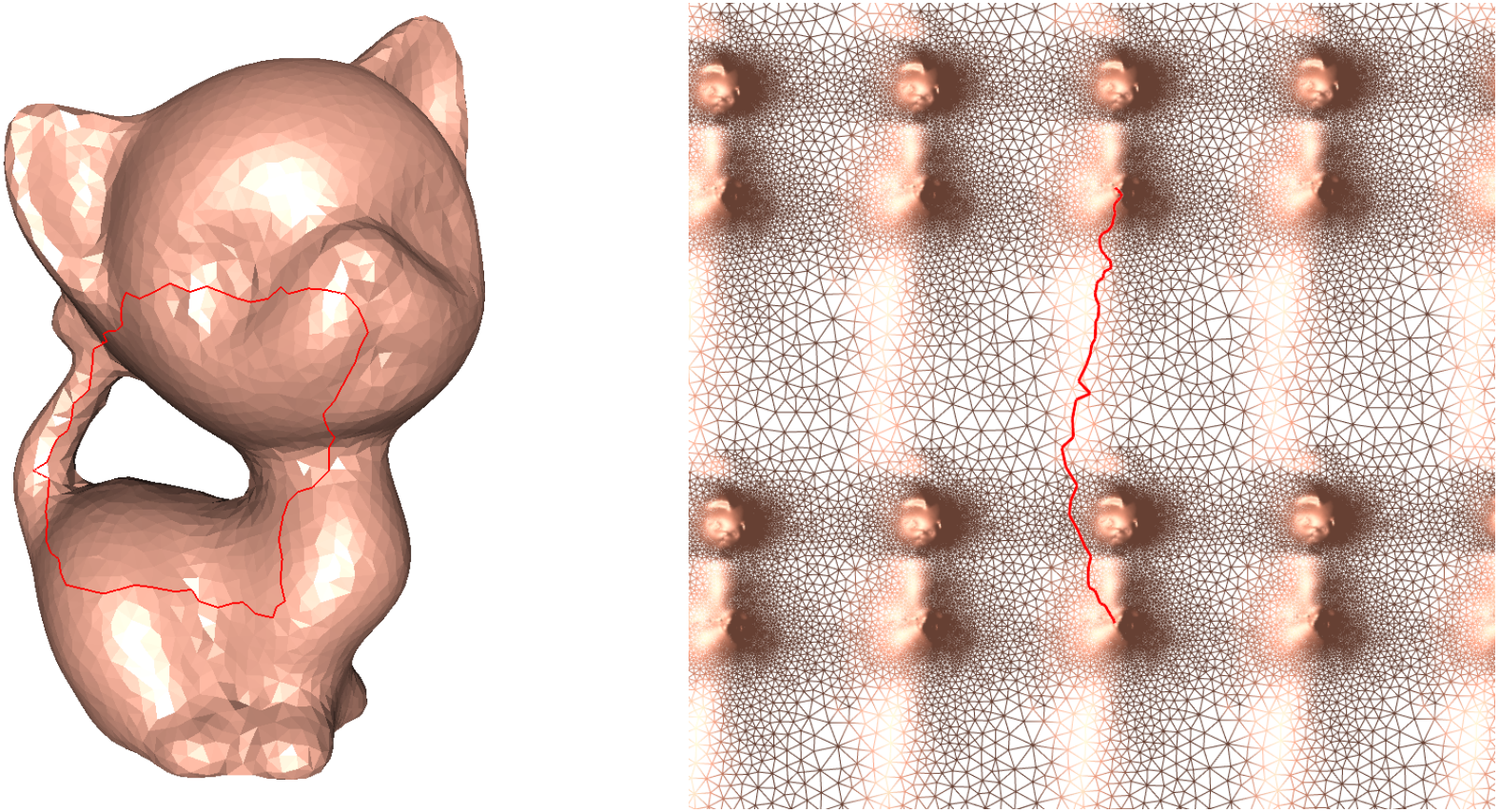


Figure: Universal Covering Space

Lifting to Universal Covering Space

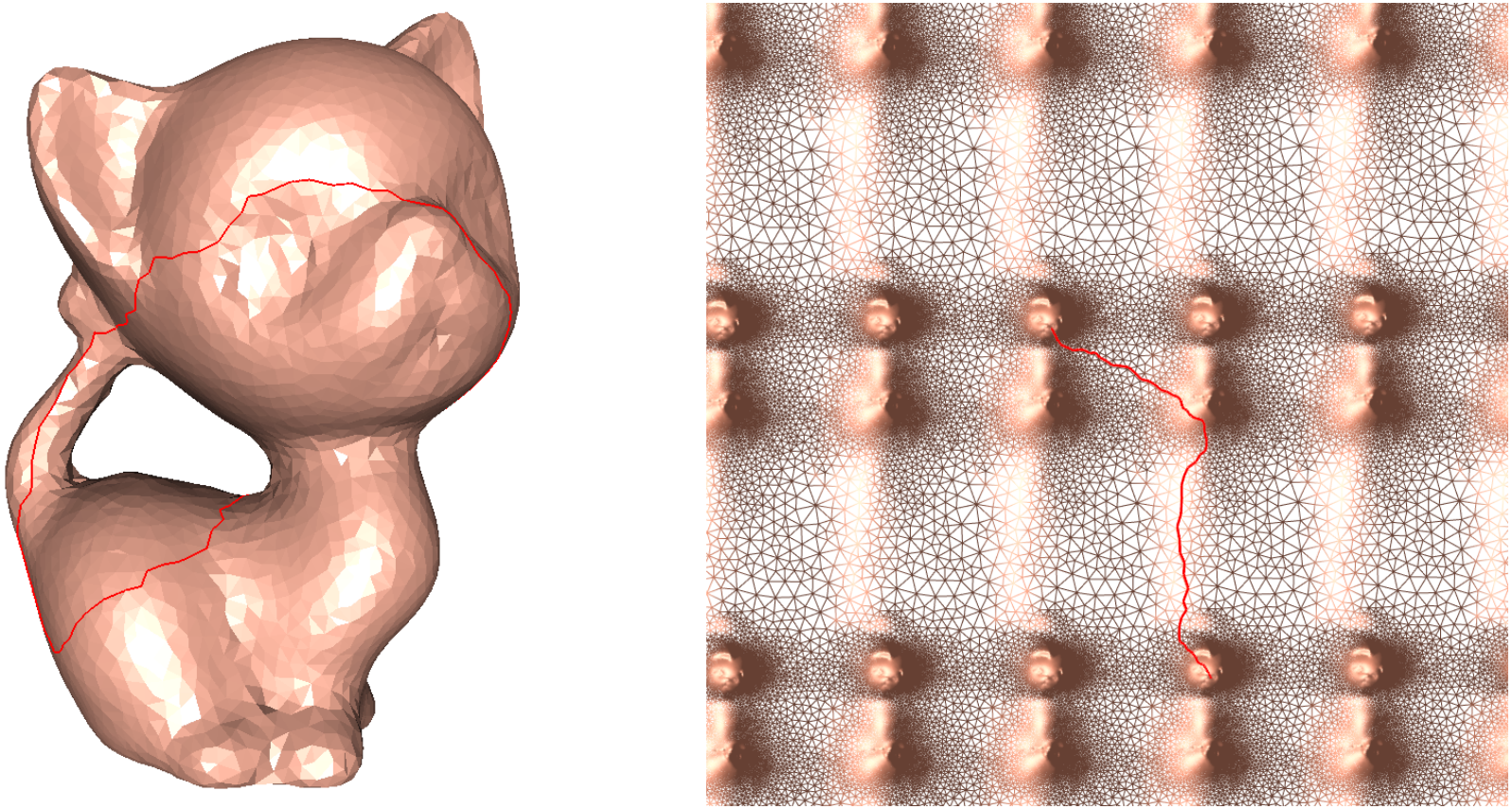


Figure: Universal Covering Space

Lifting to Universal Covering Space

Let $(\tilde{S}, p)$ be the universal covering space of S, q be the base point. The orbit of base is $p^{-1}(q) = \{\tilde{q}_k\}$. Given a loop through q, there exists a unique lift of γ, $\tilde{\gamma} \subset \tilde{S}$, starting from $\tilde{q}_0$.

Lemma

γ_1 and γ_2 are two loops through the base point, their lifts are $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. $\gamma_1 \sim \gamma_2$ if and only if the end points of $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ coincide.

$$\begin{tikzcd}
{[0,1]} \arrow[r, "\tilde{\gamma}"] \arrow[d, "id"] & \tilde{M} \arrow[d, "p"] \\
{[0,1]} \arrow[r, "\gamma"] & M
\end{tikzcd}$$

CW-cell decomposition

Definition (CW-cell decomposition)

A k dimensional cell D_k is a k dimensional topological disk. Suppose M is a n-dimensional manifold.

1. 0-skeleton S_0 is the union of a set of 0-cells.
2. k-skeleton S_k
$$S_k = S_{k-1} \cup D_k^1 \cup D_k^2 \cdots \cup D_k^{n_k},$$
such that
$$\partial D_k^i \subset S_{k-1}.$$
The k-skeleton is constructed by gluing k-cells to the $k-1$ skeleton, all the boundaries of the cells are in the $k-1$ skeleton.
3. $S_n = M$.

Persistent Homology

Definition (Persistent Homology)

Define Z_k^l, B_k^l be the K-th cycle group and k-th boundary group respectively, of the l-complex K^l in a filtration. The p-persistent k-th homology group K^l is

$$H_k^{l,p} := \frac{Z_k^l}{B_k^{l+p} \bigcap Z_k^l}.$$

The p-persistent k-th Betti number $\beta_k^{l,p}$ of K^l is the rank of $H_k^{l,p}$.

Lemma

Consider the homomorphism $\eta_k^{l,p} : H_k^l \to H_k^{l+p}$, *then*

$$img\ \eta_k^{l,p} \cong H_k^{l,p}$$

Homology Group

Closed k-chains form the kernel space of the boundary operator ∂_k. Exact k-chains form the image space of ∂_{k+1}.

Definition (Homology Group)

The k dimensional homology group $H_k(\Sigma, \mathbb{Z})$ is the quotient space of $ker\partial_k$ and the image space of $img\partial_{k+1}$.

$$H_k(\Sigma, \mathbb{Z}) = \frac{ker\partial_k}{img\partial_{k+1}}.$$

Two k-chains γ_1,γ_2 are homologous, if they boundary a $(k+1)$-chain σ,

$$\gamma_1 - \gamma_2 = \partial_{k+1}\sigma.$$

Cohomology

The difference between exact forms and closed forms indicates the topology of the manifold.

Definition (Cohomology Group)

The k-dimensional cohomology group of Σ is defined as

$$H^n(\Sigma, \mathbb{Z}) = \frac{ker\delta^n}{img\delta^{n-1}}.$$

Two 1-forms ω_1, ω_2 are cohomologous, if they differ by a gradient of a 0-form f,

$$\omega_1 - \omega_2 = \delta_0 f.$$

Simplicial Mapping

Definition (simplicial mapping)

Suppose M and N are simplicial complexes, $f : M \to N$ is a continuous map, $\forall \sigma \in M$, σ is a simplex, $f(\sigma)$ is a simplex.

For each simplex, we can add its gravity center, and subdivide the simplex to multiple ones. The resulting complex is called the gravity center subdivision.

Theorem

Suppose M and N are simplicial complexes embedded in $\mathbb{R}^n$, $f : M \to N$ is a continuous mapping. Then for any $\epsilon > 0$, there exists gravity subdivisions $\tilde{M}$ and $\tilde{N}$, and a simplicial mapping $\tilde{f} : \tilde{M} \to \tilde{N}$, such that

$$\forall p \in |M|, |f(p) - \tilde{f}(p)| < \epsilon.$$

Degree of a mapping

Suppose M and N are two closed surfaces. $H_2(M,\mathbb{Z}) = \mathbb{Z}$, $H_2(N,\mathbb{Z}) = \mathbb{Z}$, suppose $[M]$ is the generator of $H_2(M)$, which is the union of all faces. similarly, $[n]$ is the generator of $H_2(N)$. $f : M \to N$ is a continuous map. Then

$$f_* : \mathbb{Z} \to \mathbb{Z},$$

must has the form $f_*(z) = cz, c \in \mathbb{Z}$.

Definition (Mapping Degree)

$f_*([M]) = c[N]$, then the integer c is the degree of the map.

map degree is the algebraic number of pre-images $f^{-1}(q)$ for arbitrary point $q \in N$, which is independent of the choice the point q.

Degree of a mapping

Example (Gauss-Bonnet)

$G : S \to \mathbb{S}^2$ is the Gauss map, which maps the point p to its normal $\mathbf{n}(p)$, then $deg(G) = 1 - g$. The total area of the image is $4\pi \deg(G) = 2\pi\chi(S)$.

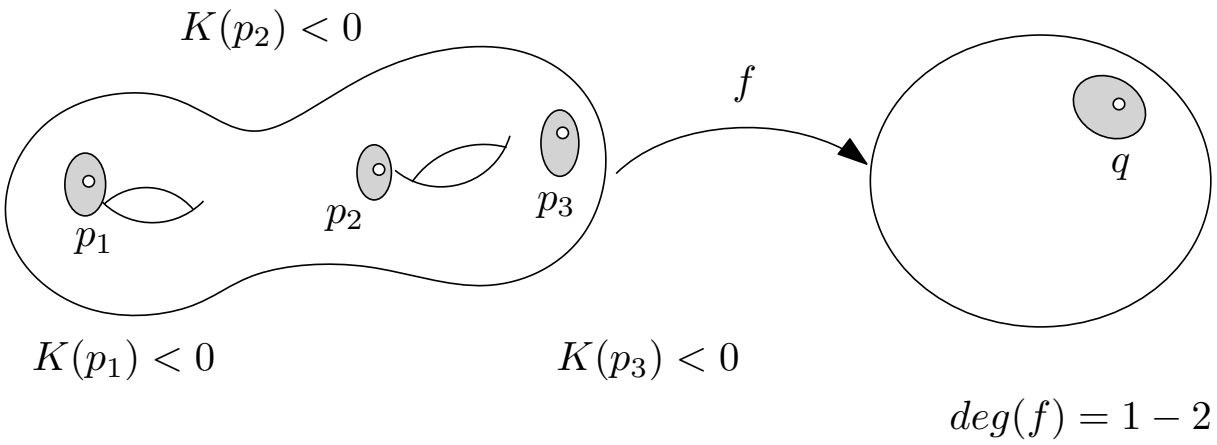


Figure: Map degree and Gauss-Bonnet theorem proof.

High dimensional Gauss-Bonnet theorem was first proved by Allendoerfer and Weil, Prof. Chern used different method to reprove it.

Brouwer Fixed Point

Theorem (Brouwer Fixed Point)

Suppose $\Omega \subset \mathbb{R}^n$ is a compact convex set, $f : \Omega \to \Omega$ is a continous map, then there exists a point $p \in \Omega$, such that $f(p) = p$.

Proof.

Assume $f : \Omega \to \Omega$ has no fixed point, namely $\forall p \in \Omega$, $f(p) \neq p$. We construct $g : \Omega \to \partial\Omega$, a ray starting from $f(p)$ through p and intersect $\partial\Omega$ at $g(p)$, $g|_{\partial\Omega} = id$. i is the inclusion map, $(g \circ i) : \partial\Omega \to \partial\Omega$ is the identity,

$$\partial\Omega \xrightarrow{i} \Omega \xrightarrow{g} \partial\Omega$$

$(g \circ i)_\# : H_{n-1}(\partial\Omega, \mathbb{Z}) \to H_{n-1}(\partial\Omega, \mathbb{Z})$ is $z \mapsto z$. But $H_{n-1}(\Omega, \mathbb{Z}) = 0$, then $g_\# = 0$. Contradiction. $\square$

Lefschetz Fixed Point

Theorem (Lefschetz Fixed Point)

Given a continuous automorphism of a compact topological space $f : M \to M$, if its Lefschetz number is non-zero, then there is a point $p \in M$, $f(p) = p$.

Proof.

Triangulate M, use a simplicial map to approximate f, then

$$\sum_k (-1)^k Tr(f_k|C_k) = \sum_k (-1)^k Tr(f_k|H_k) = \Lambda(f). \tag{1}$$

If $\Lambda(f) \neq 0$, $\exists \sigma \in C_k$, $f_k(\sigma) \subset \sigma$, from Brouwer fixed point theorem, there is a fixed point $p \in \sigma$. $\square$

Poincaré-Hopf

Theorem (Poincaré-Hopf Index)

Assume S is a compact, oriented smooth surface, v is a smooth tangent vector field with isolated zeros. If S has boundaries, then v point along the exterior normal direction, then we have

$$\sum_{p \in Z(v)} Index_p(v) = \chi(S),$$

where $Z(v)$ is the set of all zeros, $\chi(S)$ is the Euler characteristic number of S.

Cut Graph

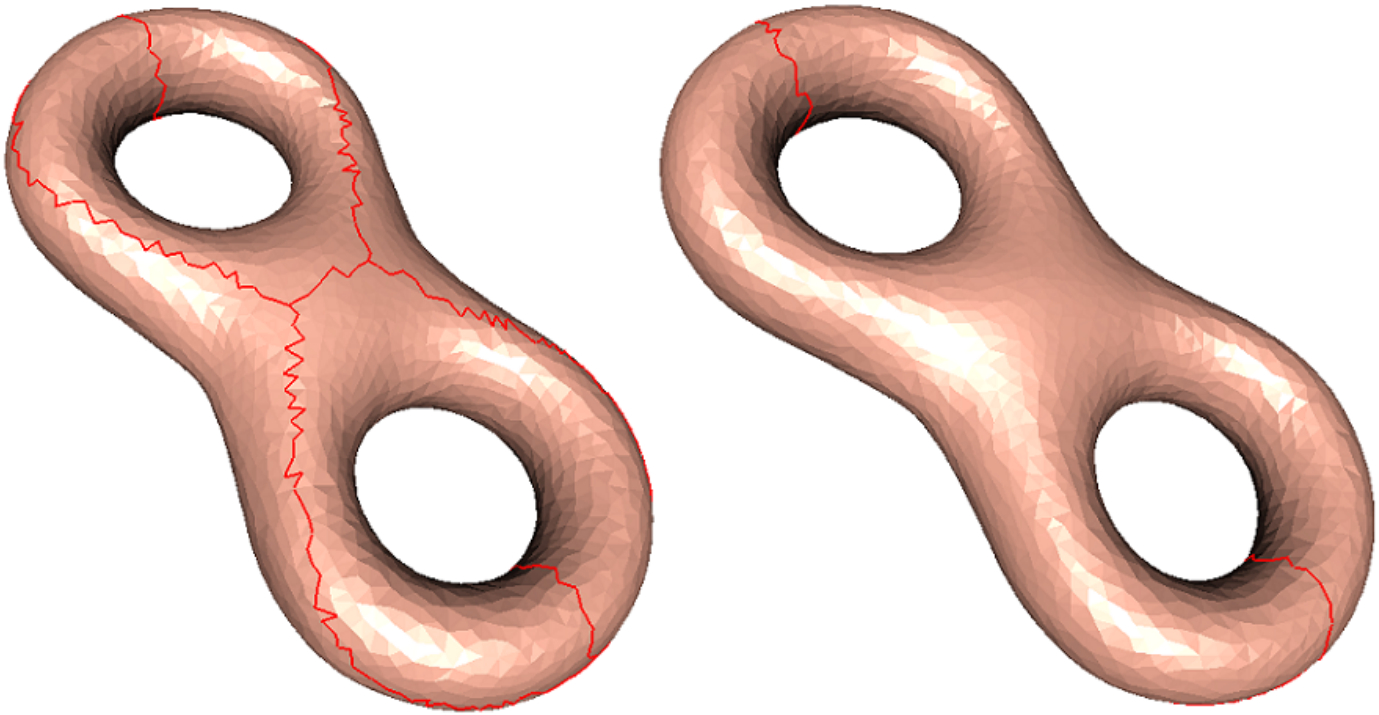


Figure: Cut graph of a genus two surface.

Definition (Cut Graph)

Γ is a graph on the surface S, such that $S \setminus \Gamma$ is a topological disk, then Γ is a cut graph of S.

Algorithm for Cut Graph

Cut Graph Algorithm

Input : A closed triangle mesh M;
Output: A cut graph Γ of M.

1. Compute the dual mesh $\bar{M}$ of the input mesh M;
2. Compute a spanning tree $\bar{T}$ of $\bar{M}$;
3. The cut graph is given by

$$\Gamma := \{e \in M | \bar{e} \notin \bar{T}\}.$$

Fundamental Group Generators

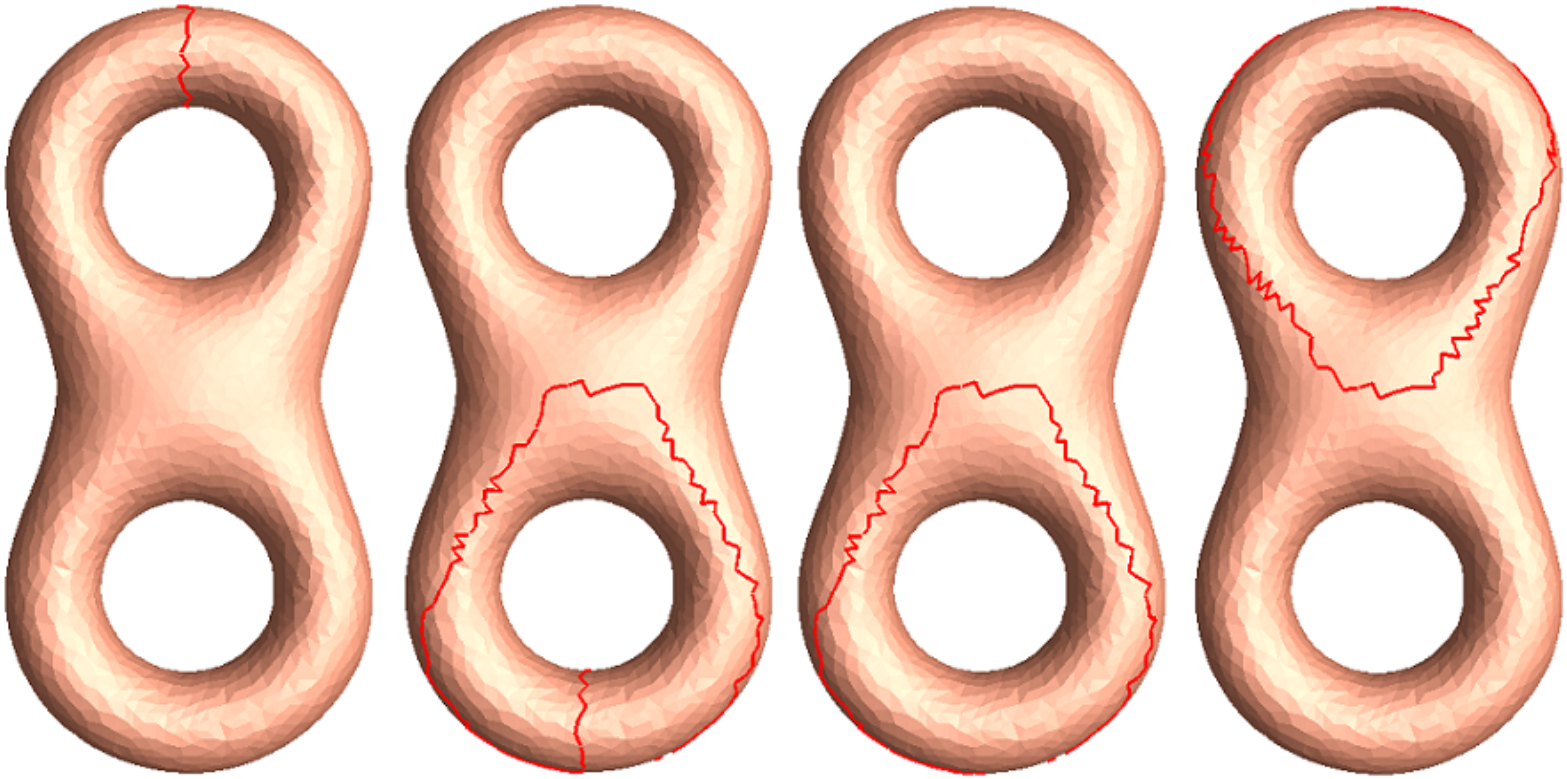


Figure: Foundamental group generators of a genus two surface.

Algorithm for Fundamental Group Generators

Fundamental Group Generators Algorithm

Input : A closed triangle mesh M;
Output: A set of generators of $\pi_1(M, p)$.

1. Compute a cut graph Γ of the input mesh M;
2. Compute a spanning tree T of Γ;
3. Select an edge $e_i \in \Gamma \setminus T$, $e_i \cup T$ has a unique loop γ_i;
4. $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$ is a set of generators of the fundamental group of M.

Algorithm for Fundamental Group Relations

Fundamental Group Relations Algorithm

Input : A closed triangle mesh M;
Output: The relations in $\pi_1(M, p)$.

1. Compute a cut graph Γ of the input mesh M;
2. Compute a spanning tree T of Γ, $\Gamma \setminus T = \{e_1, e_2, \cdots, e_k\}$;
3. For each oriented edge, $e_i \cup T$ has an oriented loop γ_i, $\{\gamma_1, \gamma_2, \cdots, \gamma_k\}$;
4. Cut the mesh M along Γ to obtain $\bar{M}$;
5. Set Let $\gamma = \partial \bar{M}$, traverse γ. Set $w = \emptyset$, once $e_i^{\pm 1}$ is encountered, append $\gamma_i^{\pm 1}$ to w, $w \leftarrow w\gamma_i^{\pm 1}$.

Algorithm for Universal Covering Space

Universal Covering Space Algorithm

Input : A closed triangle mesh M;
Output: A finite portion of the universal covering space $\tilde{M}$.

1. Compute a cut graph Γ of M, divide Γ into nodes and oriented segments, $\{s_1, s_2, \cdots, s_k\}$;
2. Slice M along Γ to obtain one fundamental domain $\bar{M}$;
3. Initialize $\tilde{M} \leftarrow \bar{M}$
4. Choose an oriented segment s_i on the boundary of $\tilde{M}$, glue a copy of $\bar{M}$ with $\tilde{M}$ along s_i,

$$\tilde{M} \leftarrow \tilde{M} \cup_{\partial\tilde{M} \supset s_i \sim s_i^{-1} \subset \partial\bar{M}} \bar{M}$$

5. Trace the boundary of $\tilde{M}$, if there are two adjacent segments $s_i, s_{i+1} \subset \partial\tilde{M}$, such that $s_i^{-1} = s_{i+1}$, then glue them together;
6. Repeat step 4 and step 5, until $\tilde{M}$ is large enough.

Algorithm for Homotopy Detection

Homotopy Detection Algorithm

Input : A closed triangle mesh M, two loops γ_1 and γ_2 through a base point p;
Output: Verify whether $\gamma_1 \sim \gamma_2$.

1. Compute a finite portion of the universal covering space $\tilde{M}$ of M;
2. Lift $\gamma_1 \cdot \gamma_2^{-1}$ to $\tilde{M}$, the lifted path is $\tilde{\gamma}$;
3. If $\tilde{\gamma}$ is a closed loop, then return Yes; otherwise, return No.

Persistent Homology: Combinatorial Pairing Algorithm

Pair(σ)

1. $c = \partial_p \sigma$
2. τ is the youngest positive $(p-1)$-simplex in c.
3. **while** τ is paired and c is not empty **do**
4. find (τ, d), d is the p-simplex paired with τ;
5. $c \leftarrow \partial_p d + c$
6. Update τ to be the youngest positive $(p-1)$-simplex in c
7. **end while**
8. **if** c is not empty **then**
9. σ is negative p-simplex and paired with τ
10. **else**
11. σ is a positive p-simplex
12. **endif**

Persistent Homology: Algebraic Pairing Algorithm

Algorithm: Incidence matrix reduction

1. $R \leftarrow D$
2. **for** $j = 1$ **to** n **do**
3. **while** $\exists j' < j$ **with** $\mathrm{low}_R(j') = \mathrm{low}_R(j)$ **do**
4. add column j' to column j
5. **endwhile**
6. **endfor**.

The pairing is given by

$$\boxed{(\sigma_i, \sigma_j) \iff i = \mathrm{low}_R(j)}.$$

σ_i is positive, it generates a homology class; σ_j is negative, it kills a homolog class.

Algorithm for Handle Loop and Tunnel Loop

1. The simplices on the surface M are added into the filtration in any arbitrary order. Since $H_1(M)$ is of rank $2g$, the algorithm Pair generates $2g$ number of unpaired positive edges.
2. The simplices up to dimension 2 in I are added into the filtration. Since $H_1(I)$ of rank g, half of $2g$ positive edges generated in step 1 get paired with the negative triangles in I. Each pair correponds to a killed loop, these g loops are handle loops.
3. Or the simplices up to dimension 2 in O are added into the filtration. Since $H_1(O)$ of rank g, half of $2g$ positive edges genrated in step 2 get paried with the negative triangles in O. Each pair corresponds to a killed loop, these g loops are tunnel loops.

Algorithm forHomology Basis

Cominatorial Laplace Operator

Construct linear operator $\Delta_k : C_k \to C_k$,

$$\Delta_k := \partial_k^T \partial_k + \partial_{k+1} \partial_{k+1}^T,$$

the eigen vectors of zero eigen values of Δ_k form the basis of $H_k(M, \mathbb{Z})$.

Smith Norm

The eigen vectors can be found using Smith norm of integer matrix. The computational cost is very high.

Algorithm for Cohomology Group

Algorithm for $H^1(M, \mathbb{R})$

Input: A genus g closed triangle mesh M;
Output: A set of basis of $H^1(M, \mathbb{R})$

1. Compute a set of basis of $H_1(M, \mathbb{Z})$, denoted as
$$\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\},$$
2. for each γ_i, slice M along $gamma_i$, to obtain a mesh with two boundaries M_i, $\partial M_i = \gamma_i^+ - \gamma_i^-$;
3. set a 0-form τ_i on M_i, such that $\tau_i(v) = 1$ for all $v \in \gamma_i^+$ and $\tau_i(w) = 0$, for all $w \in \gamma_i^-$; set $\omega_i = d\tau_i$;
4. All $\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$ form a basis of $H^1(M, \mathbb{R})$.

Application

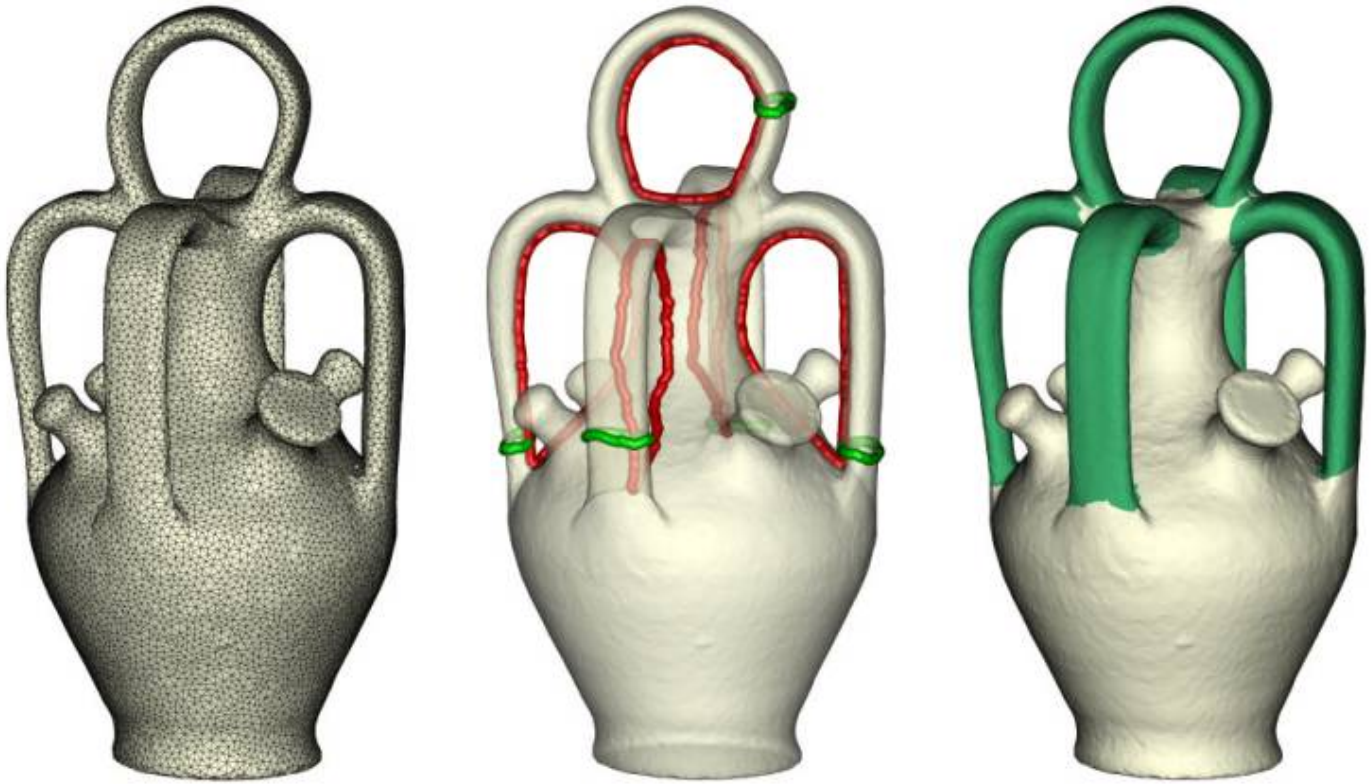


Figure: Handle detection by finding the handle loops and the tunnel loops.

Application

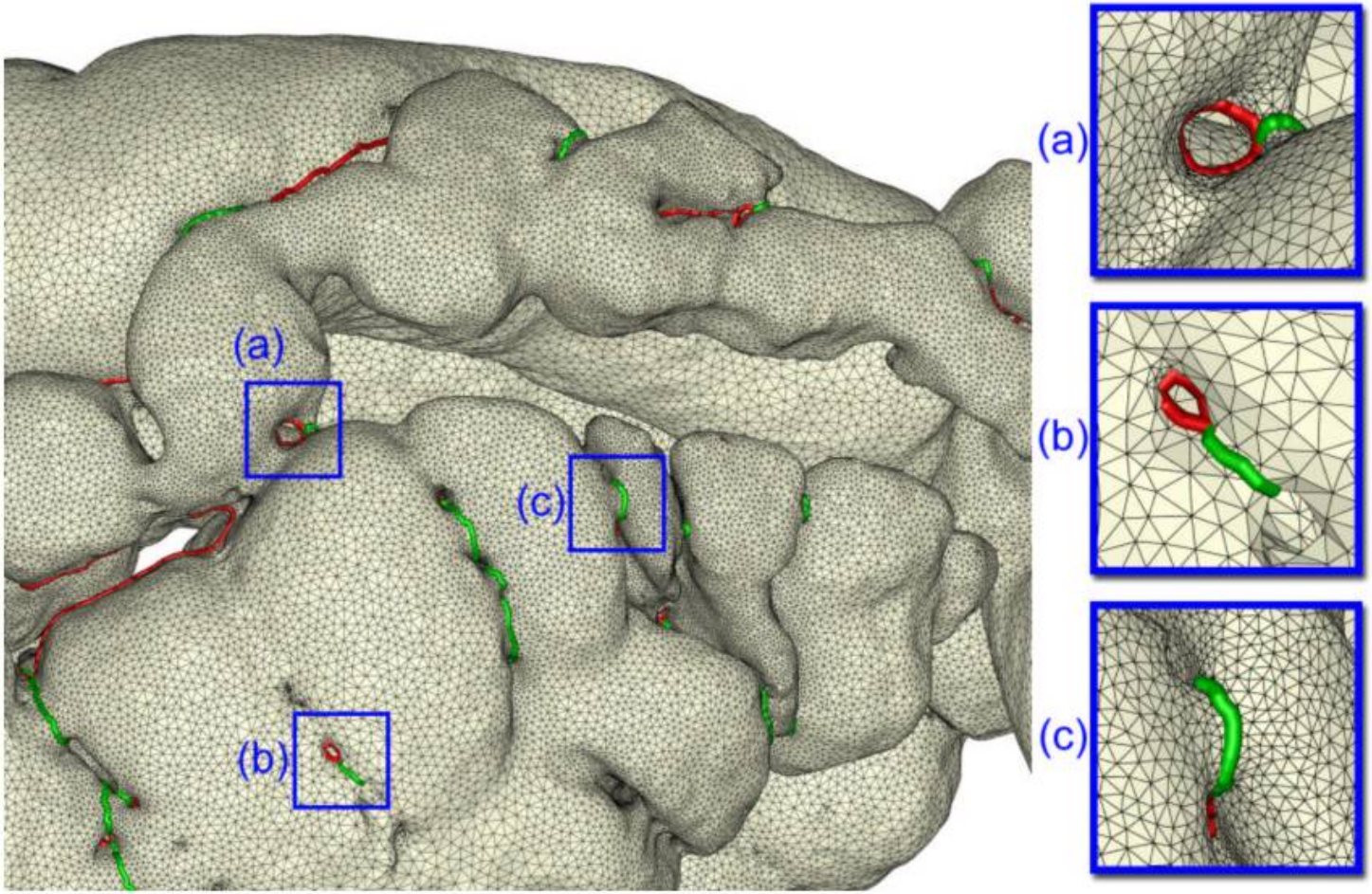
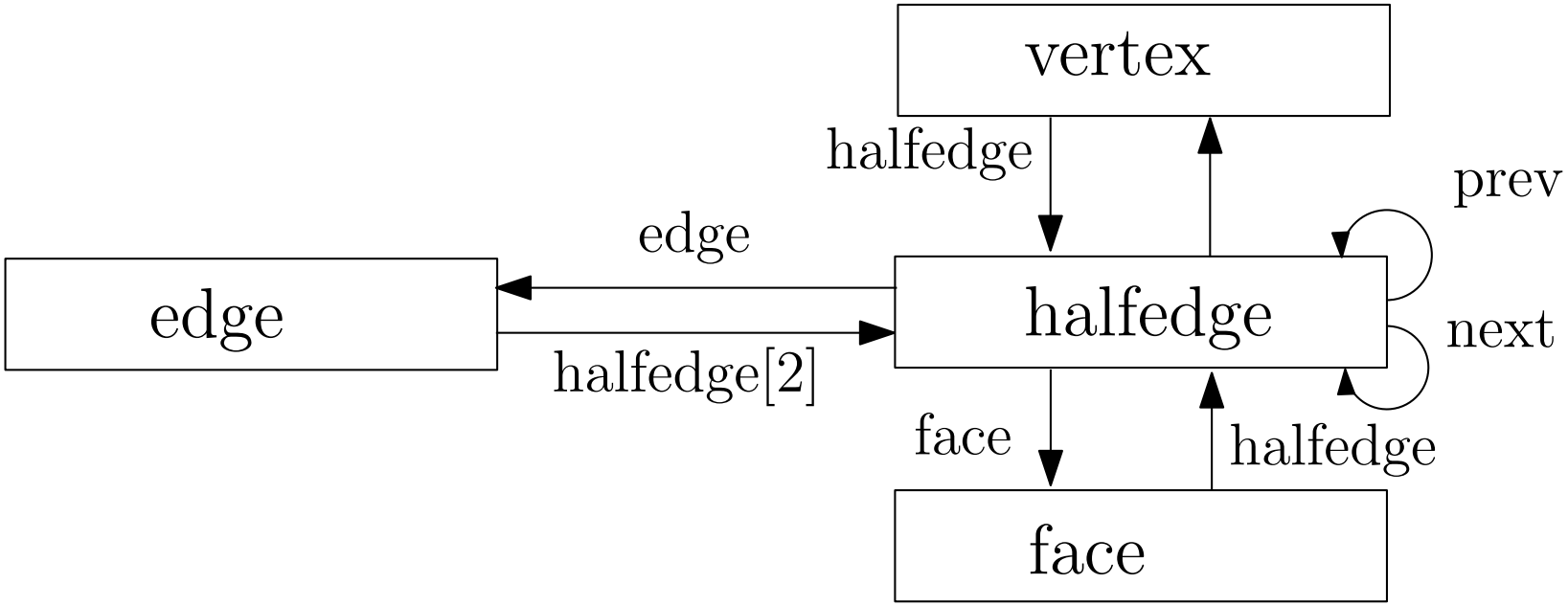


Figure: Topological Denoise in medical imaging.

Assignment One - Halfedge Data Structure



Assignment One- Cut Graph

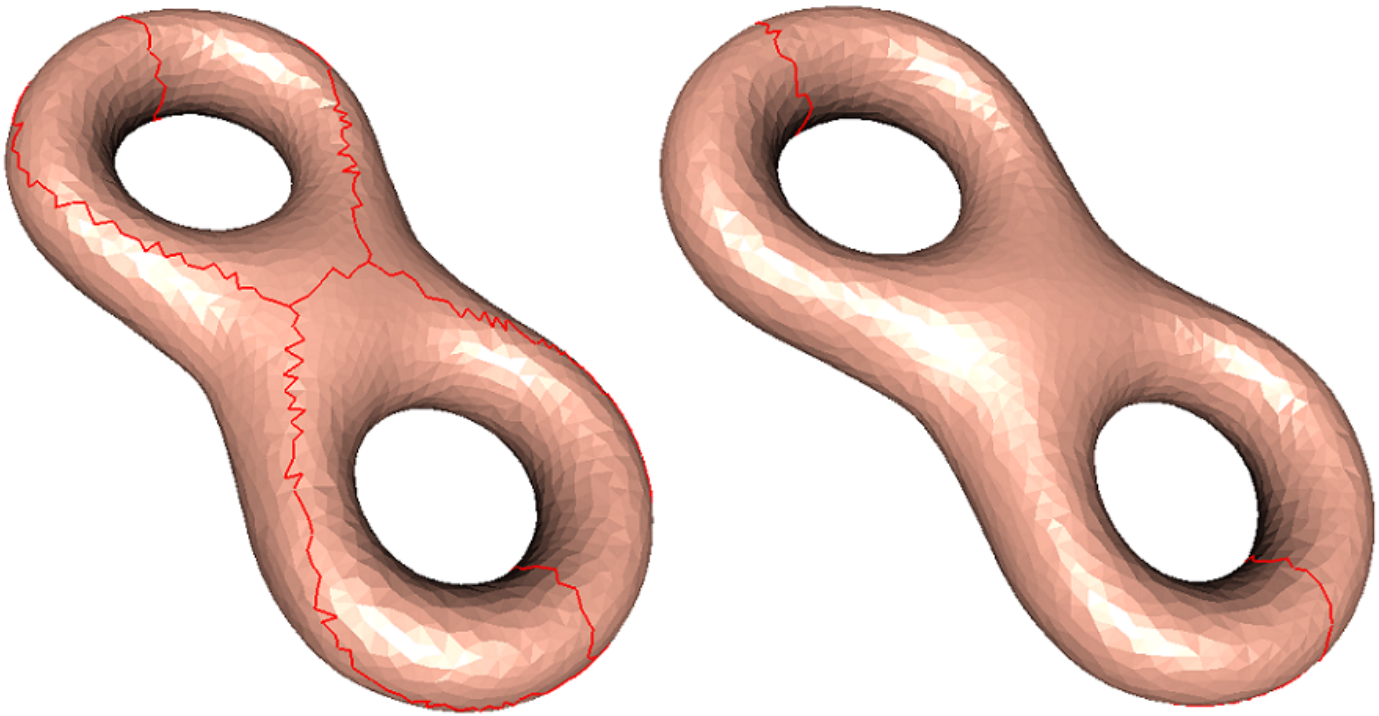


Figure: Cut graph of a genus two surface.

Compute cut graph for high genus surfaces.

Assignment Seven - Dart Data Structure

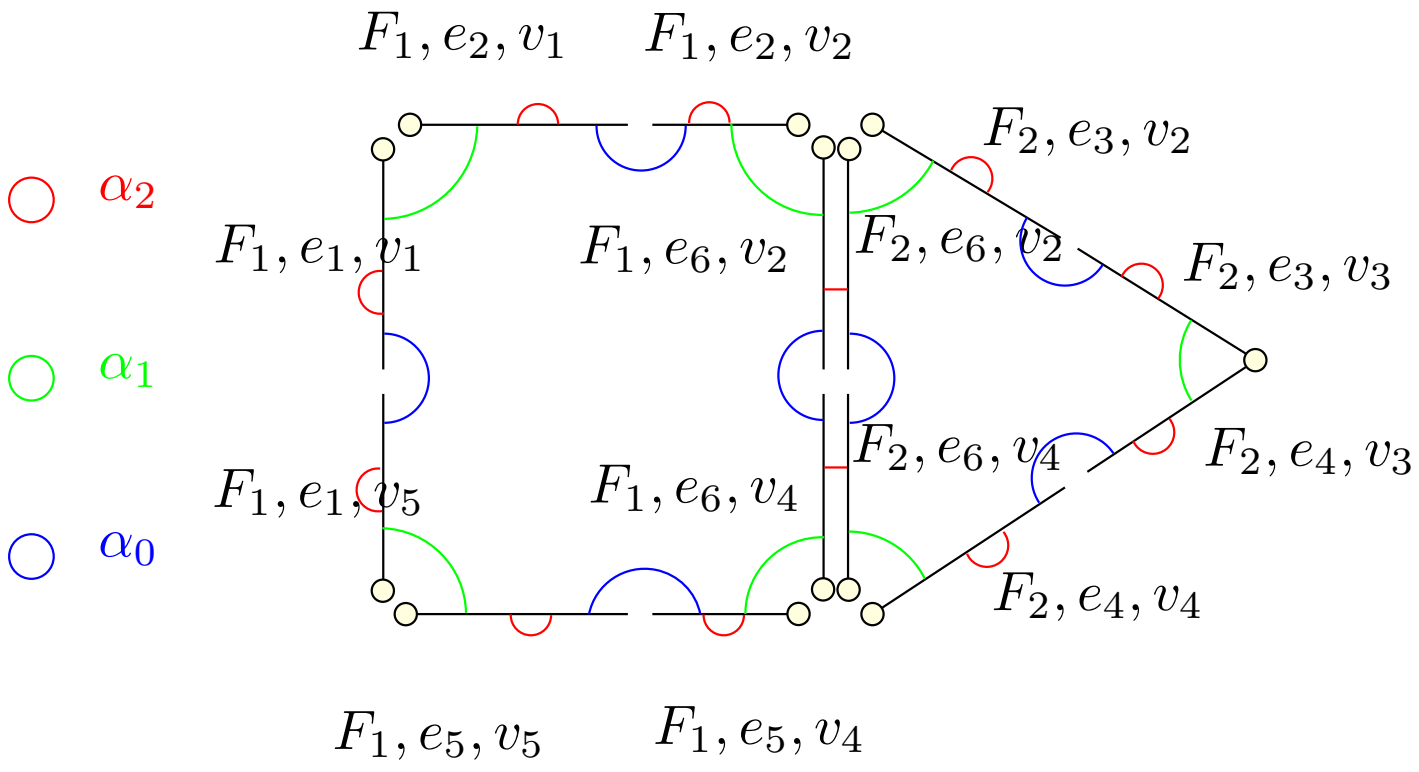


Figure: Combinatorial map, $\alpha_0, \alpha_1, \alpha_2$ for a dart (F, E, V).

Assignment Seven - Handle Loops and Tunnel Loops

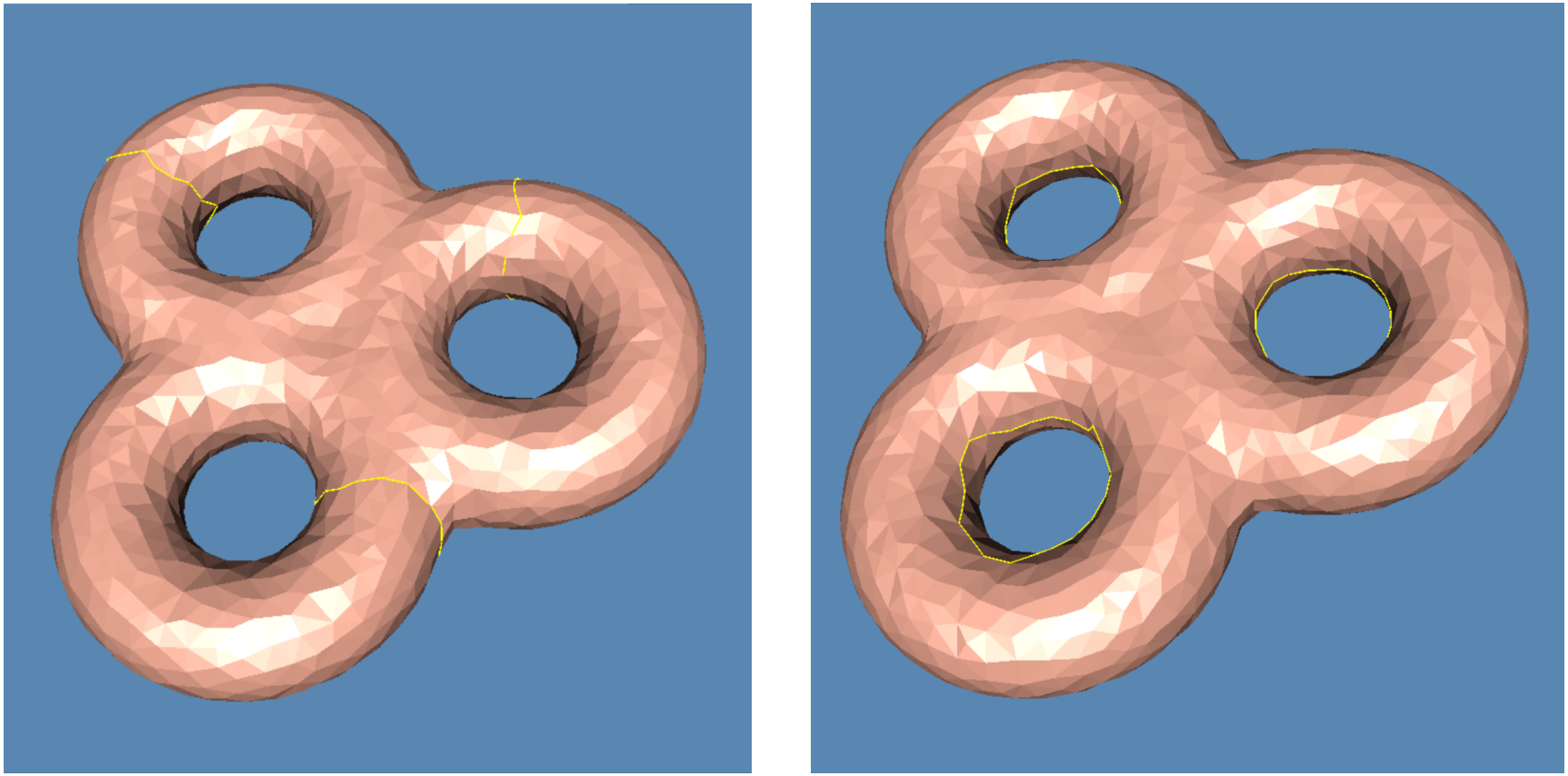


Figure: Handle and tunnel loops.

Assignment Seven - Handle Loops and Tunnel Loops

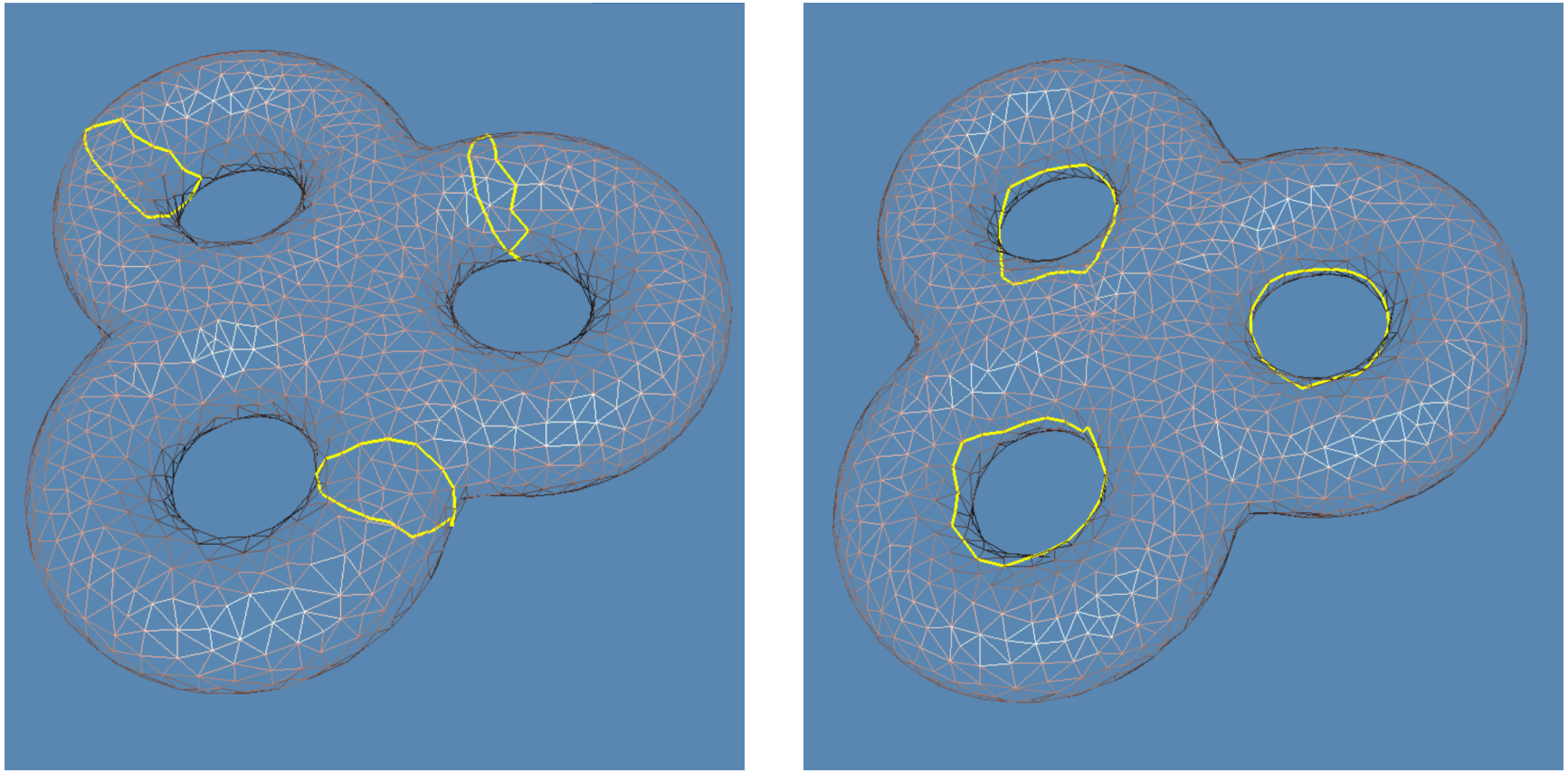


Figure: Handle and tunnel loops.

Assignment Seven - Handle Loops and Tunnel Loops

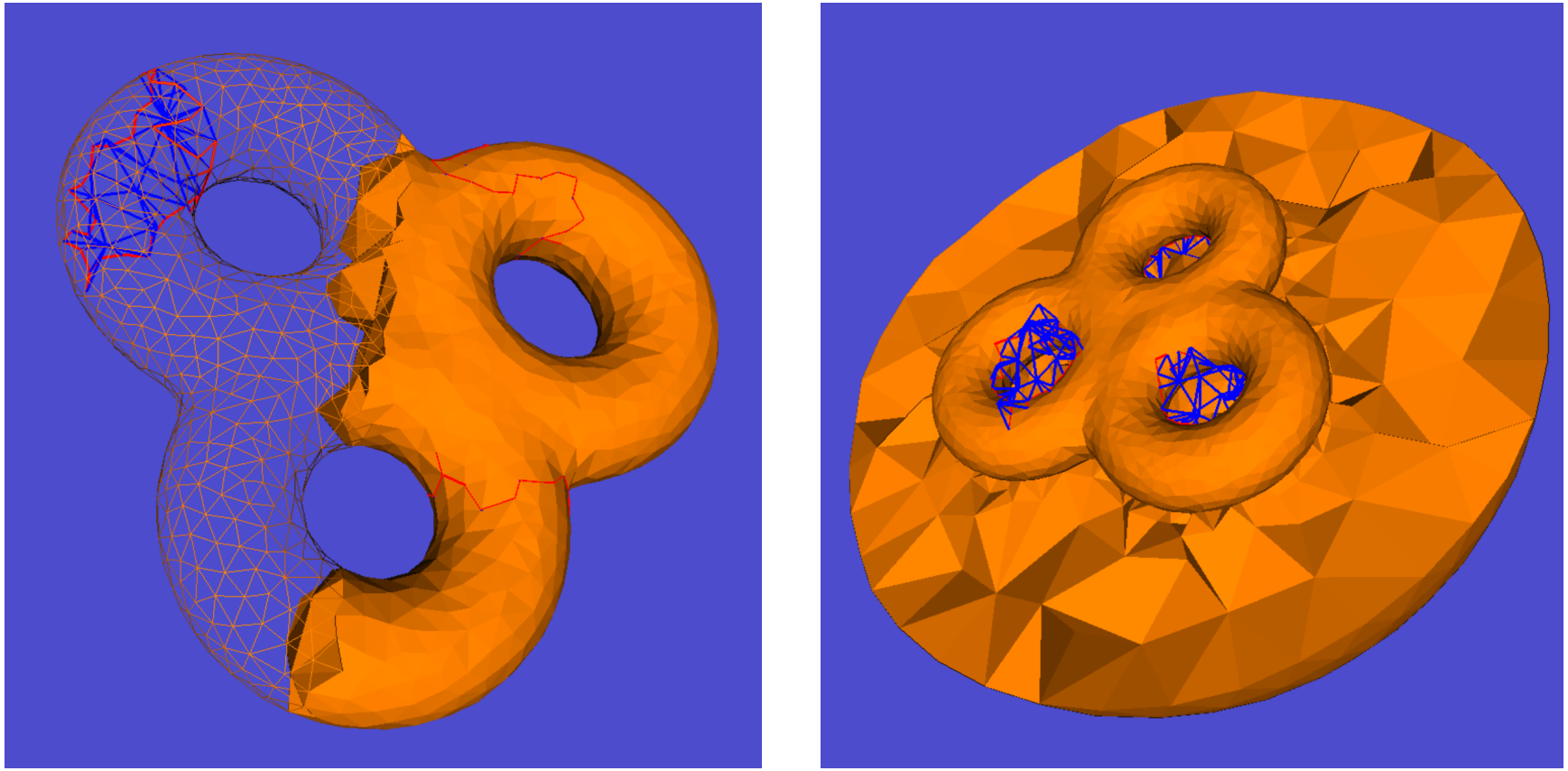


Figure: Interior and exterior volumes.

Differential Topology

Smooth Manifold

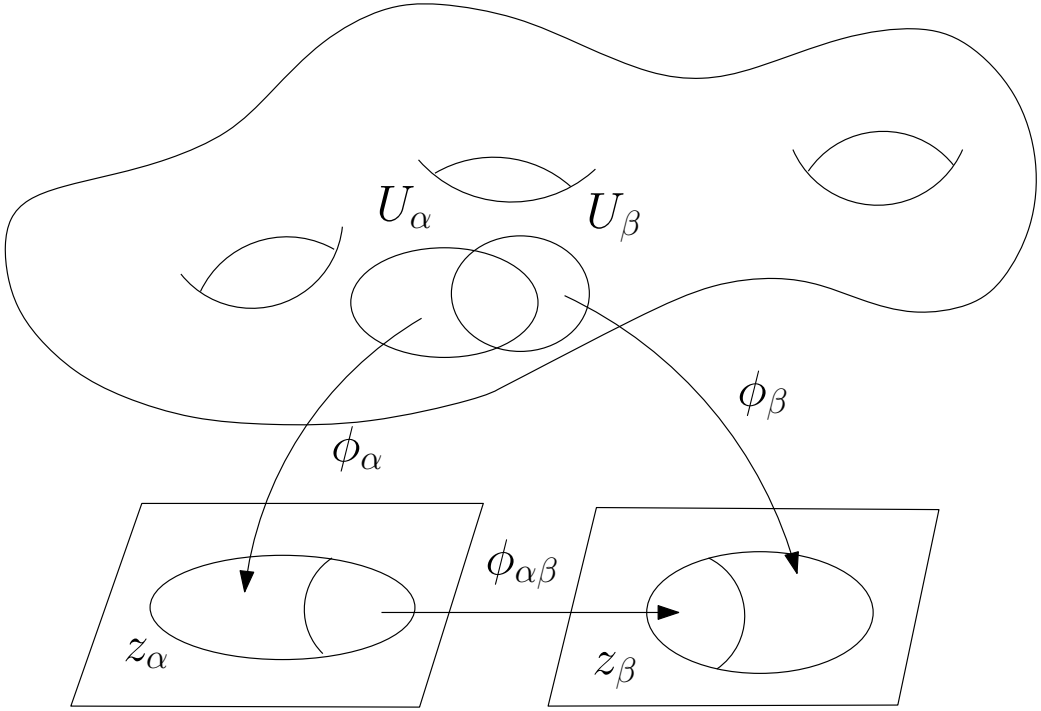


Figure: A manifold.

Smooth Manifold

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

If all transition maps $\phi_{\alpha\beta} \in C^\infty(\mathbb{R}^n)$ are smooth, then the manifold is a differential manifold or a smooth manifold.

Tangent Space

Definition (Tangent Vector)

A tangent vector ξ at the point p is an association to every coordinate chart $(x^1, x^2, \cdots, x^n)$ at p an n-tuple $(\xi^1, \xi^2, \cdots, \xi^n)$ of real numbers, such that if $(\tilde{\xi}^1, \tilde{\xi}^2, \cdots, \tilde{\xi}^n)$ is associated with another coordinate system $(\tilde{x}^1, \tilde{x}^2, \cdots, \tilde{x}^n)$, then it satisfies the transition rule

$$\tilde{\xi}^i = \sum_{j=1}^{n} \frac{\partial \tilde{x}^i}{\partial x^j}(p)\xi^j.$$

A smooth vector field ξ assigns a tangent vector for each point of M, it has local representation

$$\xi(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \xi_i(x^1, x^2, \cdots, x^n)\frac{\partial}{\partial x_i}.$$

$\{\frac{\partial}{\partial x_i}\}$ represents the vector fields of the velocities of iso-parametric curves on M. They form a basis of all vector fields.

Poincaré-Hopf

Theorem (Poincaré-Hopf Index)

Assume S is a compact, oriented smooth surface, v is a smooth tangent vector field with isolated zeros. If S has boundaries, then v point along the exterior normal direction, then we have

$$\sum_{p \in Z(v)} Index_p(v) = \chi(S),$$

where $Z(v)$ is the set of all zeros, $\chi(S)$ is the Euler characteristic number of S.

Unit Tangent Bundle

Definition (UTM)

The unit tangent bundle of the unit sphere is the manifold

$$UTM(S) := \{(p, v) | p \in S, v \in T_p(S), |v|_{\mathbf{g}} = 1\}.$$

The unit tangent bundle of a surface is a 3-dimensional manifold. We want to compute its triangulation and its fundamental group.

Obstruction Class

The topological obstruction for the existence of global section $\varphi : \mathbb{S}^2 \to UTM(\mathbb{S}^2)$ is constructed as follows:

1. Construct a triangulation $\mathcal{T}$, which is refined enough such that the fiber bundle of each face is trivial (direct product).
2. For each vertex v_i, choose a point on its fiber, $\varphi(v_i) \in F(v_i)$
3. For each edge $[v_i, v_j]$, choose a curve connecting $\varphi(v_i)$ and $\varphi(v_j)$ in the restiction of the UTM on $[v_i, v_j]$, which is annulus;
4. For each face Δ, $\varphi(\partial\Delta)$ is a loop in the fiber bundle of Δ, $[\varphi(\partial\Delta)]$ is an integer, an element in $\pi_1(UTM(\Delta))$, this gives a 2-form Ω on the original surface M,
$$\Omega(\Delta) = [\varphi(\partial\Delta)].$$
5. If Ω is zero, then global section exists. Otherwise doesn't exists.
6. Different constructions get different Ω's, but all of them are cohomological. Therefore $[\Omega] \in H^2(M, \mathbb{R})$ is the characteristic class of fiber bundle.

differential forms

Definition (Differential 1-form)

The tangent space T_pM is an n-dimensional vector space, its dual space T_p^*M is called the cotangent space of M at p. Suppose $\omega \in T_p^*M$, then $\omega : T_pM \to \mathbb{R}$ is a linear function defined on T_pM, ω is called a differential 1-form at p.

A differential 1-form field has the local representation

$$\omega(x^1, x^2, \cdots, x^n) = \sum_{i=1}^{n} \omega_i(x^1, x^2, \cdots, x^n) dx_i,$$

where $\{dx_i\}$ are the differential forms dual to $\{\frac{\partial}{\partial x_j}\}$, such that

$$\langle dx_i, \frac{\partial}{\partial x_i} \rangle = dx_i \left(\frac{\partial}{\partial x_j} \right) = \delta_{ij}.$$

Stokes Theorem

Theorem (Stokes)

let M be an n-manifold with boundary ∂M and ω be a differentialble $(n-1)$-form with compact support on M, then

$$\int_{\partial M} \omega = \int_M d\omega.$$

de Rham Cohomology

Definition (de Rham Cohomology)

Assume Σ is a differntial manifold, then de Rham complex is

$$\Omega^0(\Sigma,\mathbb{R}) \xrightarrow{d^0} \Omega^1(\Sigma,\mathbb{R}) \xrightarrow{d^1} \Omega^2(\Sigma,\mathbb{R}) \xrightarrow{d^2} \Omega^3(\Sigma,\mathbb{R}) \xrightarrow{d^3} \cdots$$

$$H^k_{dR}(\Sigma,\mathbb{R}) := \frac{\operatorname{Ker} d^k}{\operatorname{Img} d^{k-1}}$$

Theorem

The de Rham cohomology group $H^m_{dR}(M)$ is isomorphic to the cohomology group $H^m(M,\mathbb{R})$

$$H^m_{dR}(M) \cong H^m(M,\mathbb{R}).$$

Hodge Star Operator - First Definition

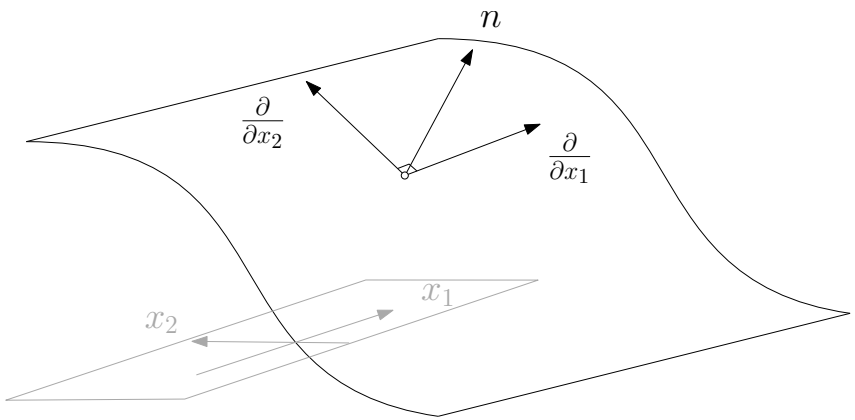
Suppose M is a Riemannian manifold, we can locally find oriented orthonormal basis of vector fields, and choose parameterization, such that

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right\}$$

form an oriented orthonormal basis. let

$$\{dx_1, dx_2, \cdots, dx_n\}$$

be the dual 1-form basis.



Hodge star operator

Definition (Hodge Star Operator)

The Hodge star opeartor $^* : \Omega^k(M) \to \Omega^{n-k}(M)$ is a linear operator

$$^*(dx_1 \wedge dx_2 \wedge \cdots \wedge dx_k) = dx_{k+1} \wedge dx_{k+2} \wedge \cdots \wedge dx_n.$$

Hodge Star Operator

Let $\sigma = (i_1, i_2, \cdots, i_n)$ be a permutation, then the hoedge star operator

$$* \left(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}\right) = (-1)^\sigma dx_{i_{k+1}} \wedge dx_{i_{k+2}} \wedge \cdots \wedge dx_{i_n}.$$

Codifferential operator

Definition

The codifferential operator $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined as

$$\delta = (-1)^{kn+n+1} * d^*,$$

where d is the exterior derivative.

Lemma

The codifferential is the adjoint of the exterior derivative, in that

$$(\delta\zeta, \eta) = (\zeta, d\eta).$$

Laplace Operator

Definition (Laplace Operator)

The Laplace operator $\Delta : \Omega^k(M) \to \Omega^k(M)$,

$$\Delta = d\delta + \delta d.$$

Lemma

The Laplace operator is symmetric

$$(\Delta\zeta, \eta) = (\zeta, \Delta\eta)$$

and non-negative

$$(\Delta\eta, \eta) \geq 0.$$

Proof.

$$(\Delta\zeta, \eta) = (d\zeta, d\eta) + (\delta\zeta, \delta\eta).$$

□

Harmonic Forms

Definition (Harmonic forms)

Suppose $\omega \in \Omega^k(M)$, then ω is called a k-harmonic form, if

$$\Delta\omega = 0.$$

Lemma

ω is a harmonic form, if and only if

$$d\omega = 0, \delta\omega = 0.$$

Proof.

$$0 = (\Delta\omega, \omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

□

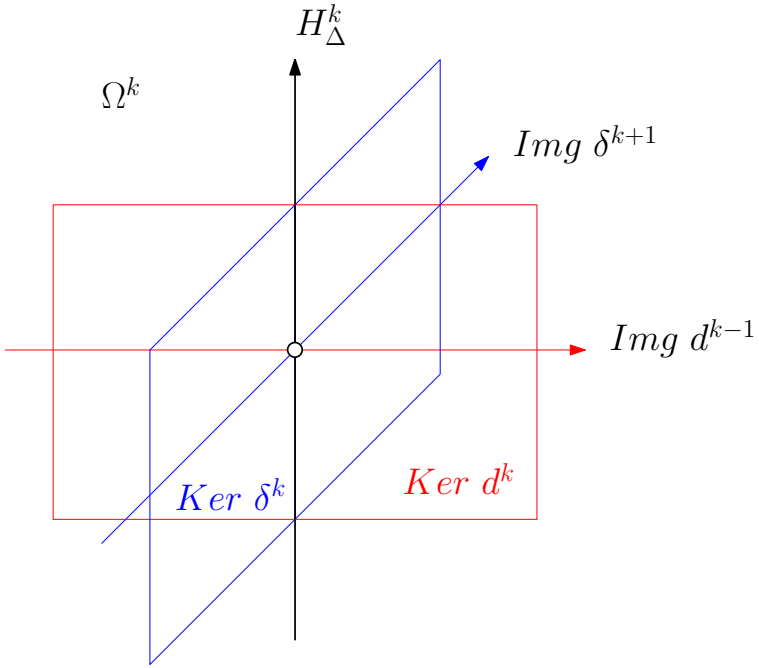
Hodge Decomposition

Definition (Harmonic form group)

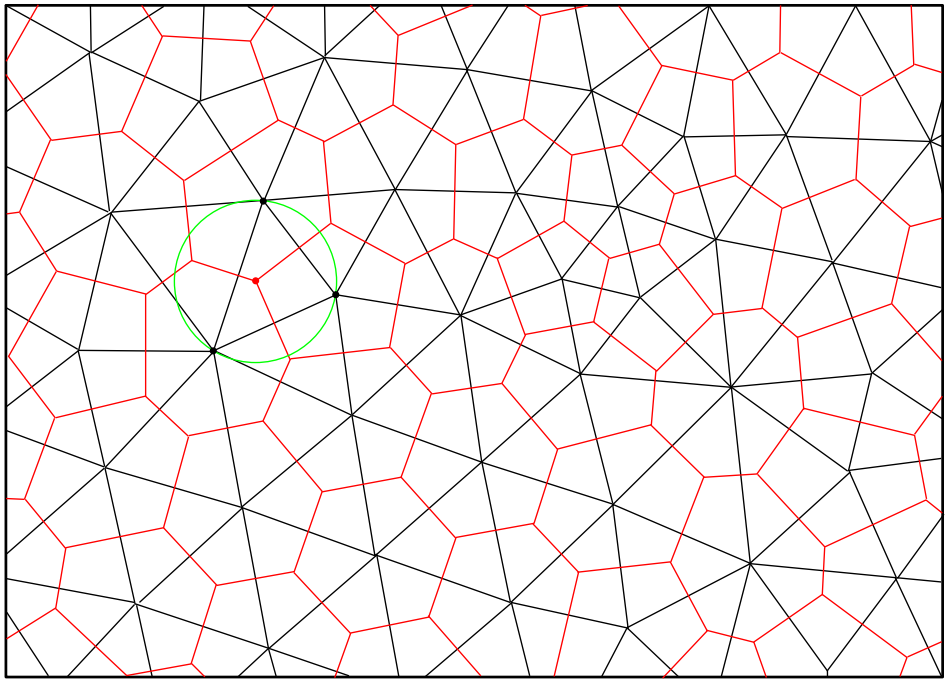
All harmoic k-forms form a group, denoted as $H^k_\Delta(M)$.

Theorem (Hodge Decomposition)

$$\Omega_k = img d^{k-1} \bigoplus img \delta^{k+1} \bigoplus H^k_\Delta(M).$$



Algorithm - Dual Mesh



Poincaré's duality, equivalent to Delaunay triangulation and Voronoi diagram. The Delaunay triangulation is the primal mesh, the Voronoi diagram is the dual mesh.

Algorithm for Random Harmonic One-form

Input:A closed genus one mesh M;
output: A basis of harmonic one-form group;

1. Generate a random one form ω, assign each $\omega(e)$ a random number;
2. Compute cotangent edge weight;
3. Compute the coexact form δF;
4. Compute the exact form df;
5. Harmonic 1-form is obtained by $h = \omega - d\eta - \delta\Omega$;

Algorithm for Holomorphic 1-form Basis

Input: A set of harmonic 1-form basis $\omega_1, \omega_2, \ldots, \omega_{2g}$;
Output: A set of holomorphic 1-form basis $\omega_1, \omega_2, \ldots, \omega_{2g}$;

1. Compute the integration of the wedge of ω_i and ω_j, $\int_M \omega \wedge \omega_j$;
2. Compute the integration of the wedge of ω_i and ${}^*\omega_j$, $\int_M \omega \wedge {}^*\omega_j$;
3. Solve linear equation group, obtain the linear combination coefficients, get conjugate harmonic 1-forms, ${}^*\omega_i = \sum_{j=1}^{2g} \lambda_{ij}\omega_j$
4. Form the holomorphic 1-form basis $\{\omega_i + \sqrt{-1}{}^*\omega_i, \quad i = 1, 2, \ldots, 2g\}$.

Algorithm - Exact Harmonic One-form on Annulus

Input: A topological annulus M;
Output: Exact harmonic one-form ω;

1. Trace the boundary of the mesh $\partial M = \gamma_0 - \gamma_1$;
2. Set boundary condition:

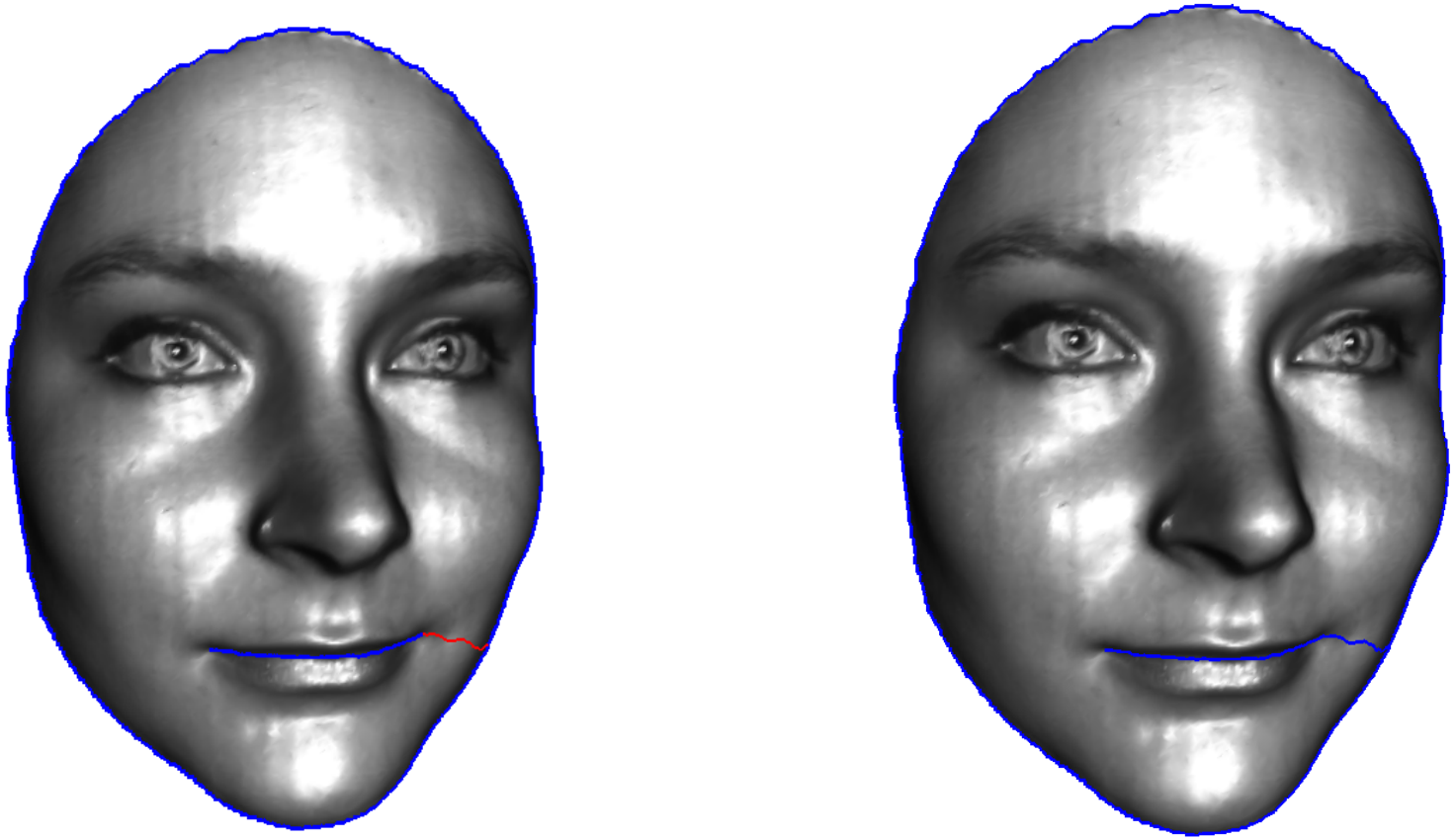
$$f|_{\gamma_0} = 0, \quad {}_{\gamma_1} = -1;$$

3. Compute cotangent edge weight;
4. Solve Laplace equation $\Delta f \equiv 0$ with Dirichlet boundary condition, for all interior vertex,

$$\sum_{v_i \sim v_j} w_{ij}(f_j - f_i) = 0;$$

5. $\omega = df$.

Algorithm for Topological Fundamental Domain



Find the shortest path τ connecting γ_0 and γ_1, slice the mesh along τ to get a topological disk $\bar{M}$.

Algorithm - Holomorphic One-Form on Annulus

holomorphic 1-form

Input: A topological annulus M;
Output: A holomorphic 1-form on M;

1. Use the algorithm for random harmonic One-form algorithm to compute a closed but non-exact harmonic one-form ω_1;
2. Use holomorphic 1-form basis algorithm with $\{\omega, \omega_1\}$ as input to compute a holomorphic 1-form $\omega + \sqrt{-1}^*\omega$.

Algorithm - Integration

Input: A topological disk $\bar{M}$, a holomorphic 1-form;
Output: Integration

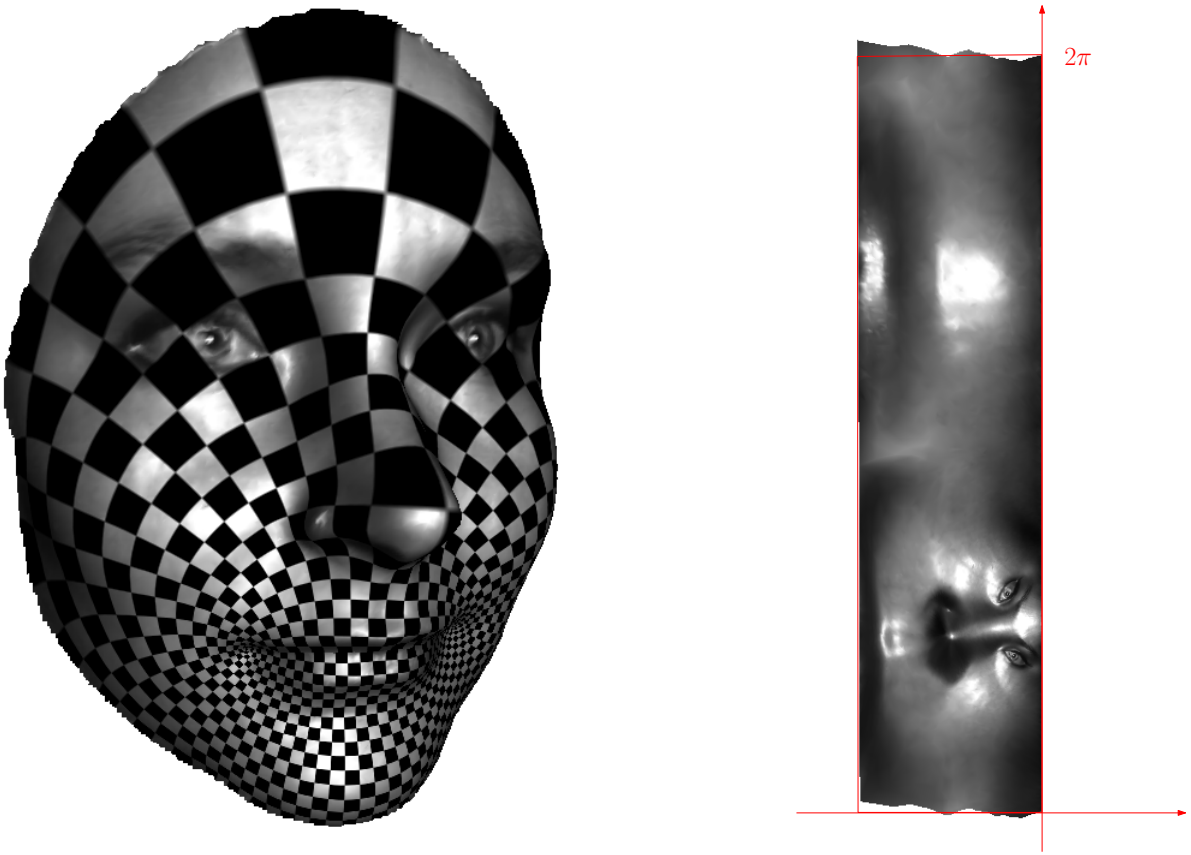
$$\varphi(q) := \int_p^q \omega + \sqrt{-1}^* \omega$$

1. Choose a base point p, set $\varphi(p) = (0, 0)$. $p \to touched() = true$, put p to the queue Q;
2. while Q is non-empty, $v_i \leftarrow Q.pop()$;
3. for each adjacent vertex $v_j \sim v_i$, if v_j hasn't been touched, $v_j \to touched() = true$, enqueue v_j to Q;

$$\varphi(v_j) = \varphi(v_i) + (\omega, {}^*\omega)([v_i, v_j]);$$

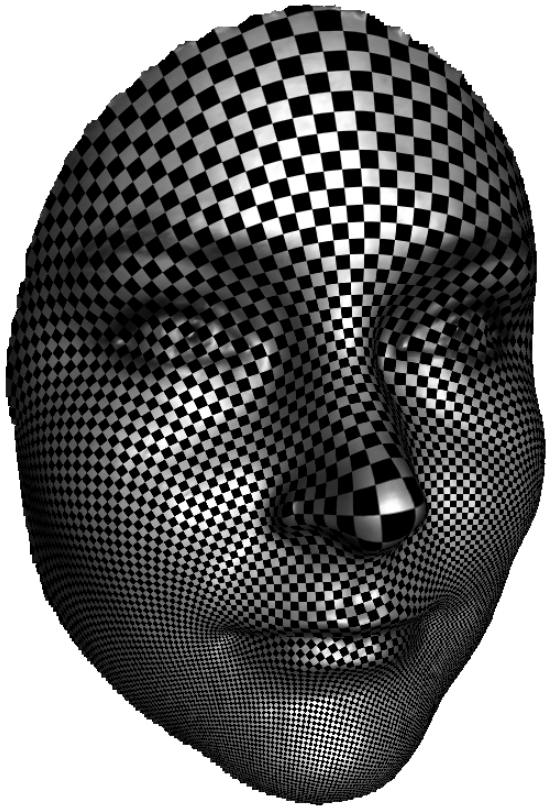
4. repeat step 3,4 until all vertices have been touched.

Algoritm - Integration



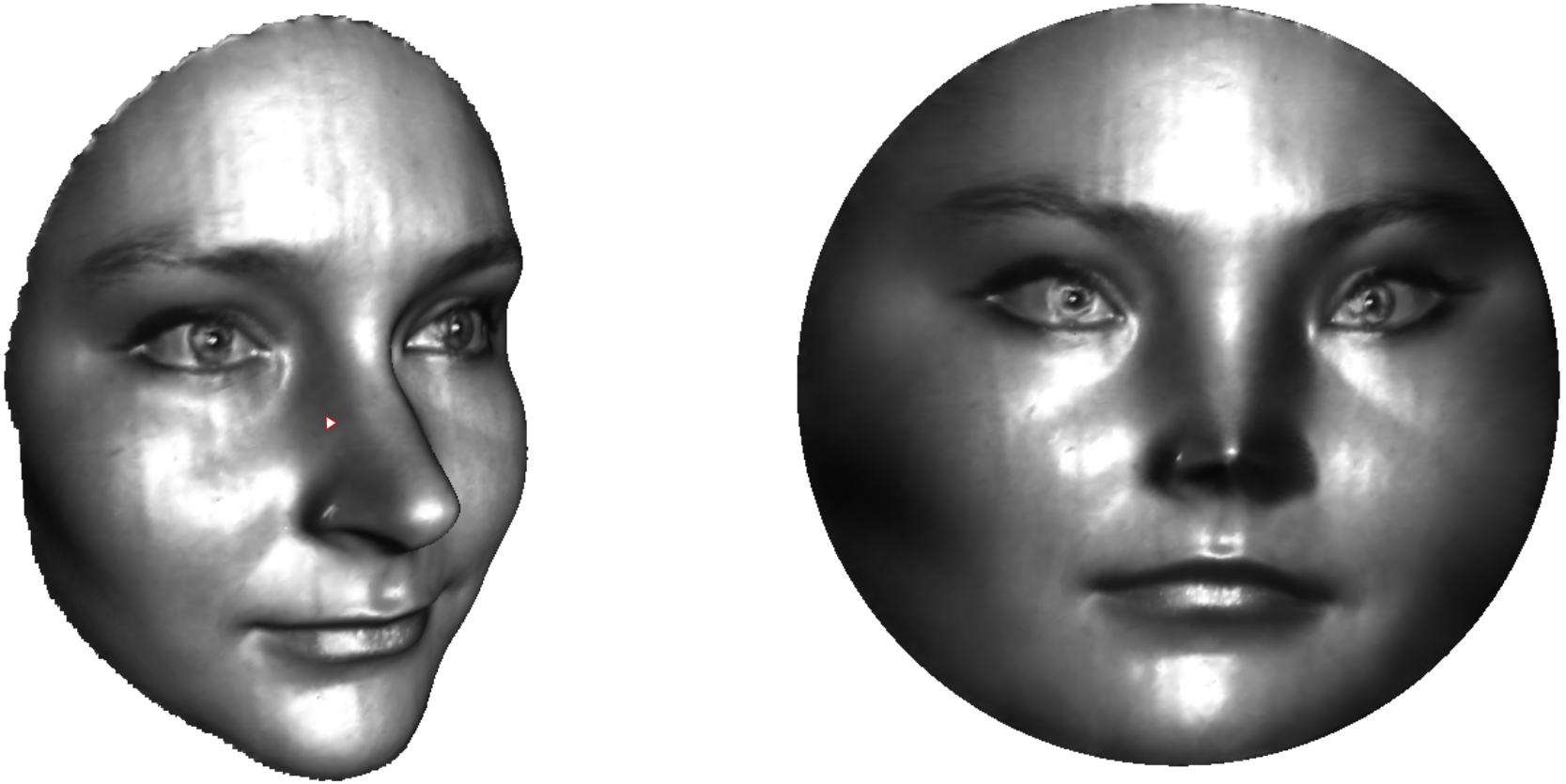
Integrating $\omega + \sqrt{-1}^*\omega$ on $\bar{M}$, normalize the rectangular image $\varphi(\bar{M})$, such that $\varphi(\gamma_0)$ is along the imaginary axis, the height is 2π, $\varphi(\gamma_1)$ is $x = -c$, $c > 0$ is a real number.

Algorithm - Polar map



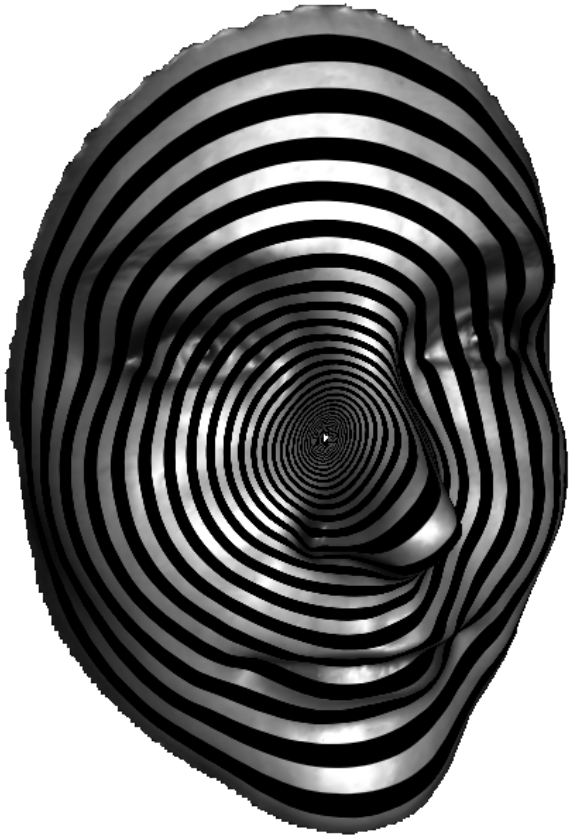
Compute the polar map e^{φ}, which maps $\varphi(\bar{M})$ to an annulus.

Algorithm - Riemann Mapping

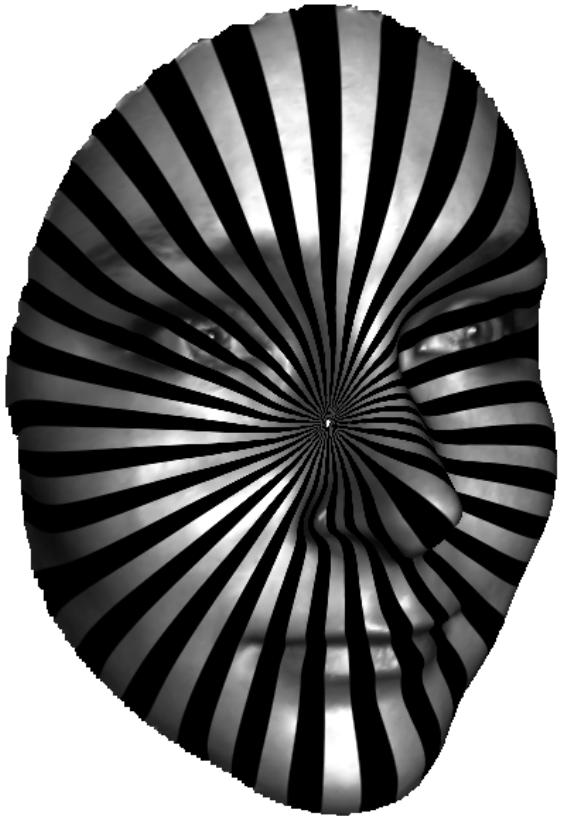


Step 1. Riemann mapping can be obtained by puncturing a small hole on the surface, then use topological annulus conformal mapping algorithm.

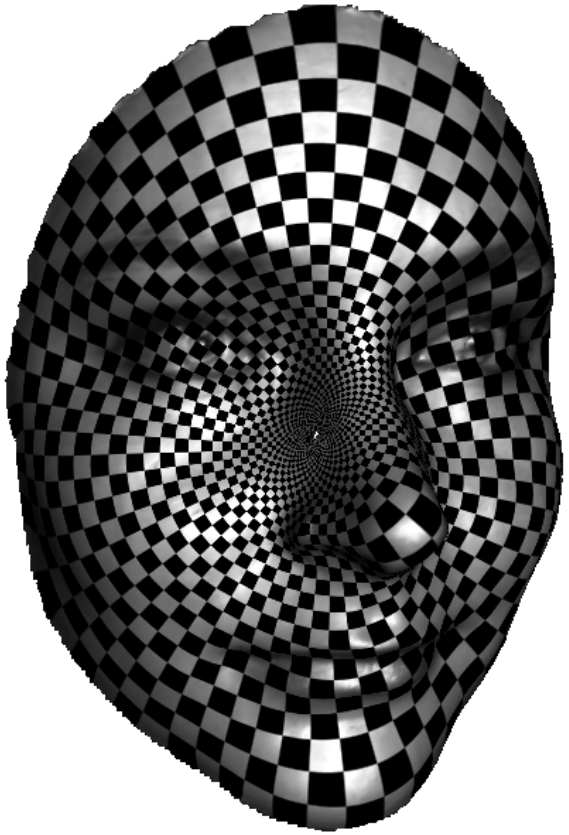
Algorithm - Riemann Mapping



ω



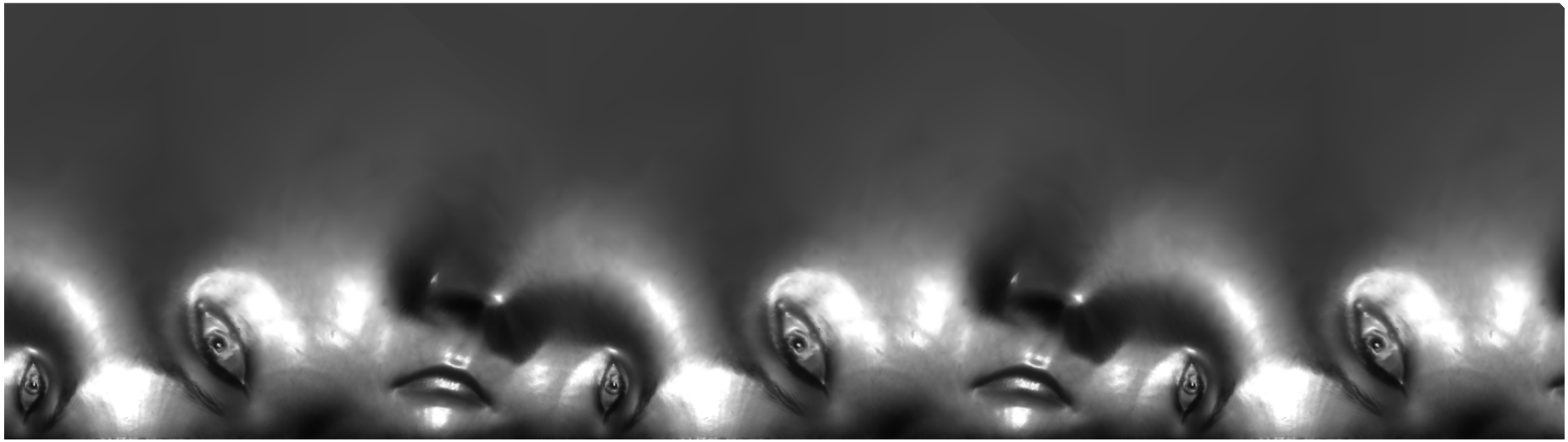
$^*\omega$



$\omega + \sqrt{-1}^*\omega$

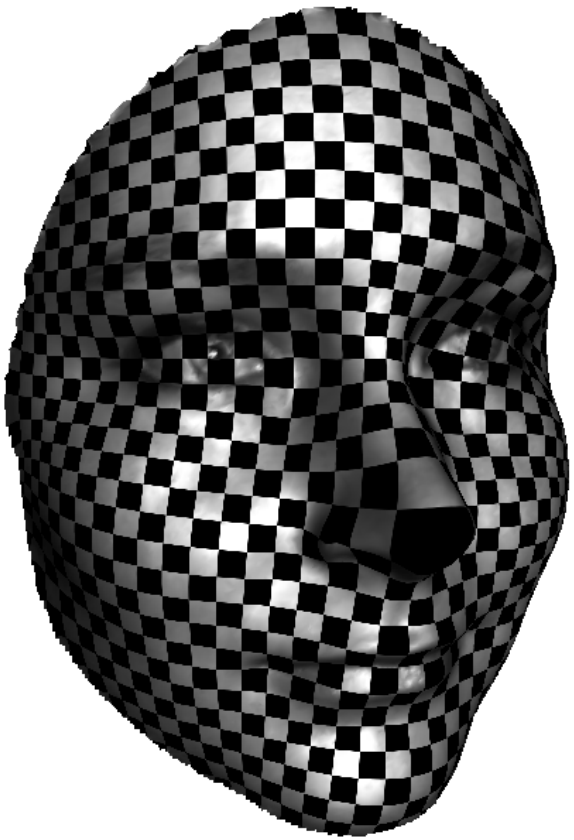
Step 2. Compute exact harmonic 1-form and closed, non-exact harmonic 1-form and holomorphic 1-form.

Algorithm - Riemann Mapping



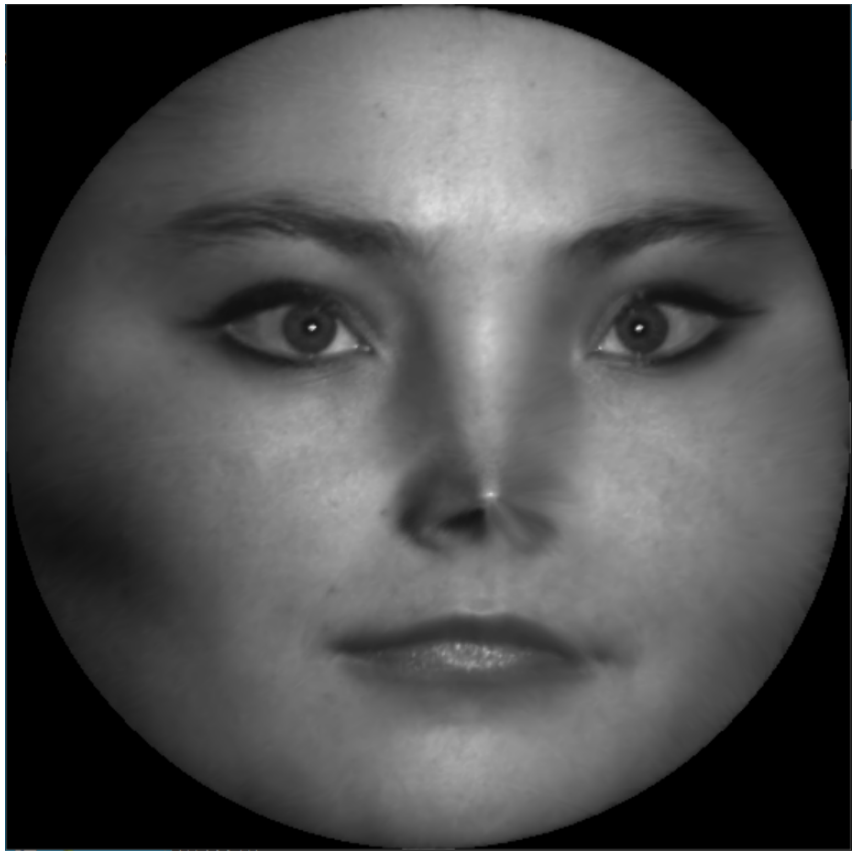
Step 3. Integration to get periodic conformal mapping image $\varphi(M)$.

Algorithm - Riemann Mapping



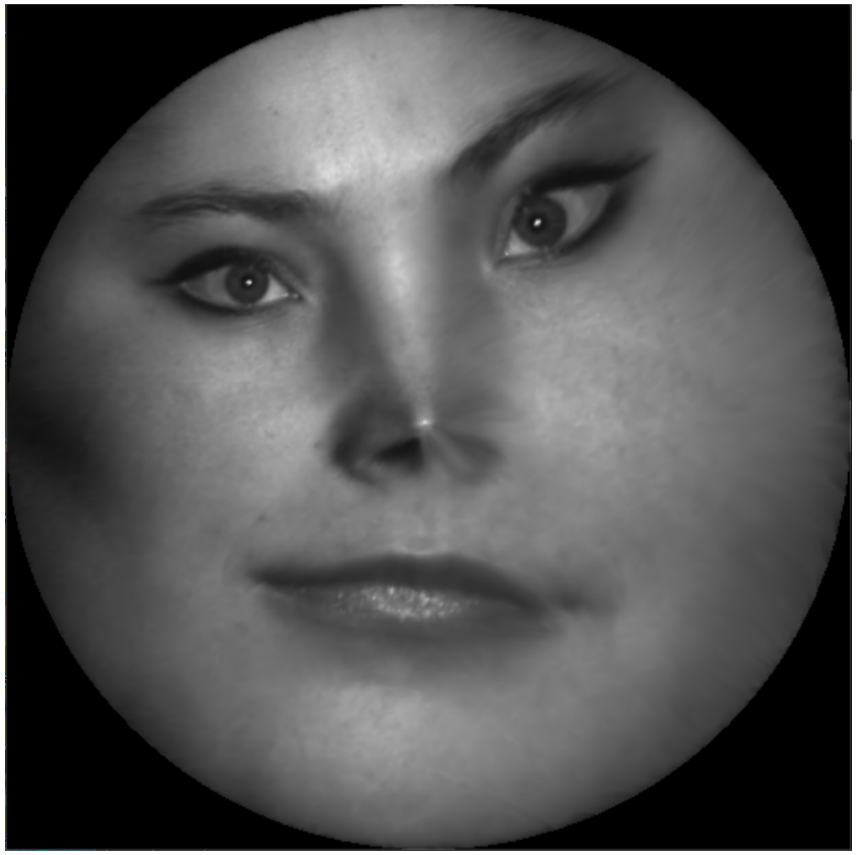
Step 4. Polar map $e^{\varphi(p)}$ induces the Riemann mapping.

Alogrithm - Riemann Mapping



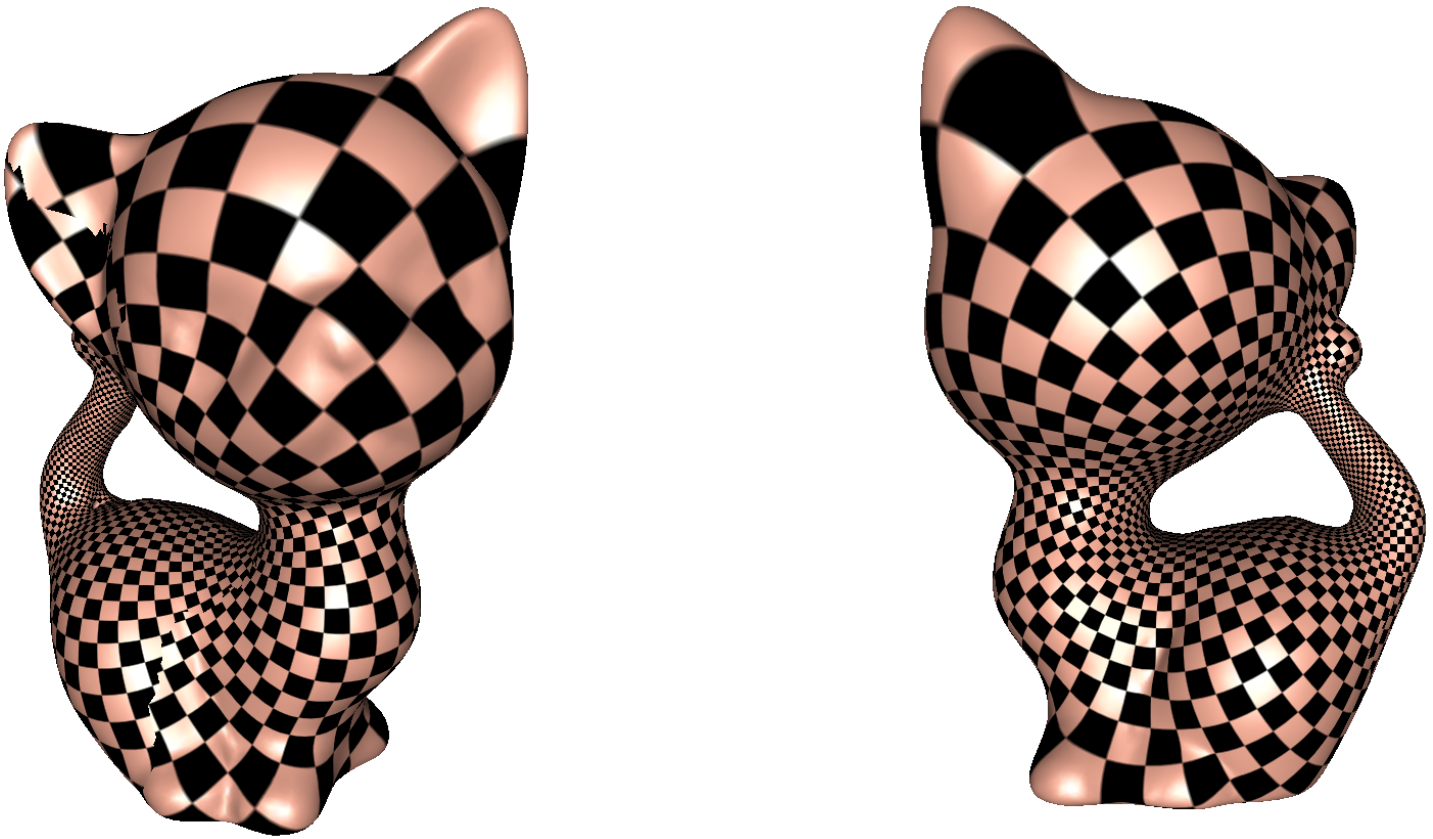
Step 5. The choice of the central puncture, and the rotation determine a Möbius transformation.

Algorithm - Riemann Mapping

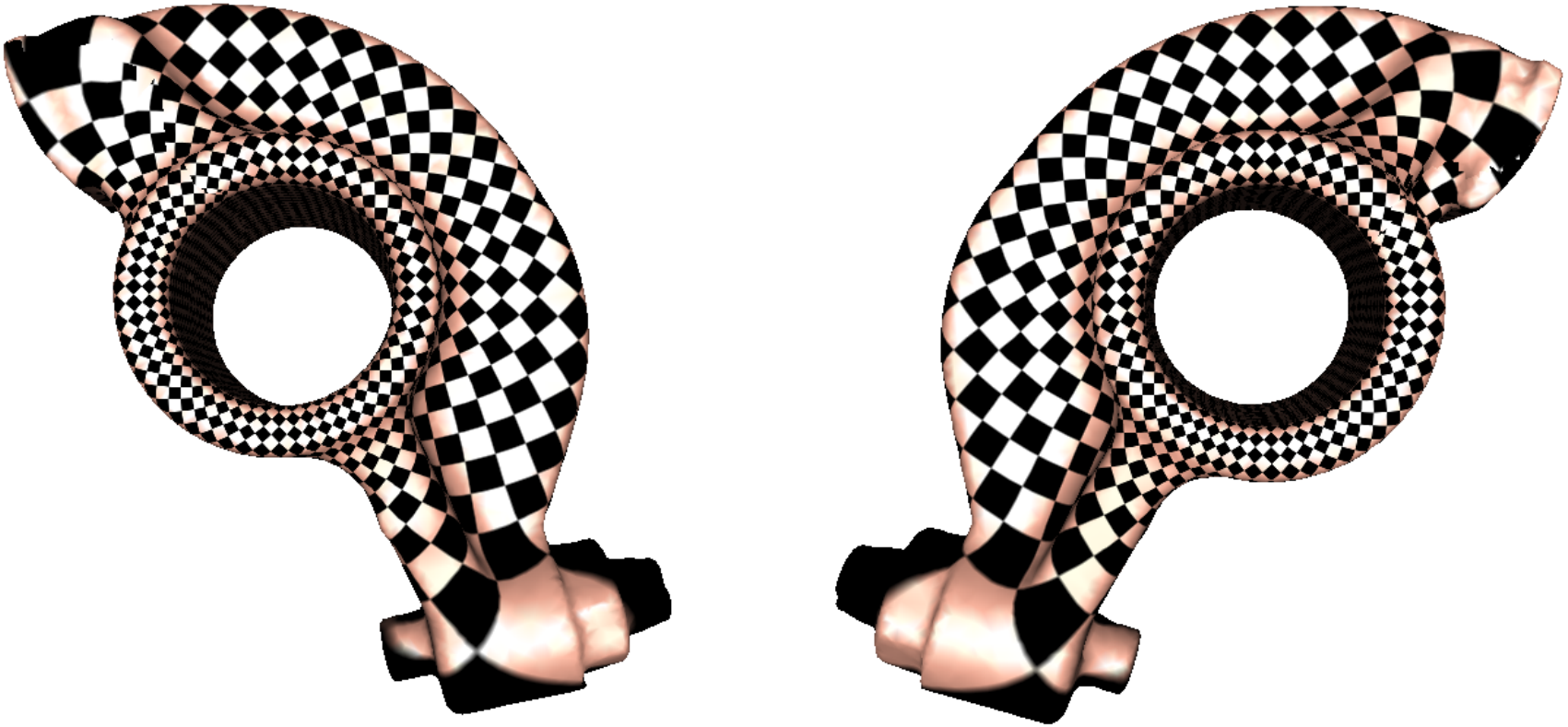


Step 5. The conformal automorphism of the unit disk is the Möbius transformation group.

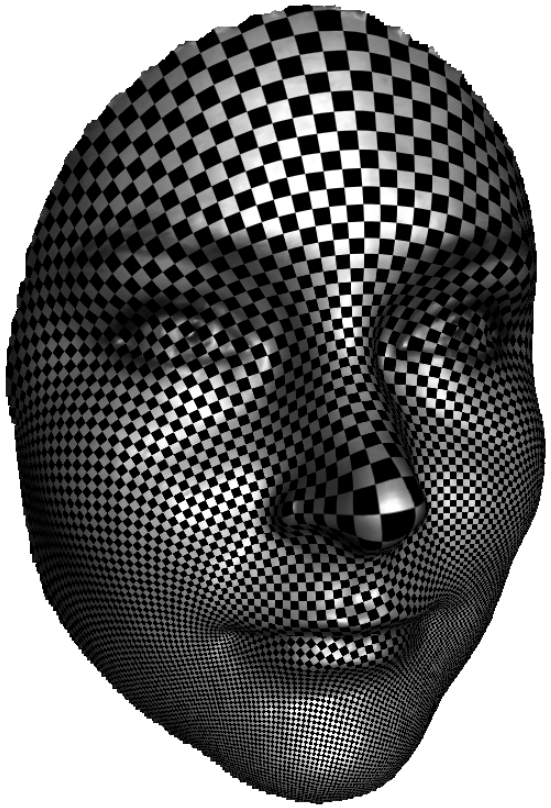
Assignment Three - Holomorphic One-form



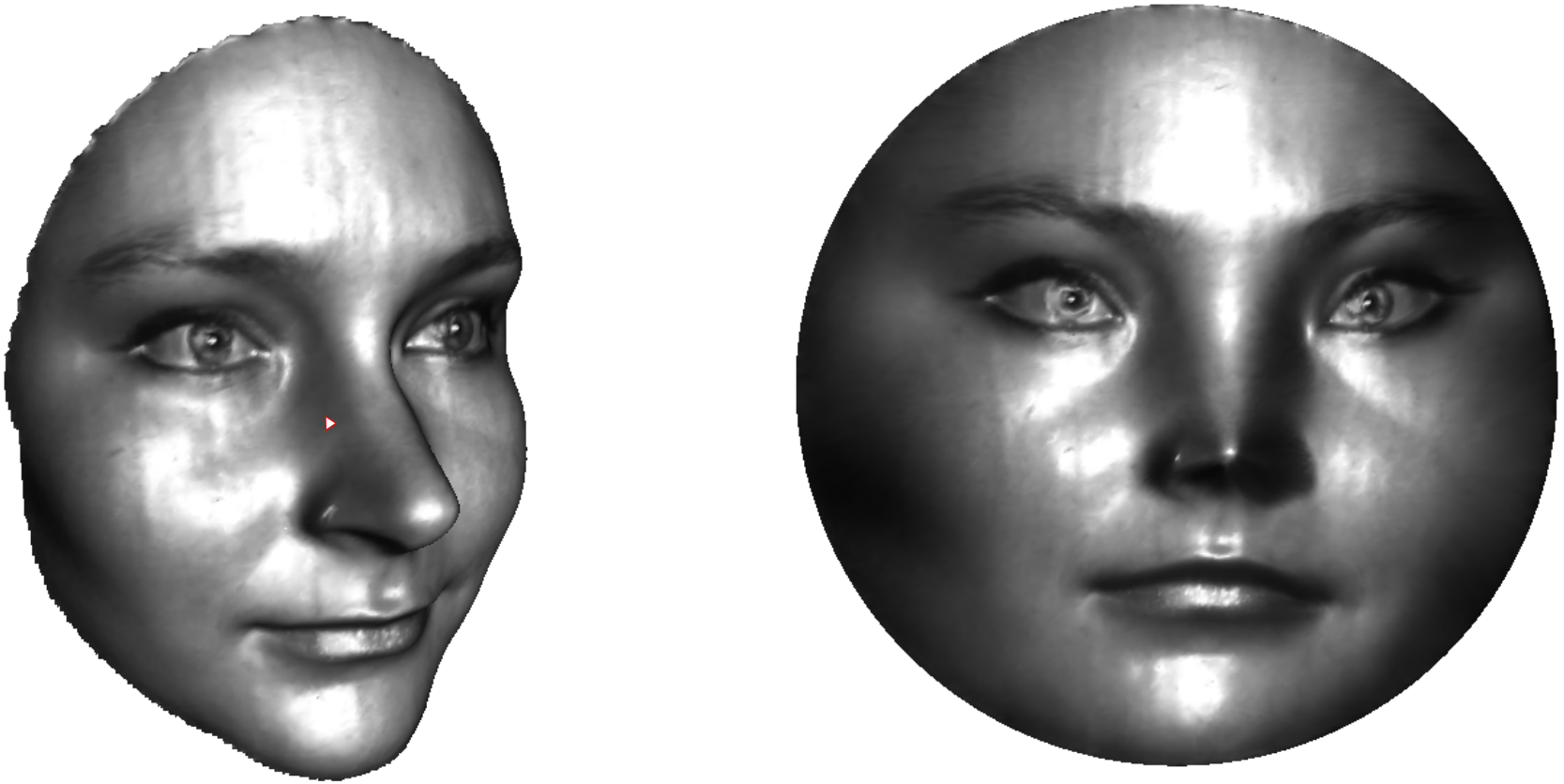
Assignment Three - Holomorphic One-form



Assignment Three - Conformal Annulus Map



Assignment Three - Riemann Mapping



Riemann mapping.

Application: Graphics

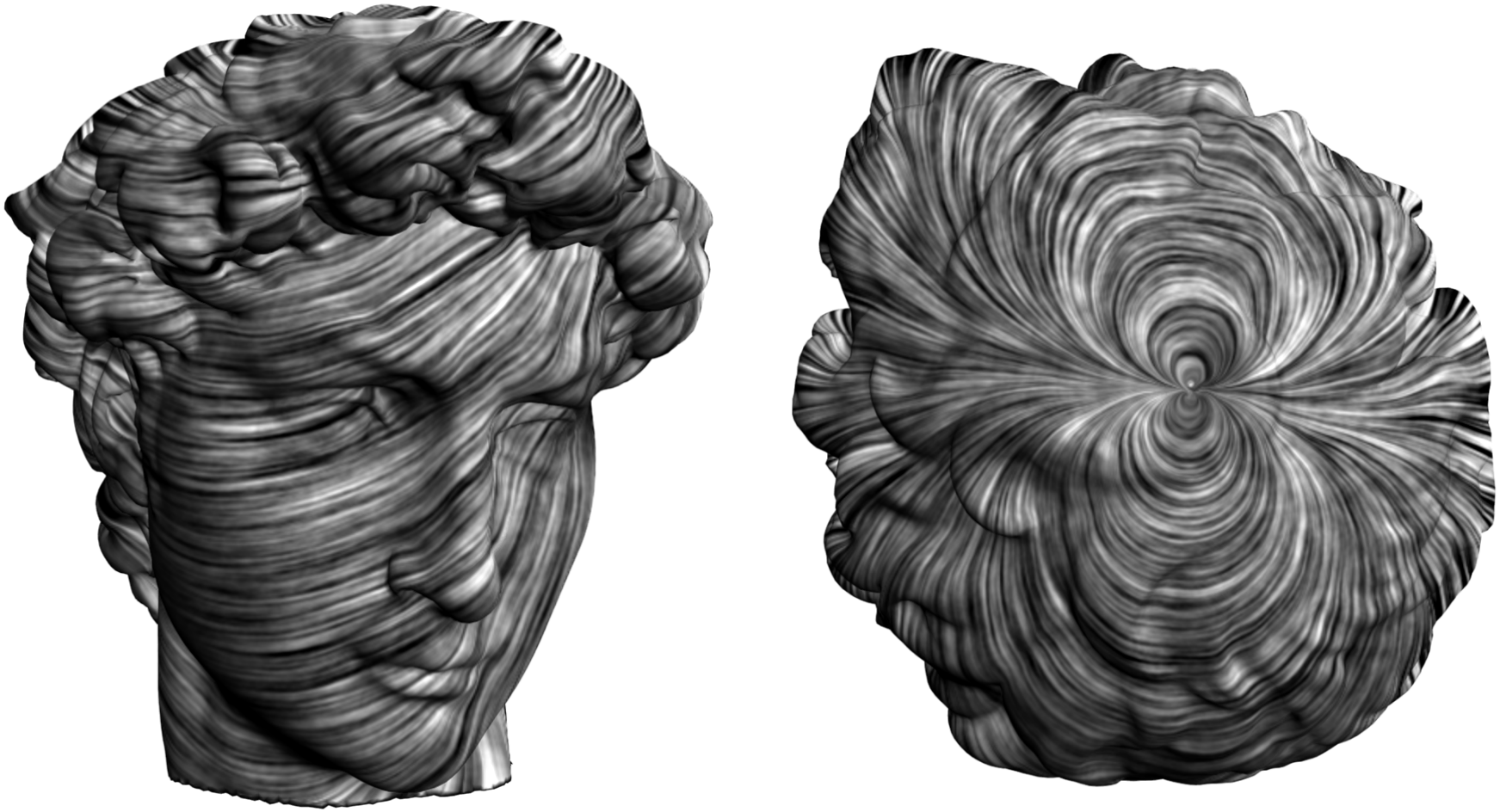


Figure: Vector field design.

Application: Graphics

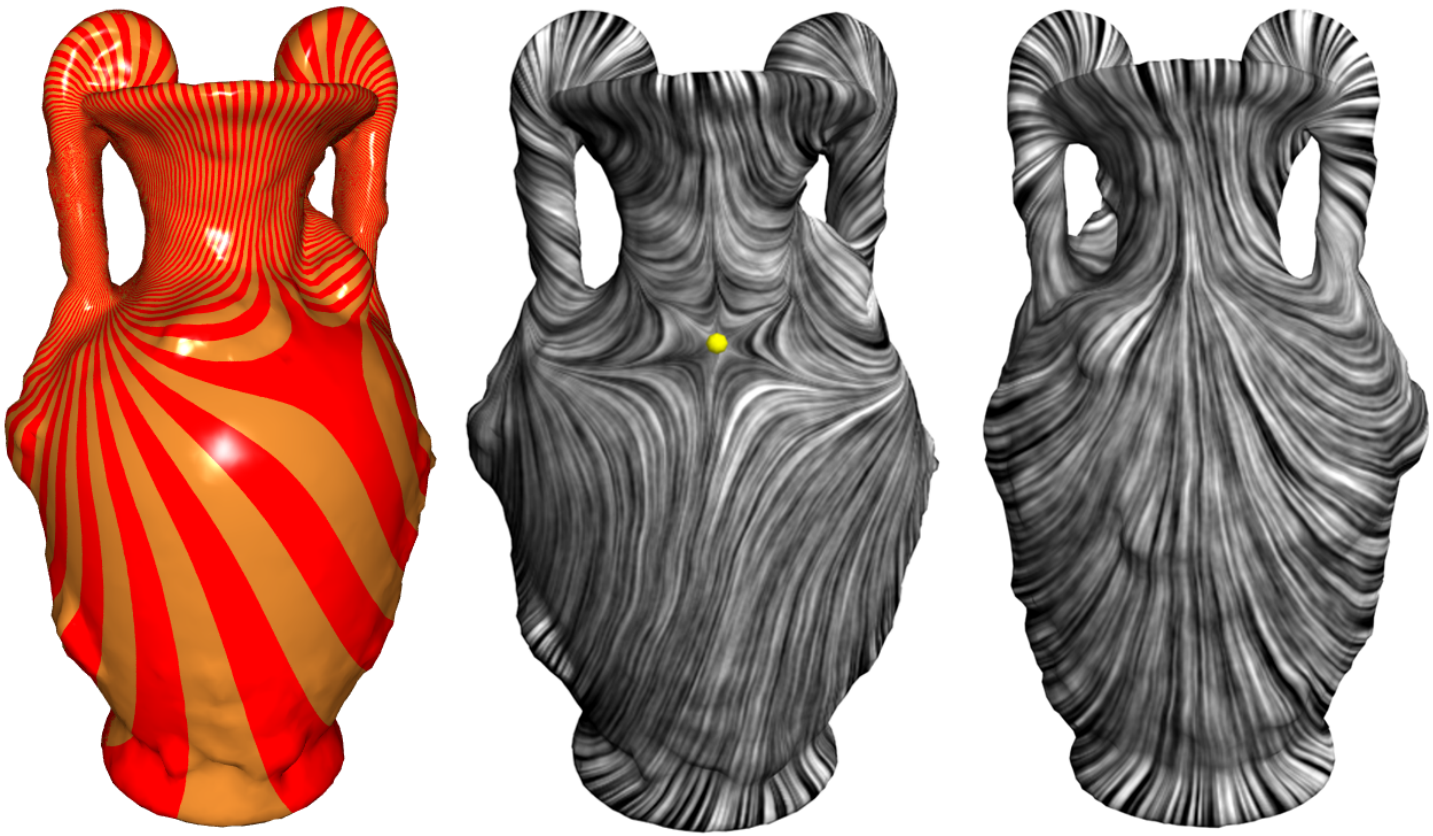
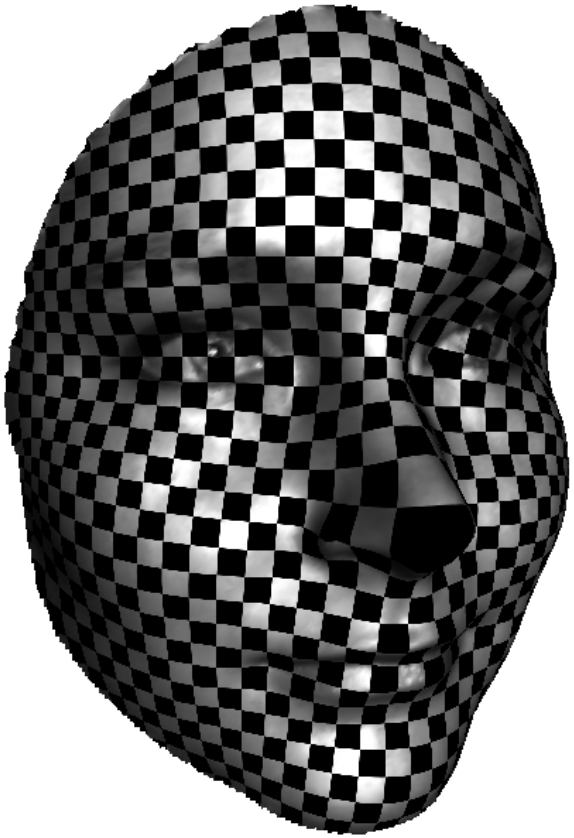


Figure: Graphics: vector field design.

Application: Graphics



Graphics: mesh parameterization.

Surface Differential Geometry

Movable Frame

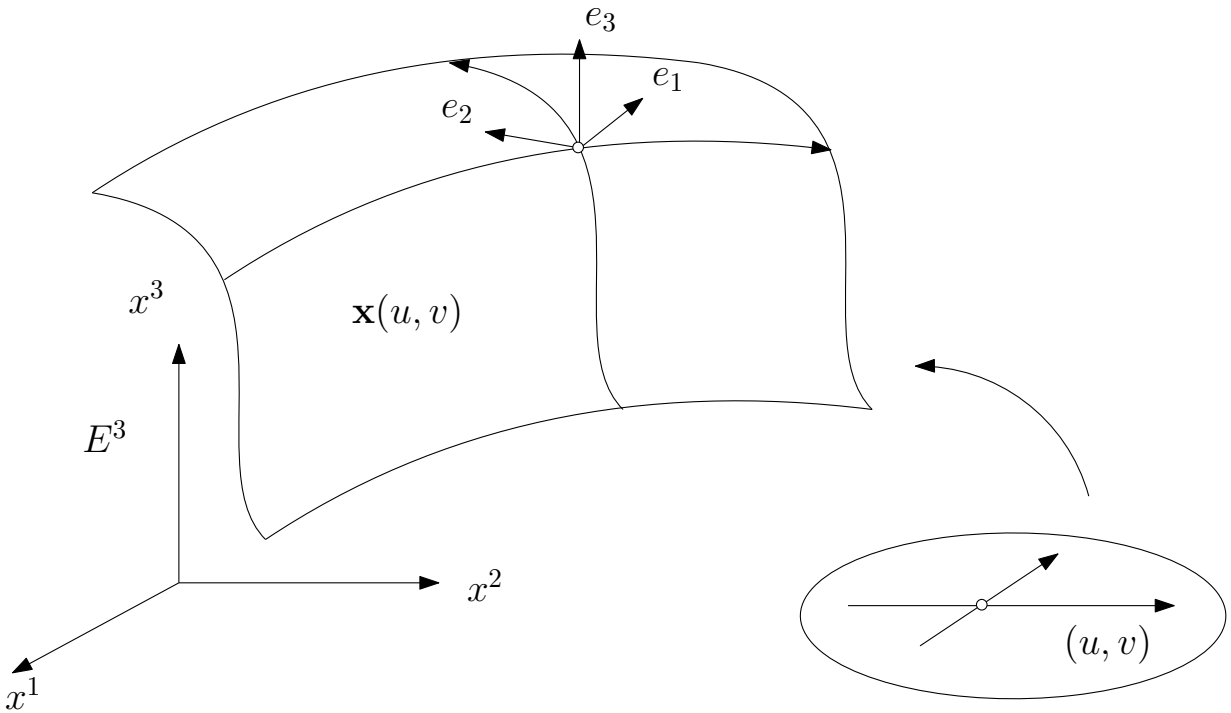


Figure: A parametric surface.

Orthonormal Movable frame

Movable Frame

Suppose a regular surface S is embedded in $\mathbb{R}^3$, a parametric representation is $\mathbf{r}(u, v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let $\mathbf{e}_3$ be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orhonormal frame field* of the surface.

Motion Equation

Motion Equation

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1 \omega_1 + \omega_2 \omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1 \omega_{31} - \omega_2 \omega_{32} = \omega_1 \omega_{13} + \omega_2 \omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \to \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \to d\mathbf{e}_3, \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 \to \omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K \omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gauss's theorem Egregium

Lemma

The connection is given by the Riemannian metric:

$$\boxed{\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2}\omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2}\omega_2}$$

Proof.

$$\begin{aligned}
0 &= d^2\mathbf{r} \\
&= d(\omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2) \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2\mathbf{e}_2 - \omega_w \wedge d\mathbf{e}_2 \\
&= d\omega_1\mathbf{e}_1 - \omega_1 \wedge (\omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3) + \\
&\quad d\omega_2\mathbf{e}_2 - \omega_2 \wedge (\omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3) \\
&= (d\omega_1 - \omega_2 \wedge \omega_{21})\mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12})\mathbf{e}_2 + \\
&\quad -(\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23})\mathbf{e}_3.
\end{aligned}$$

Levy-Civita Connection

Definition (Levy-Civita Connection)

The connection D is the Levy-Civita connection with respect to the Riemannianmetic $\mathbf{g}$, it it satisfies:

1. compatible with the metric

$$\mathbf{x}\langle \mathbf{y}, \mathbf{z}\rangle_{\mathbf{g}} = \langle D_{\mathbf{x}}\mathbf{y}, \mathbf{z}\rangle_{\mathbf{g}} + \langle \mathbf{y}, D_{\mathbf{x}}\mathbf{z}\rangle_{\mathbf{g}}$$

2. free of torsion

$$D_{\mathbf{v}}\mathbf{w} - D_{\mathbf{w}}\mathbf{v} = [\mathbf{v}, \mathbf{w}]$$

Suppose $\mathbf{v}$ and $\mathbf{w}$ are two vector fields parallel along γ, then

$$\frac{d}{dt}\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{g}} = \dot{\gamma}\langle \mathbf{v}, \mathbf{w}\rangle_{\mathbf{g}} = \langle D_{\dot{\gamma}}\mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{v}, D_{\dot{\gamma}}\mathbf{w}\rangle \equiv 0.$$

Namely, parallel transportation preserves inner product.

Geodesic Curvature

Definition (Geodesic Curvature)

Assume $\gamma : [0,1] \to S$ is a C^2 curve on a surface S, s is the arc length parameter. Construct orthonormal frame field along the curve $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e}_3\}$, where $\mathbf{e_1}$ is the tangent vector field of γ, $\mathbf{e}_3$ is the normal field of the surface,

$$k_g := \frac{D\mathbf{e_1}}{ds} = k_g \mathbf{e_2}$$

is called geodesic curvature vector,

$$k_g = \langle \frac{D\mathbf{e_1}}{ds}, \mathbf{e_2} \rangle = \frac{\omega_{12}}{ds}$$

is called geodesic curvature.

Compute Geodesics

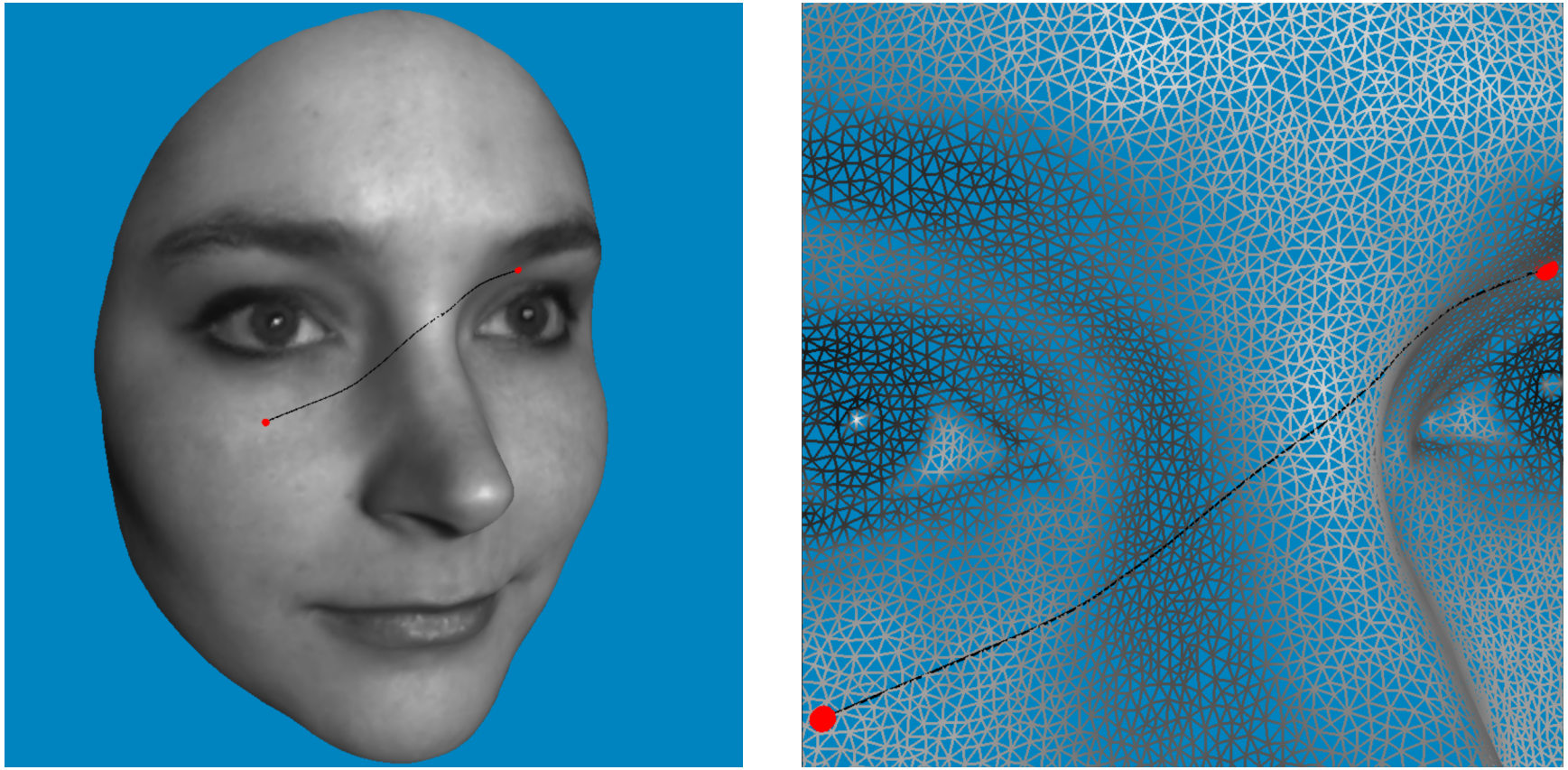


Figure: Geodesic on polyhedral surfaces.

Geodesic on a surface $\gamma : [0,1] \to (S, \mathbf{g})$:

$$D_{\dot{\gamma}}\dot{\gamma} \equiv 0.$$

Gauss-Bonnet

Theorem

Suppose $(S, \mathbf{g})$ *is an oriented metric surface with boundaries, then*

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S).$$

Proof.

Construct a vector field with isolated zeros $\{p_i\}$, $\mathbf{e}_1$ is tangent to ∂S, small disks $D(p_i, \varepsilon)$. Define $\bar{S} := S \setminus \bigcup_i D(p_i, \varepsilon)$,

$$\begin{aligned}
\int_{\bar{S}} K dA &= -\int_{\bar{S}} \frac{d\omega_{12}}{\omega_1 \wedge \omega_2} dA = -\int_{\bar{S}} d\omega_{12} = -\int_{\partial \bar{S}} \omega_{12} \\
&= -\int_{\partial S - \bigcup_i \partial D(p_i, \varepsilon)} \omega_{12} = -\int_{\partial S} \frac{\omega_{12}}{ds} ds + \sum_i \int_{\partial D(p_i, \varepsilon)} \omega_{12} \\
&= -\int_{\partial S} k_g ds + 2\pi \sum_i \mathrm{Index}(p_i) = -\int_{\partial S} k_g ds + 2\pi\chi(S).
\end{aligned}$$

Minimal Surface

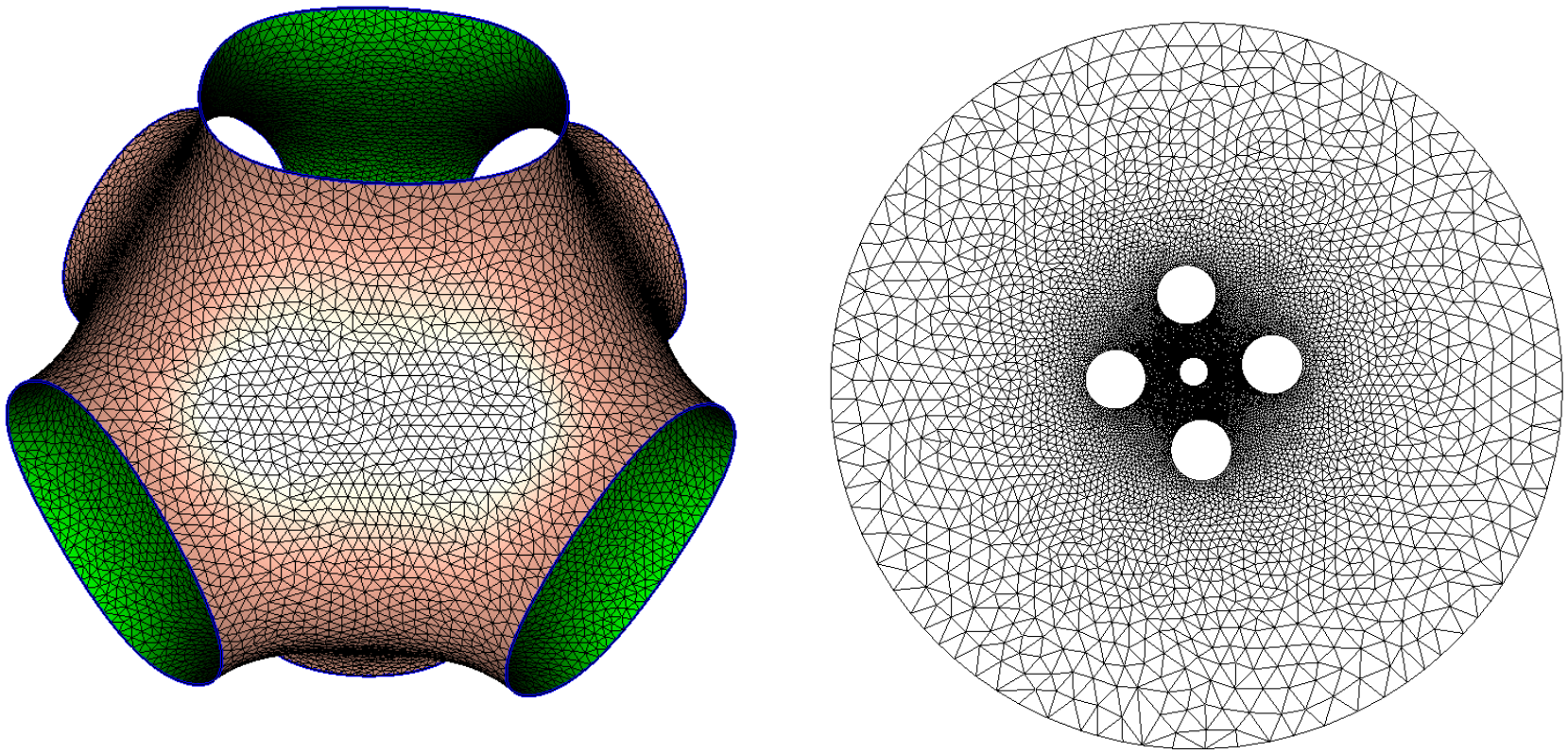


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$.

Weierstrass-Ennerper Representation

Theorem (Weierstrass-Ennerper)

If f is holomorphic on a domain Ω, g is meromorphic in Ω, and fg^2 is holomorphic on Ω, then a minimal surface is defined by $\mathbf{x}(z,\bar{z}) = (x^1(z,\bar{z}), x^2(z,\bar{z}), x^3(z,\bar{z}))$, where

$$x^1(z,\bar{z}) = \Re\left(\int f(1-g^2)dz\right)$$

$$x^2(z,\bar{z}) = \Re\left(\int \sqrt{-1}f(1+g^2)dz\right)$$

$$x^3(z,\bar{z}) = \Re\left(\int 2fgdz\right)$$

Isothermal Coordinates

Lemma (Isothermal Coordinates)

Let $(S, \mathbf{g})$ *be a metric surface, use isothermal coordinates*

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then we obtain

$$\boxed{\omega_1 = e^u dx \quad \omega_2 = e^u dy}$$

and the orthonormal frame is

$$\boxed{\mathbf{e_1} = e^{-u}\partial_x \quad \mathbf{e_2} = e^{-u}\partial_y}$$

and the connection

$$\boxed{\omega_{12} = -u_y dx + u_x dy}$$

Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvautre is given by

$$\boxed{K = -\frac{1}{e^{2u}}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u.}$$

Proof.

From

$$\omega_{12} = -u_y dx + u_x dy$$

we get

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u} dx \wedge dy} = -\frac{1}{e^{2u}}\Delta u.$$

□

Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \to e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, *then*

$$\boxed{\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda)}$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{-2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{-2\tilde{u}}\Delta\tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}
\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u+\lambda) \\
&= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\
&= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).
\end{aligned}$$

Geodesic Equation

Lemma (Geodesic Equation on a Riemann Surface)

Suppose S is a Riemann surface with a metric, $\rho(z)dzd\bar{z} = e^{2u(z)}dzd\bar{z}$, then a geodesic γ with local representation $z(t)$ satisfies the equation:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0.}$$

equivalently,

$$\ddot{\gamma} + 4u_\gamma\dot{\gamma}^2 \equiv 0.$$

Geodesics

Lemma (geodesic)

If γ is the shortest curve connecting p and q, then γ is a geodesic.

Proof.

Consider a family of curves, $\Gamma : (-\varepsilon, \varepsilon) \to S$, such that $\Gamma(0, t) = \gamma(t)$, and

$$\Gamma(s, 0) = p, \Gamma(s, 1) = q, \frac{\partial \Gamma(s, t)}{\partial s} = \varphi(t)\mathbf{e_2}(t),$$

where $\varphi : [0, 1] \to \mathbb{R}$, $\varphi(0) = \varphi(1) = 0$. Fix parameter s, curve $\gamma_s := \Gamma(s, \cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right| dt, \quad \frac{\partial L(s)}{\partial s} = -\int_0^1 \varphi k_{\mathbf{g}}(\tau) d\tau.$$

□

First Variation of arc length

Lemma (First variation of arc length)

If The length of γ_v is given by

$$L(\gamma_v) := \int_a^b |\mathbf{u}(\gamma_v(u))| du.$$

γ_0 *is parameterized by arc length, that is,* $|\mathbf{u}(\gamma_0(u))| \equiv 1$, *then*

$$\frac{d}{dv}\Big|_{v=0} L(\gamma_v) = -\int_a^b \langle D_{\mathbf{u}}\mathbf{u}, \mathbf{v}\rangle du + \langle \mathbf{u}, \mathbf{v}\rangle\Big|_a^b.$$

If we choose $\mathbf{u} = \mathbf{e_1}$, the tangent vector of γ, $\mathbf{v} = \mathbf{e_2}$ orthogonal to $\mathbf{e_1}$, and fix the starting and ending points of paths, then

$$\frac{d}{dv} L(\gamma_v) = -\int_a^b k_g ds.$$

Geodesics

Lemma (Uniqueness of geodesics)

Suppose $(S, \mathbf{g})$ *is a closed oriented metric surface,* $\mathbf{g}$ *induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.*

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ, by Gauss-Bonnet,

$$\int_{\Sigma} K dA + \int_{\partial\Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. □

Hyperbolic Geodesics

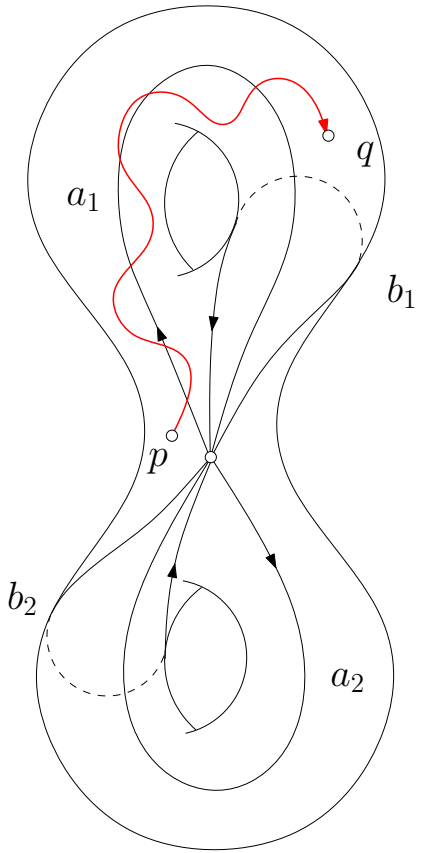
Lemma

Let Σ be a compact hyperbolic Riemann surface, $K \equiv -1$, $p, q \in \Sigma$, then there exists a unique geodesic in each homotopy class, the geodesic depends on p and q continuously.

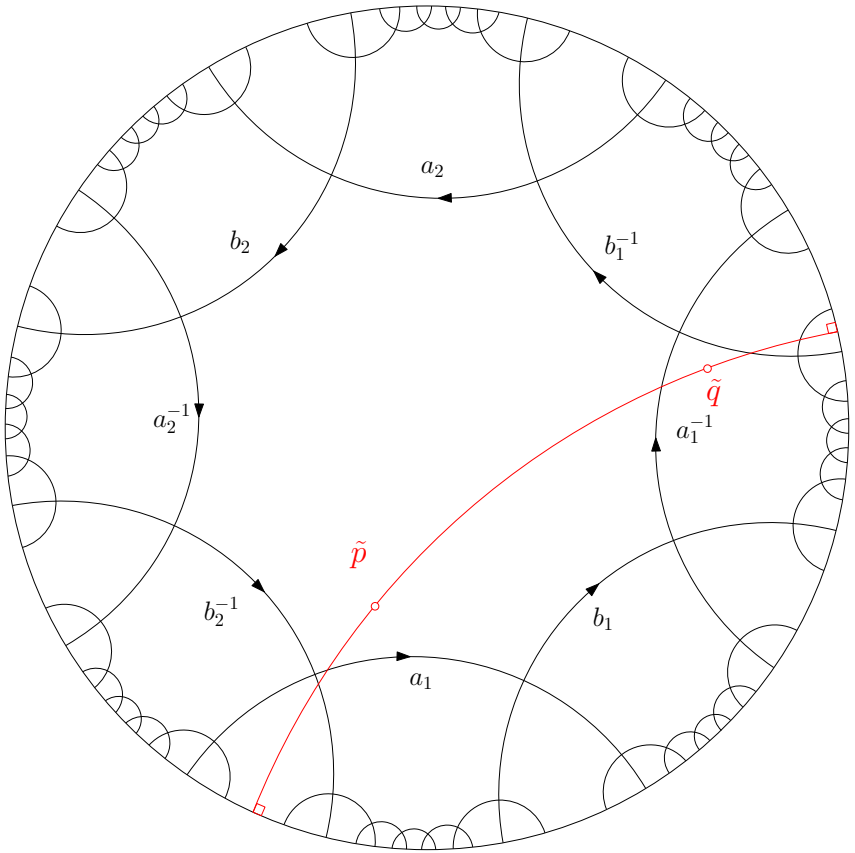
Proof.

Given a path $\gamma : [0,1] \to \Sigma$ connecting p and q. Let $\pi : \mathbb{H}^2 \to \Sigma$ be the universal covering space of Σ. Fix one point $\tilde{p} \in \pi^{-1}(p)$, then there exists a unique lifting of γ, $\tilde{\gamma} : [0,1] \to \mathbb{H}^2$, $\tilde{\gamma}(0) = \tilde{p}$ and $\tilde{\gamma}(1) = \tilde{q}$. On the hyperbolic plane, the geodesic between $\tilde{p}$ and $\tilde{q}$ exists and is unique, $\tilde{\gamma}$ depends on $\tilde{p}$ and $\tilde{q}$ continuously. $\square$

Hyperbolic Geodesic

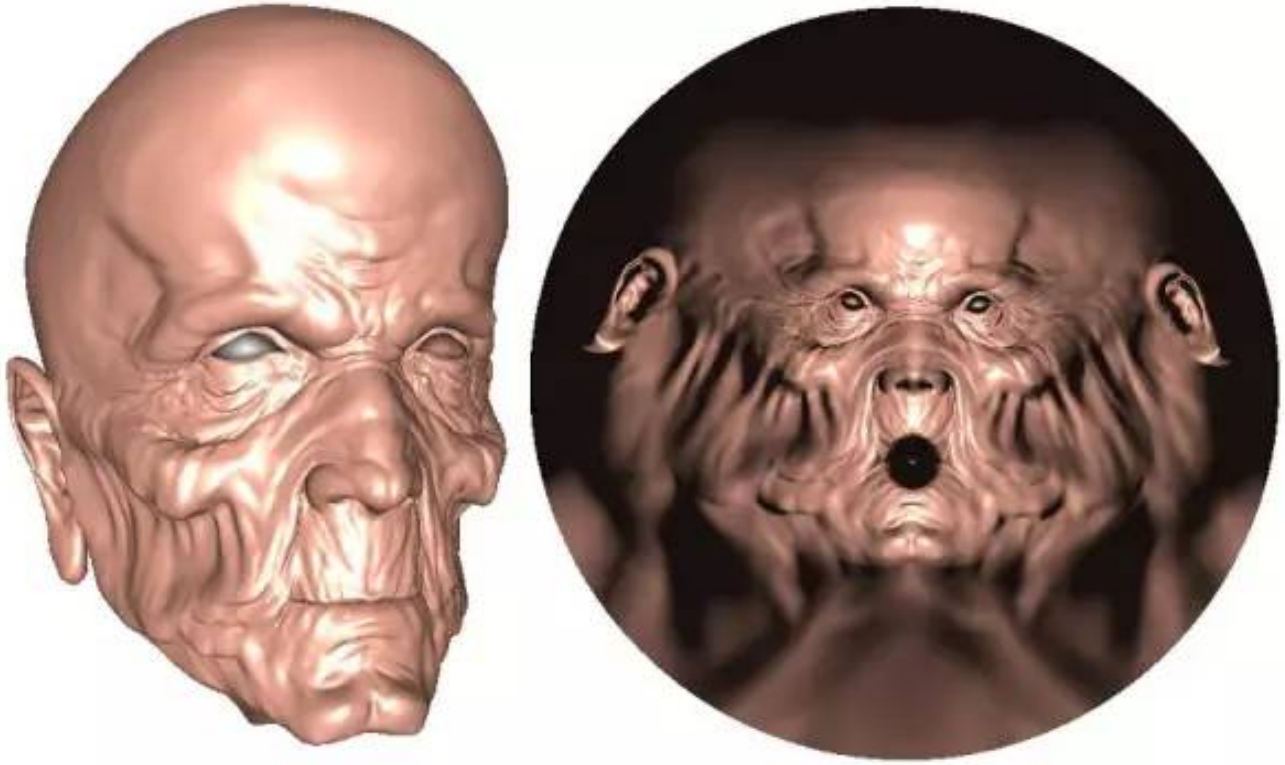


geodesic on surface



Poincaré's disk model

Algorithm - Finite Element Method



Finite element method to solve elliptic PDE.

Iterative Algorithm for Harmonic Map

Input: A topological disk mesh M;
Output: Harmonic map result, stored at the vertex uv coordinates;

1. Trace the boundary of M counter clockwisely, set the uv to be on the unit circle, the angle for each vertex is proportional to the arc length;
2. Set all the interior vertices uv to be at the original $(0, 0)$;
3. Compute all the corner angles;
4. Compute edge cotangent edge weight, w_{ij} for edge $[v_i, v_j]$;
5. For each vertex v_i, move it uv to the weighted center of its neighbors,

$$uv(v_i) \leftarrow \frac{\sum_j w_{ij} \; uv(v_j)}{\sum_j w_{ij}}$$

6. Repeat step 5, until it converges.

Direct Algorithm for Harmonic Map

Input: A topological disk mesh M;
Output: Harmonic map, $\varphi : V \to \mathbb{R}^2$;

1. Trace the boundary of M counter clockwisely, set the uv to be on the unit circle, the angle for each vertex is proportional to the arc length;
2. Compute all the corner angles;
3. Compute edge cotangent edge weight, w_{ij} for edge $[v_i, v_j]$;
4. For each interior vertex v_i, construct a linear equation

$$\sum_{v_j \sim v_i} w_{ij}(\varphi(v_j) - \varphi(v_i)) = 0.$$

with Dirichlet boundary condition.

Algorithm to Compute Minimal Surface

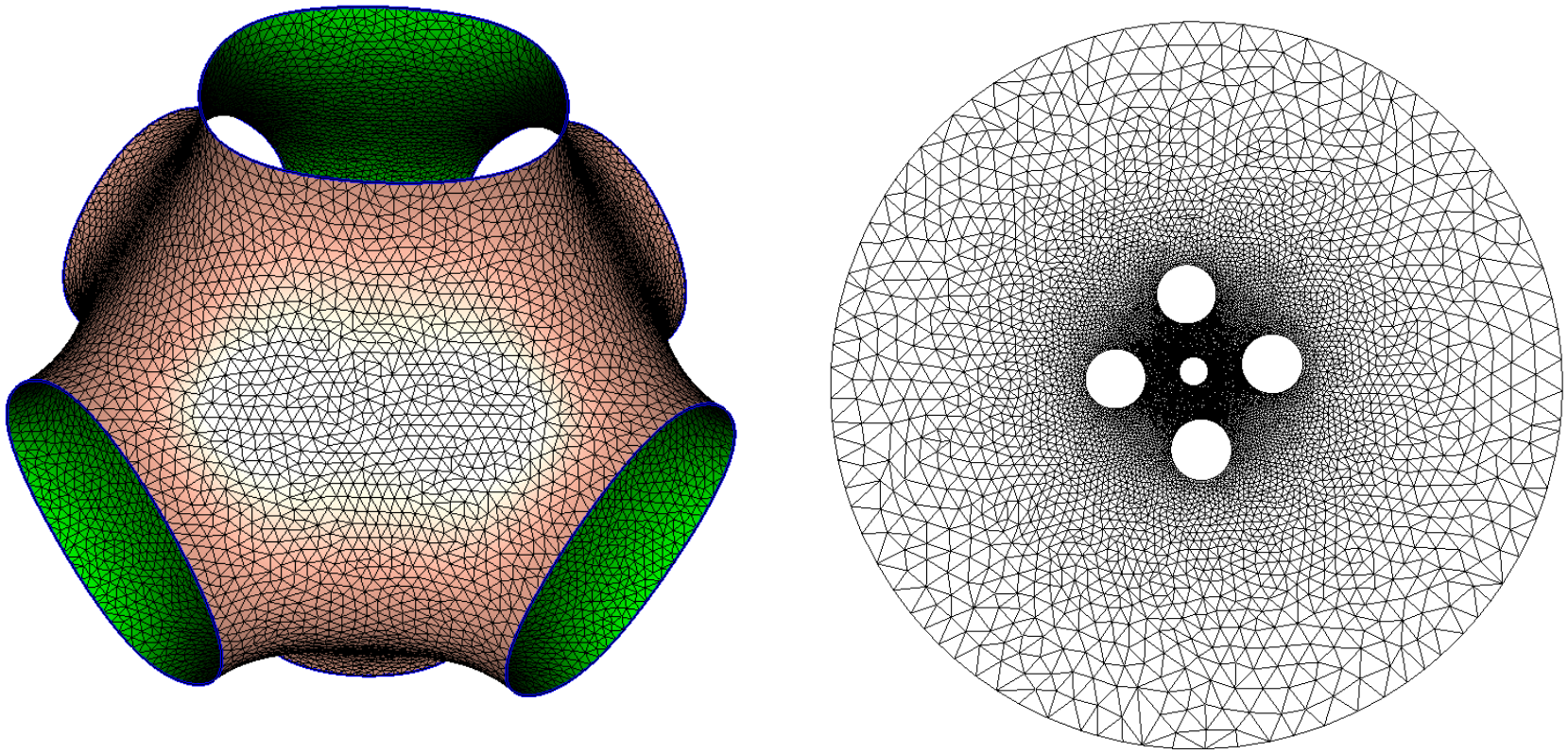


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0$, $\forall v_i \notin \partial M$.

Algorithm: Homotopy Detection

Input: A high genus closed mesh M, two loops γ_1 and γ_2;
Output: Whether $\gamma_1 \sim \gamma_2$;

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ_k to geodesics, $k = 1, 2$;
3. if two geodesics coincide, return true; otherwise, return false;

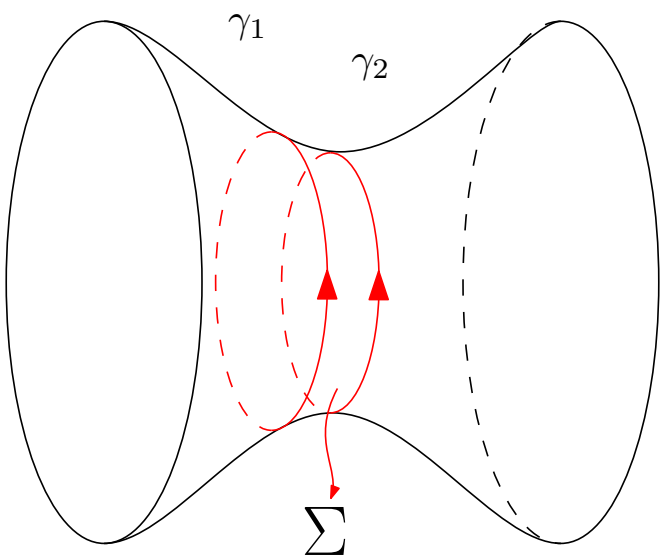


Figure: Geodesics uniqueness.

Algorithm: Shortest Word

Input: A high genus closed mesh M, one loop γ

1. Compute a hyperbolic metric of M, using Ricci flow;
2. Homotopically deform γ to a geodesic;
3. Compute a set of canonical fundamental group basis;
4. Embed a finite portion of the universal covering space onto the Poincaré disk;
5. Lift γ to the universal covering space $\tilde{\gamma}$. If $\tilde{\gamma}$ crosses $b_i^{\pm}$, append $a_i^{\pm}$; crosses $a_i^{\pm}$, append $b_i^{\mp}$.

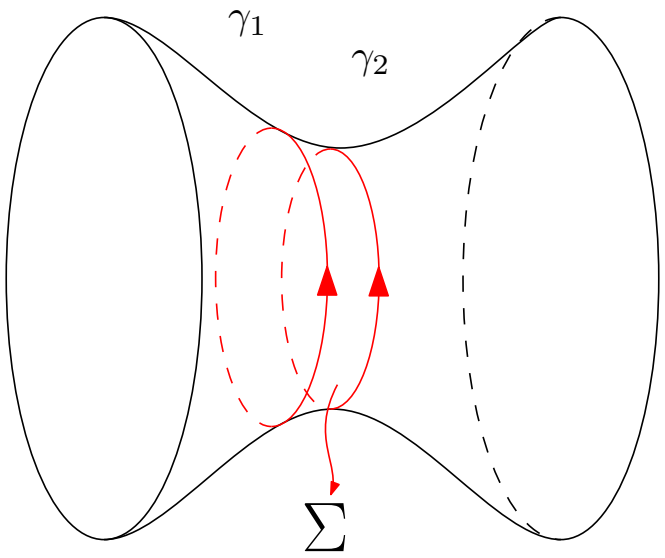
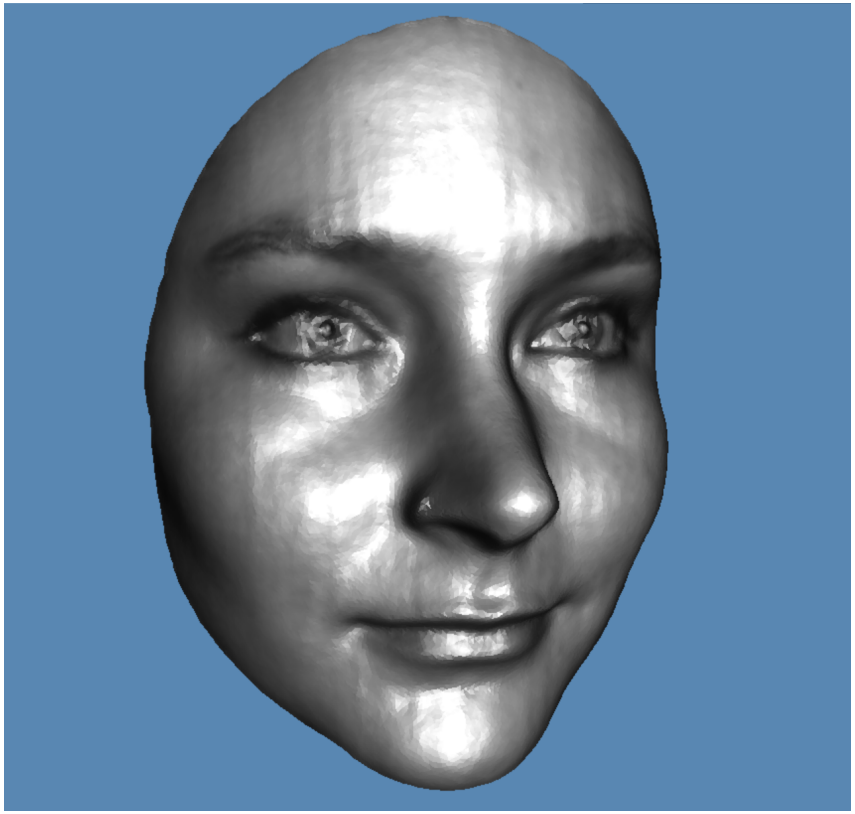
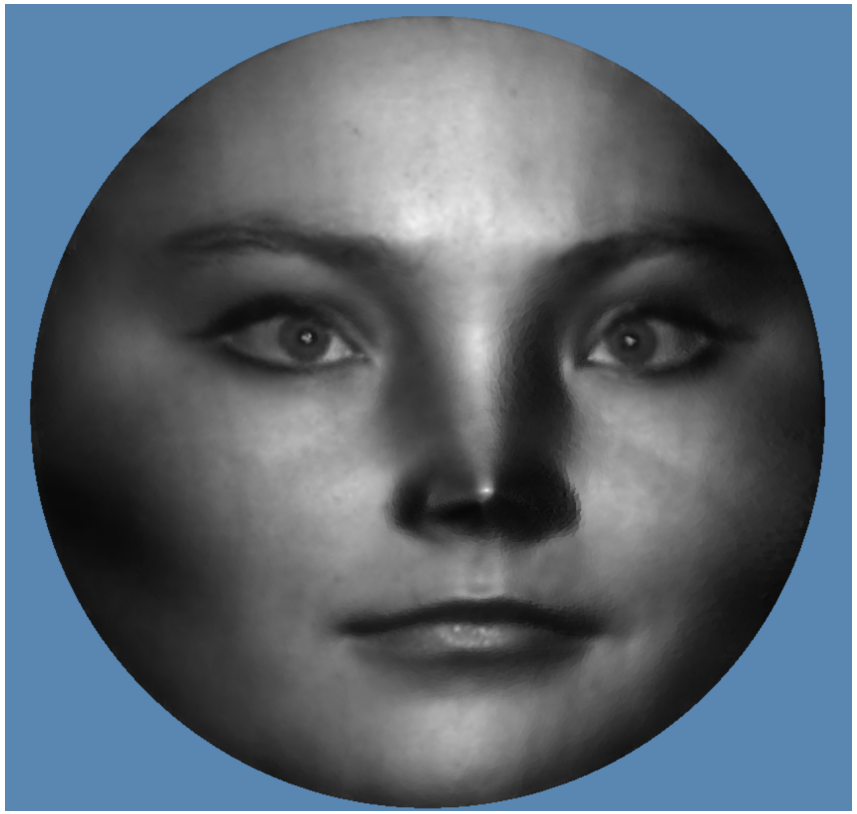


Figure: Geodesics uniqueness

Assignment Two - Harmonic Maps for Topologicl Disk



input mesh



harmonic map image

Harmonic Maps

Harmonic Function

Theorem (Mean Value)

Assume $\Omega \subset \mathbb{R}^2$ is a planar open set, $u : \Omega \to \mathbb{R}$ is a harmonic function, then for any $p \in \Omega$

$$u(p) = \frac{1}{2\pi\varepsilon} \oint_\gamma u(q) ds, \tag{2}$$

where γ is a circle centered at p, with radius ε.

Proof.

u is harmonic, du is a harmonic 1-form, its Hodge star $^*du$ is also harmonic. Define the conjugate function v, $dv = {}^*du$, then $\varphi(z) := u + \sqrt{-1}v$ is holomorphic. By Cauchy integration formula,

$$\varphi(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\varphi(\zeta)}{\zeta - z} dz \tag{3}$$

Hence, we obtain the mean value property of harmonic function. □

Harmonic Function

Corollary (Maximal value principle)

Assume $\Omega \subset \mathbb{R}^2$ is a planar domain, and $u : \overline{\Omega} \to \mathbb{R}$ is a non-constant harmonic function, then u can't reach extremal values in the interior of Ω.

Proof.

Assume p is an interior point in Ω, p is a maximal point of u, $u(p) = C$. By mean value property, we obtain for any point q on the circle $B(p, \varepsilon)$, $u(q) = C$, where ε is arbitrary, therefore u is constant in a neighborhood of p. Therefore $u^{-1}(C)$ is open. On the other hand, u is continuous, $u^{-1}(C)$ is closed, hence $u^{-1}(C) = \Omega$. Contradiction. $\square$

Uniqueness of Harmonic Functions

Corollary

Suppose $\Omega \subset \mathbb{R}^2$ is a planar domain, $u_1, u_2 : \Omega \to \mathbb{R}$ are harmonic functions with the same boundary value, $u_1|_{\partial\Omega} = u_2|_{\partial\Omega}$, then $u_1 = u_2$ on Ω.

Proof.

$u_1 - u_2$ is also harmonic, with 0 boundary value, therefore the maximal and minimal values of $u_1 - u_2$ must be on the boundary, namely they are 0, hence u_1, u_2 are equal in Ω. $\square$

Disk Harmonic Maps

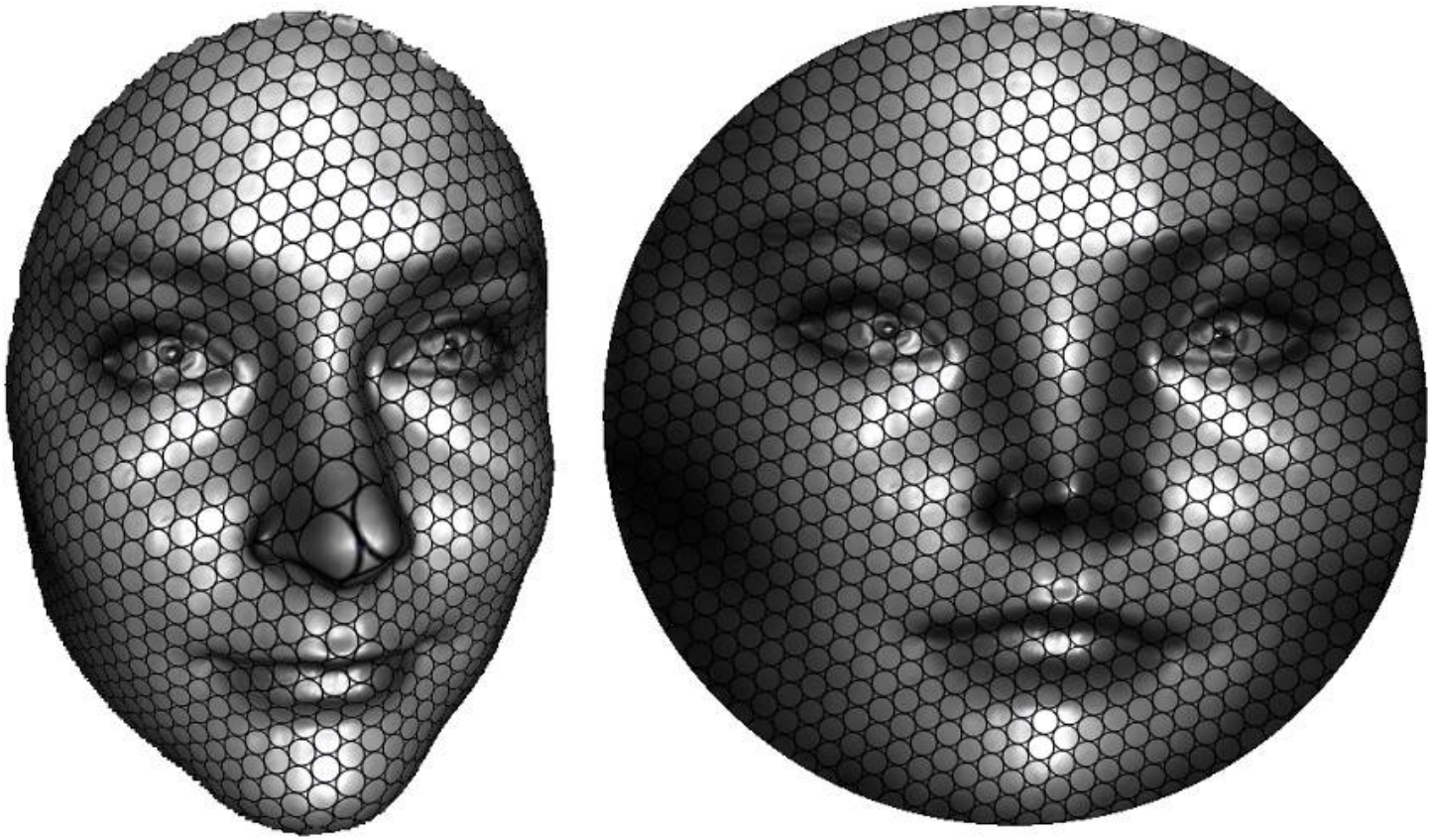


Figure: Harmonic map between topological disks.

Diffeomorphic Property of Disk Harmonic Maps

Theorem (Rado)

Suppose a harmonic map $\varphi : (S, \mathbf{g}) \to (\Omega, dx^2 + dy^2)$ *satisfies:*

1. *planar domain* Ω *is convex*
2. *the restriction of* $\varphi : \partial S \to \partial\Omega$ *on the boundary is homoemorphic,*

then u *is diffeomorphic in the interior of* S.

Proof.

By regularity theory of harmonic maps, we get the smoothness of the harmonic map. Assume $\varphi : (x, y) \to (u, v)$ is not homeomorphic, then there is an interior point $p \in \Omega$, the Jacobian matrix of φ is degenerated as p, there are two constants $a, b \in \mathbb{R}$, not being zeros simultaneously, such that

$$a\nabla u(p) + b\nabla v(p) = 0.$$

By $\Delta u = 0, \Delta v = 0$, the auxiliary function $f(q) = au(q) + bv(q)$ is also harmonic. □

General Harmonic Map

Definition (Harmonic Energy)

Let (Σ_1, z) and (Σ_2, u) be two Riemann surfaces, with Riemannian metrics $\sigma(z)dzd\bar{z}$ and $\rho(u)dud\bar{u}$. Given a C^1 map $u : \Sigma_1 \to \Sigma_2$, then the harmonic energy of u is defined as

$$E(z, \rho, u) := \int_{\Sigma_1} \rho^2(u)(u_z \bar{u}_{\bar{z}} + \bar{u}_z u_{\bar{z}}) \frac{i}{2} dzd\bar{z}$$

where $u_z := \frac{1}{2}(u_x - iu_y)$, $u_{\bar{z}} := \frac{1}{2}(u_x + iu_y)$ and $dz \wedge d\bar{z} = -2idx \wedge dy$.

Definition (Harmonic Map)

If the C^1 map $u : \Sigma_1 \to \Sigma_2$ minimizes the harmonic energy, then u is called a harmonic map.

General Harmonic Map

Theorem (Euerl-Larange Equation for Harmonic Maps)

Suppose $u : \Sigma_1 \to \Sigma_2$ is a C^2 harmonic map between Riemannian surfaces, then

$$\boxed{u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} = 0}$$

Geodesics are special harmonic maps, harmonic maps are generalized geodesics:

$$\boxed{\ddot{\gamma} + \frac{2\rho_\gamma}{\rho}\dot{\gamma}^2 \equiv 0 \quad u_{z\bar{z}} + \frac{2\rho_u}{\rho} u_z u_{\bar{z}} \equiv 0}$$

Hopf Differential of Harmonic Maps

Theorem (Hopf Diffential of Harmonic Maps)

Let $u : (\Sigma_1, \lambda^2(z)dzd\bar{z}) \to (\Sigma_2, \rho^2(u)dud\bar{u})$ *is harmonic, then the Hopf differential of the map*

$$\boxed{\Phi(u) := \rho^2 u_z \bar{u}_z dz^2}$$

is holomorphic quadratic differential on Σ_1*. Furthermore* $\Phi(u) \equiv 0$*, if and only if* u *is holomorphic or anti-holomorprhic.*

Proof.

If u is harmonic, then

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(\rho^2 u_z \bar{u}_z) &= \rho^2 u_{z\bar{z}} \bar{u}_z + \rho^2 u_z \bar{u}_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z \bar{u}_z + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} u_z \bar{u}_z \\ &= (\rho^2 u_{z\bar{z}} + 2\rho\rho_u u_{\bar{z}} u_z)\bar{u}_z + (\rho^2 \bar{u}_{z\bar{z}} + 2\rho\rho_{\bar{u}} \bar{u}_{\bar{z}} \bar{u}_z) u_z = 0.\end{aligned}$$

Therefore $\Phi(u)$ is holomorphic. □

Spherical Harmonic Map

Theorem (Spherical Harmonic Maps)

Harmonic maps between genus zero closed metric surfaces must be conformal.

Proof.

Suppose $u : \Sigma_1 \to \Sigma_2$ is a harmonic map, then $\Phi(u)$ must be a holomorphic quadratic differential. Since Σ_1 is of genus zero, therefore $\Phi(u) \equiv 0$. Hence u is holomorphic. $\square$

Spherical Harmonic Map

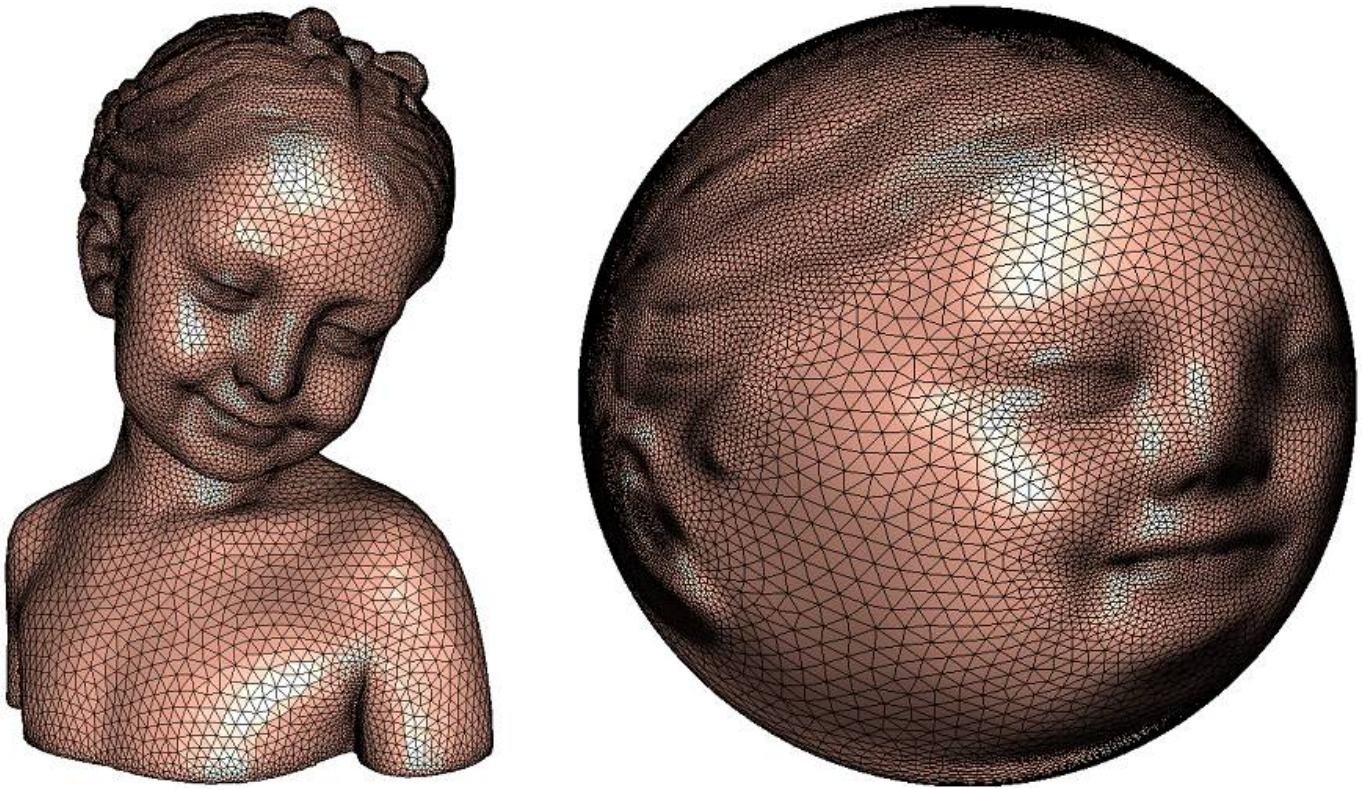


Figure: Spherical Harmonic Map

Uniqueness Spherical Harmonic Map

Definition (Möbius Transformation)

A Möbius transformation $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has the form

$$z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1.$$

Given $\{z_0, z_1, z_2\}$, there is a unique Möbius transformation, that maps them to $\{0, 1, \infty\}$,

$$z \mapsto \frac{z - z_0}{z - z_2} \frac{z_1 - z_2}{z_1 - z_0}.$$

Theorem (Uniquess of Spherical Conformal Automorphisms)

Suppose $f : \mathbb{S}^2 \to \mathbb{S}^2$ is a biholomorphic automorphism, then f must be a Möbius transformation.

Existence of Harmonic Map

Theorem (Existence of Harmonic Maps)

Assume Σ is a Riemann surface, $(N, \rho(u)dud\bar{u})$ is a metric surface, then for any smooth mapping $\varphi : \Sigma \to N$, there is a harmonic map $f : \Sigma \to N$ homotopic to φ.

The can be proven using Courant-Leesgue lemma, which controls the geodesic distance between image points by harmonic energy.

Regularity of Harmonic Map

Theorem (Regularity of Harmonic Maps)

Let $u : \Sigma_1 \to \Sigma_2$ be a (weak) harmonic map between Riemann surfaces, Σ_2 is with hyperbolic metric, the harmonic energy of u is finite, then u is a smooth map.

This is based on the regularity theory of ellptic PDEs.

Diffeomorphic Properties of Harmonic Maps

Theorem (Diffeomorphic Properties of Harmonic Maps)

Let Σ_1 and Σ_2 are compact Riemann surfaces with the same genus, $K_2 \leq 0$. If $u : \Sigma_1 \to \Sigma_2$ is a degree one harmonic map, then u is a diffeomorphism.

Uniqueness of Harmonic Map

Theorem (Uniqueness of Harmonic Map)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

Uniqueness of Harmonic Map

Theorem (Uniqueness)

Suppose Σ_1 and Σ_2 are compact Riemann surface, Σ_2 is with hyperbolic metric. $u_0, u_1 : \Sigma_1 \to \Sigma_2$ are homotopic harmonic maps. If one of the Jacobian matrix is non-degenerated at a point, then $u_0 \equiv u_1$.

Proof.

Given a homotopy connecting u_0 and u_1, $h(z,t) : \Sigma_1 \times [0,t] \to \Sigma_2$, such that $h(z,0) = u_0(z)$, $h(z,1) = u_1(z)$. Let $\psi(z,t)$ is a geodesic from $u_0(z)$ to $u_1(z)$ and homotopic to $h(z,t)$, with parameter

$$\rho(\psi(z,t))|\dot{\psi}(z,t)| \equiv \text{const}$$

then $u_t(z) := \psi(z,t)$ is also a homotopy connecting u_0 and u_1. □

Surface Riemann Mapping Theorem

Theorem (Riemann Mapping)

Suppose $(S, \mathbf{g})$ *is a topological disk with a Riemannian metric* $\mathbf{g}$*, then there exists a conformal map* $\varphi : (S, \mathbf{g}) \to \mathbb{D}^2$*. Furthermore, such kind of mappings differ by a Möbius transformation with the form*

$$z \to e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad |z_0| < 1, \theta \in [0, 2\pi). \tag{4}$$

Computational Algorithm for Disk Harmonic Maps

Input:A topological disk M;
Output:A harmonic map $\varphi : M \to \mathbb{D}^2$

1. Construct boundary map to the unit circle, $g : \partial M \to \mathbb{S}^1$, g should be a homeomorphism;
2. Compute the cotangent edge weight;
3. for each interior vertex $v_i \in M$, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_j \sim v_i} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0;$$

4. Solve the linear system, to obtain φ.

Computational Algorithm for Spherical Harmonic Map

Input: A genus zero closed mesh M;
Output: A spherical harmonic map $\varphi : M \to \mathbb{S}^2$;

1. Compute Gauss map $\varphi : M \to \mathbb{S}^2$, $\varphi(v) \leftarrow \mathbf{n}(v)$;
2. Compute the cotangent edge weight, compute Laplacian

$$\Delta\varphi(v_i) = \sum_{v_i \sim v_j} w_{ij}(\varphi(v_j) - \varphi(v_i)),$$

3. project the Laplacian to the tangent plane,

$$D\varphi(v_i) = \Delta\varphi(v_i) - \langle \Delta\varphi(v_i), \varphi(v_i) \rangle \varphi(v_i)$$

4. for each vertex, $\varphi(v_i) \leftarrow \varphi(v_i) - \lambda D\varphi(v_i)$;
5. compute the mass center $c = \sum A_i \varphi(v_i) / \sum_j A_j$; normalize $\varphi(v_i) = \varphi(v_i) - c / |\varphi(v_i) - c|$;
6. Repeat step 2 through 5, until the Laplacian norm is less than ε.

Surface Double Covering Algorithm

Input: A oriented surface with boundaries M;
Output: The double covering $\bar{M}$;

1. Make a copy of M, denoted as M';
2. Reverse the order of the vertices of each face of M';
3. Glue M and M' along their corresponding boundary edges to obtain $\bar{M}$.

Surface Double Covering Algorithm

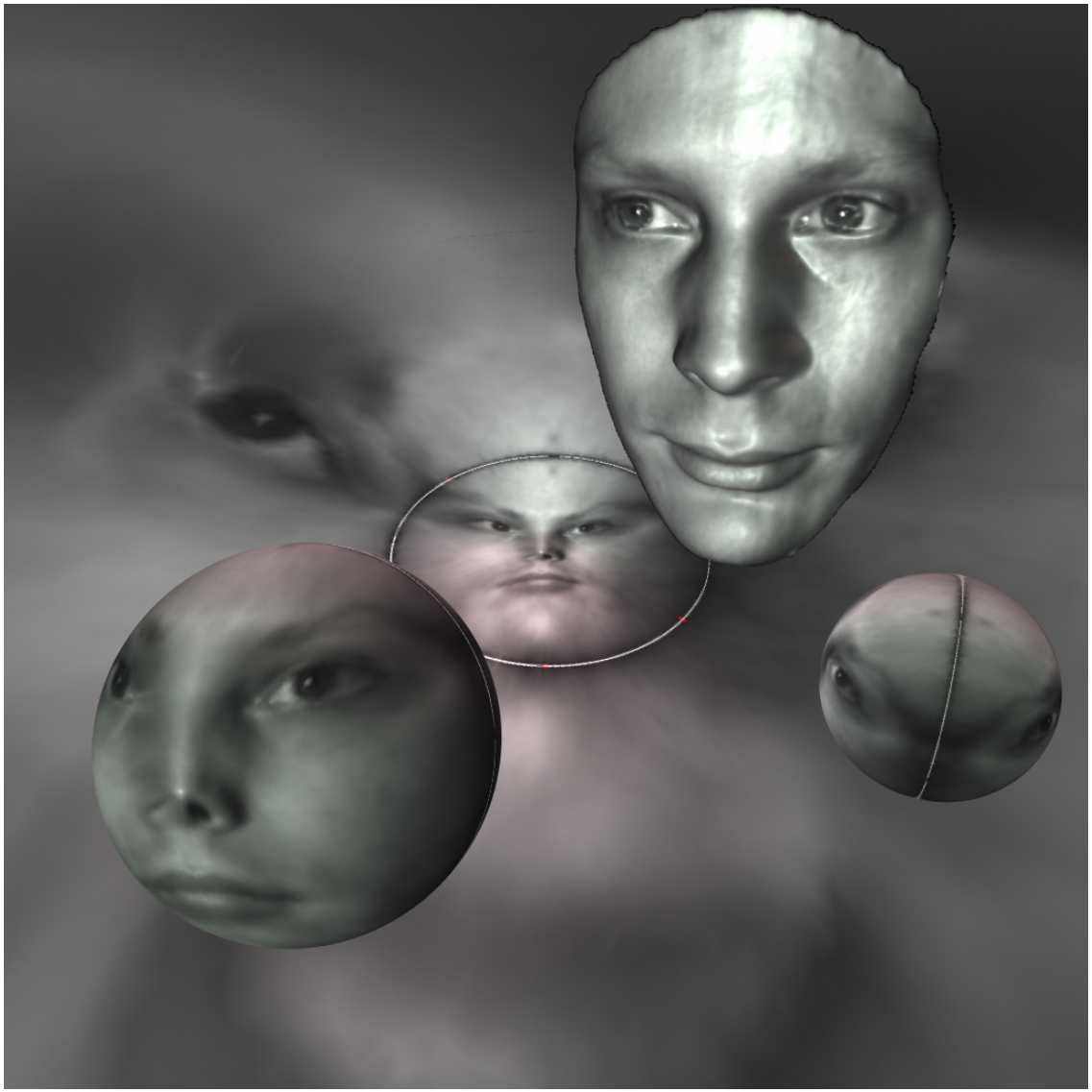


Figure: Spherical harmonic map for a double covering of a facial surface.

Riemann Mapping Algorithm I

Input: A topological disk surface M;
Output: A Riemann mapping $\varphi : M \to \mathbb{D}^2$.

1. Compute a double covering $\bar{M}$ of M;
2. Compute a harmonic map $\varphi : \bar{M} \to \mathbb{S}^2$;
3. Use a stereo-graphics projection to $\tau : \mathbb{S}^2 \hat{\mathbb{C}}$;
4. Use a Möbius transformation, to maps the hemisphere to the unit disk.

Riemann Mapping Algorithm II

Input: A topological surface $(M, \mathbf{g})$;
Output: A Riemann mapping $\varphi : M \to \mathbb{D}^2$;

1. Punch a small hole on M, to get $\bar{M}$, $\partial \bar{M} = \gamma_0 - \gamma_1$;
2. Solve Laplace equation $\Delta_{\mathbf{g}} f \equiv 0$ with Dirichlet boundary condition, $f|_{\gamma_0} = 0$ and $f|_{\gamma_1} = -1$;
3. Compute a harmonic 1-form ω, such that $\int_{\gamma_0} \omega = 2\pi$;
4. Find a constant , such that ${}^*\omega = -\lambda df$;
5. Find the shortest path τ between γ_0 and γ_1, slice $\bar{M}$ along τ to get an open surface $\hat{M}$;
6. Choose a base point $p \in \hat{M}$, compute the mapping
$$\varphi(q) := \exp\left(\int_p^q \lambda df + \sqrt{-1}\omega\right),$$
7. The mapping φ is the desired conformal mapping.

Computational Algorithm for a Random Walk

Input: A Topological Disk mesh M, a starting interior vertex v_0
Output: A random walk from v_0 to the boundary ∂M;

1. Compute cotangent edge weight w_{ij}, if $w_{ij} < 0$ replace it by an constant $\varepsilon > 0$;
2. at the current vertex v_i, generate a random number $r \in [0, 1]$,
3. sort all the neighboring vertex v_j counter-clock-wisely, find k, compute γ_k
$$\gamma_k := \frac{\sum_{j=0}^{k} w_{ij}}{\sum_{v_i \sim v_l} w_{il}}$$
4. find k, such that $r \in [\gamma_k, \gamma_{k+1})$, walk to v_k,
5. Repeat step 2 through 4, until reaches the boundary.

Computational Algorithm for Stochastic Harmonic Map

Input: A Topological Disk mesh M
Output: Harmonic map $\varphi : M \to \mathbb{D}$

1. Compute cotangent edge weight w_{ij}, if $w_{ij} < 0$ replace it by an constant $\varepsilon > 0$;
2. Set the boundary condition, maps each boundary vertex to the unit circle, $\varphi(v_k) = e^{i\theta_k}$, θ_k is proportional to the arc length parameter of ∂M;
3. for each interior vertex v_i, generate n random walks γ_i, the end vertex of γ_i is denoted as $e(\gamma_i)$, the start every $s(\gamma)$, estimate the expect value
$$\hat{E}_{s(\gamma)=v_i}(\varphi(e(\gamma))) = \frac{1}{n}\sum_{i=1}^{n}\varphi(e(\gamma_i)).$$
4. set $\varphi(v_i) \leftarrow \hat{E}_{s(\gamma)=v_i}(\varphi(e(\gamma)))$.

Assignment Two - Disk Harmonic Map

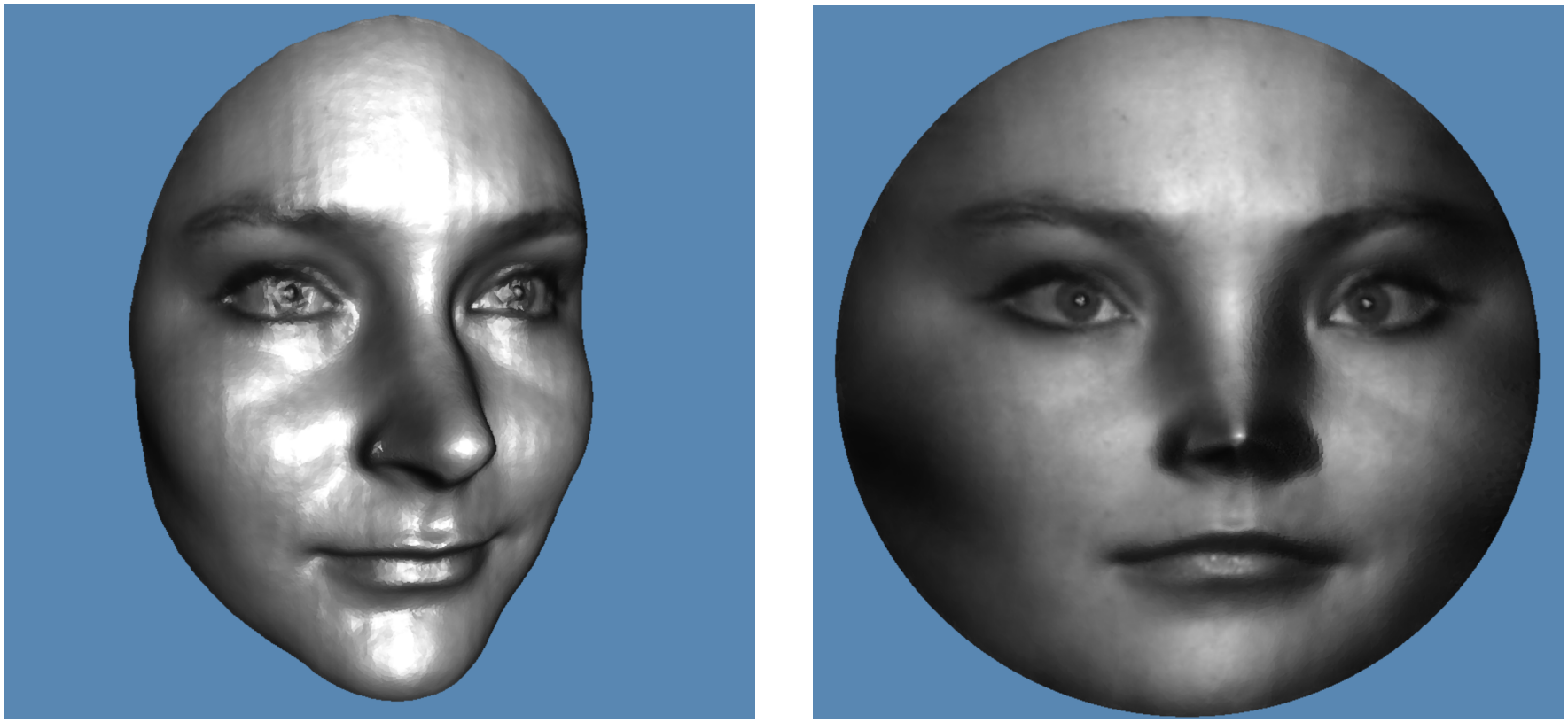


Figure: Harmonic map between topological disks.

Assignment Four - Spherical Harmonic Map

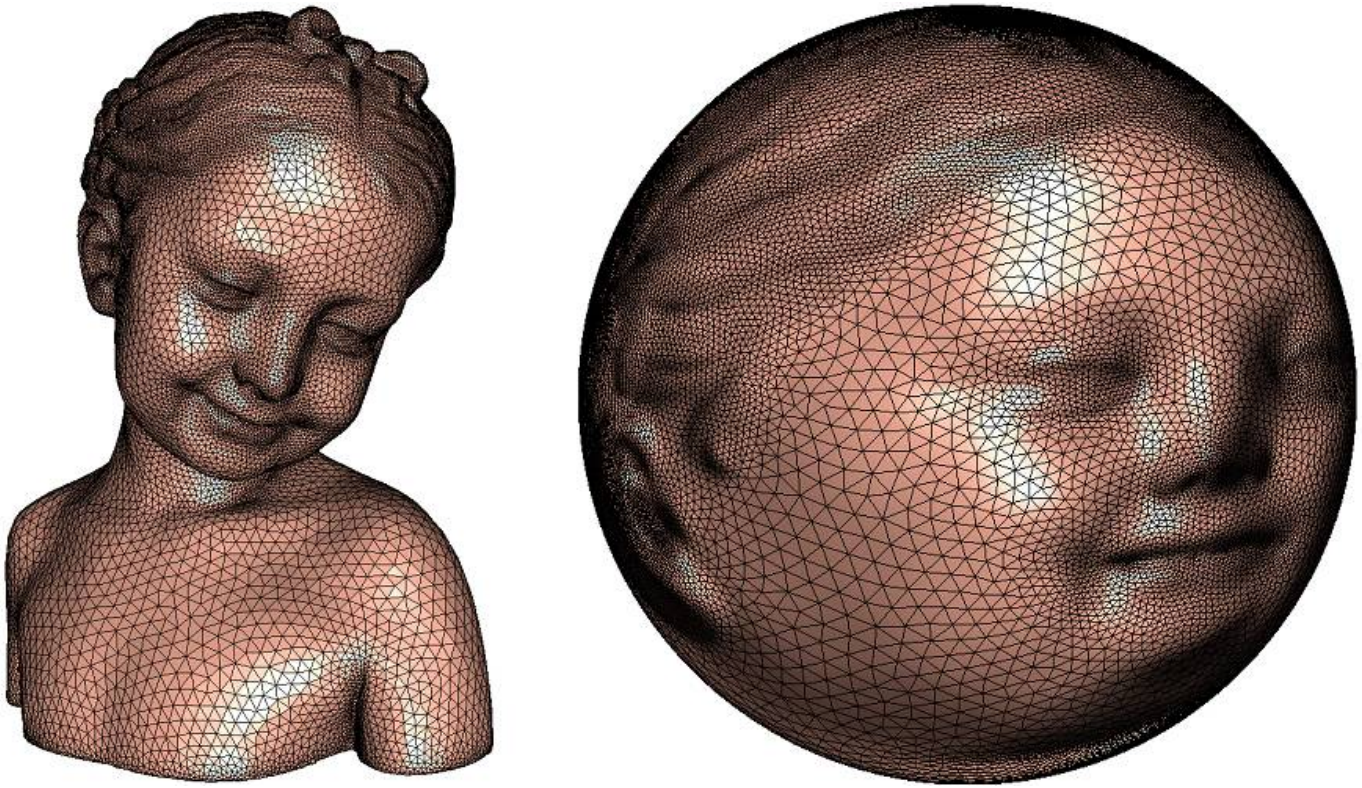


Figure: Harmonic map between topological spheres.

Complex Analysis

Normal Family Definition

Definition (Uniform Convergence)

Assume $\{f_n : \Omega \to \mathbb{C}\}$ is a sequence of holomorphic functions defined on an open set Ω. We say the functions uniformly converge to a function $f : E \to \mathbb{C}$, if for any $\varepsilon > 0$, there is a n_0, such that for any $n > n_0$ and any $z \in E$, we have

$$|f_n(z) - f(z)| < \varepsilon.$$

Definition (Normal Family)

Let $\Omega \subset \mathbb{C}$ be an open set on $\mathbb{C}$, $\mathcal{F}$ is a normal family, if any subsequence $\{f_n\}$ in $\mathcal{F}$ uniformly converge on any compact subset in Ω.

Normal Family Properties

Theorem (Weierstrass)

Let $\{f_n : \Omega \to \mathbb{C}\}$ be a sequence of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, assume $\{f_n\}$ uniformly converges to $f : \Omega \to \mathbb{C}$ on compact subsets in Ω, then f is holomorphic and $\{f_n' : \Omega \to \mathbb{C}\}$ uniformly converges to $f' : \Omega \to \mathbb{C}$.

Normal Family Properties

Definition (Univalent Map)

Let $U \subset \mathbb{C}$ be open subset on $\mathbb{C}$, if holomorphic map $f : U \to \mathbb{C}$ is injective, namely $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$, then f is called a univalent map or univalent function.

Theorem (Hurwitz)

Let $\{f_n : \Omega \to \mathbb{C}\}$ be a family of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, such that for any n and $z \in \Omega$, $f_n(z) \neq 0$. If $\{f_n\}$ uniformly converges to $f : \Omega \to \mathbb{C}$ on compact sets of Ω, then either $f \equiv 0$ or for any $z \in \Omega$, $f(z) \neq 0$.

Corollary

Let Ω be an open set in $\mathbb{C}$, let $\{f_n : \Omega \to \mathbb{C}\}$ be a holomorphic function series, and uniformly converges to $f : \Omega \to \mathbb{C}$ on compact sets. If each f_n on Ω is univalent, then either f is constant, or f is univalent on Ω.

Normal Family

Theorem (Montel)

Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, if $\mathcal{F}$ is uniformly bounded on compact sets in Ω, then

1. *$\mathcal{F}$ is equicontinuous on each compact set in Ω;*
2. *$\mathcal{F}$ is a normal family.*

1. Fix a point $p \in \Omega$, a family of univalent holomorphic functions $\mathcal{F}$ is a normal family, if for any $f \in \mathcal{F}$, $|f(p)| < M$ and $|f'(p)| < N$.
2. A family of holomorphic functions $\mathcal{F}$, if there are three points $\{z_1, z_2, z_3\}$, such that for any $f \in \mathcal{F}$, the image of f doesn't include them, then $\mathcal{F}$ is a normal family.
3. If $\mathcal{F}$ is a normal family, then

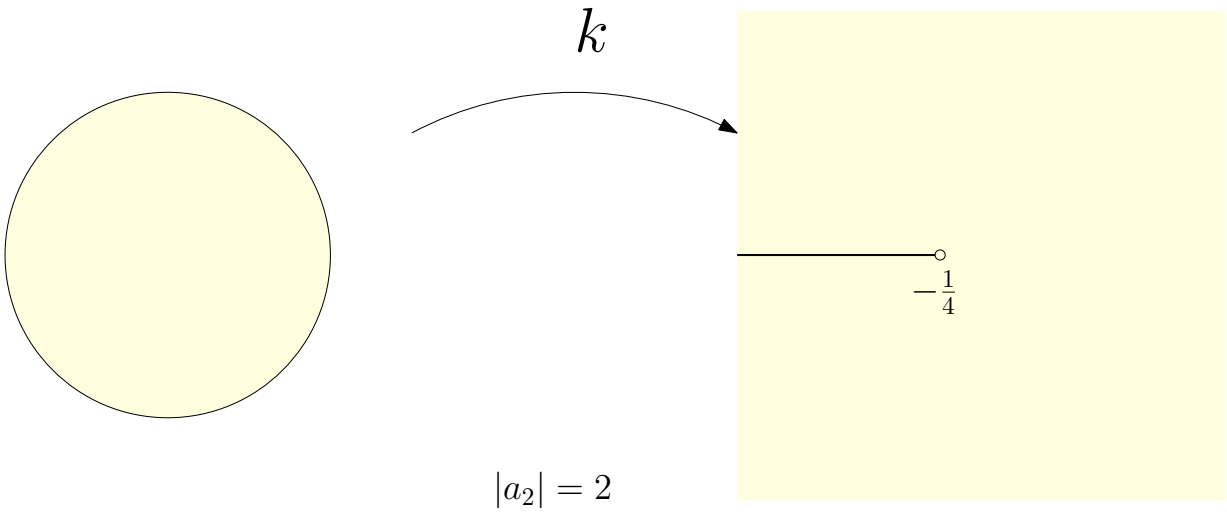
$$\mathcal{F}^{-1} = \{f^{-1} | f \in \mathcal{F}\}$$

is also a normal family.

Geometric Distortion Estimate

Theorem (Bieberbach a_2 of $\mathcal{S}$)

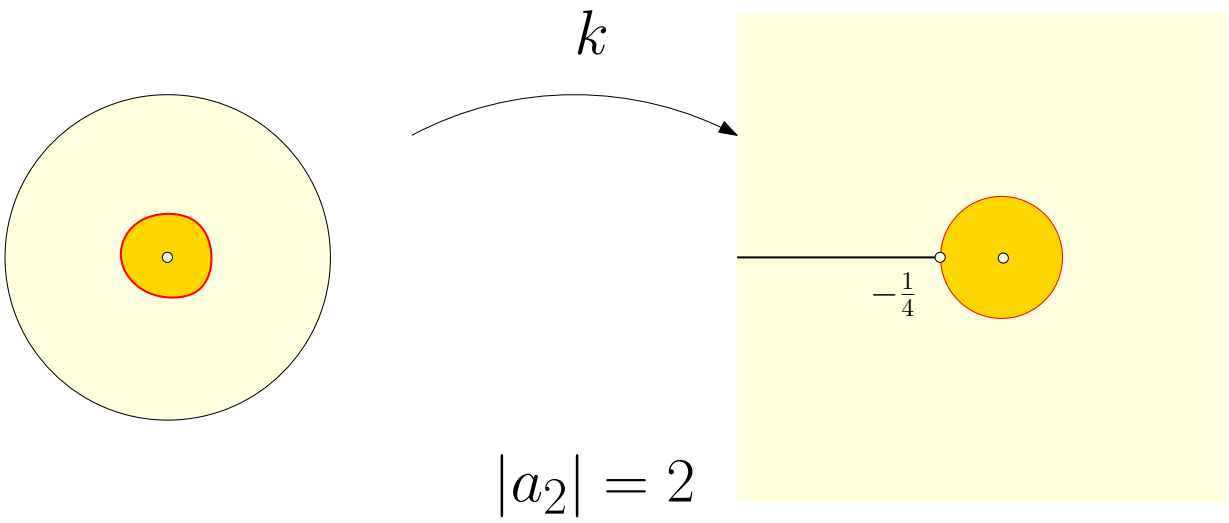
If $f \in \mathcal{S}$, then $|a_2| \leq 2$, equality holds if and only if f is a rotation of the Koebe function.



Geometric Distortion Estimate

Theorem (Koebe 1/4)

For any $f \in \mathcal{S}$, $f(\mathbb{D})$ includes an open disk $|w| < 1/4$. If there exists a $|w| = 1/4$ and $w \notin f(\mathbb{D})$, then f is a rotation of Koebe function.



Riemann Mapping

Theorem (Riemann)

Givne a non-empty, simply connected, open subset $\Omega \subset \mathbb{C}$, Ω is not the entire complex plane $\mathbb{C}$, for any point $z_0 \in \Omega$, there exits a unique biholomorphic mapping from Ω to the unit disk $\mathbb{D}$, $f : \Omega \to \mathbb{D}$, such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Riemann Mapping

Uniqueness

If we don't require $f(z_0) = 0$ and $f'(z_0) > 0$, then conformal mapping is not unique. All such kind of mappings differ by a Möbius transformation, $\varphi : \mathbb{D} \to \mathbb{D}$,

$$\varphi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad z_0 \in \mathbb{D}, \theta \in [0, 2\pi)$$

Extendibility

If Ω is a Jordan domain, the boundary $\partial\Omega$ is a piecewise anylatical curves, then the conformal mapping φ can be extended to the boundary $\varphi : \partial\Omega \to \partial\mathbb{D}$.

Riemann Mapping

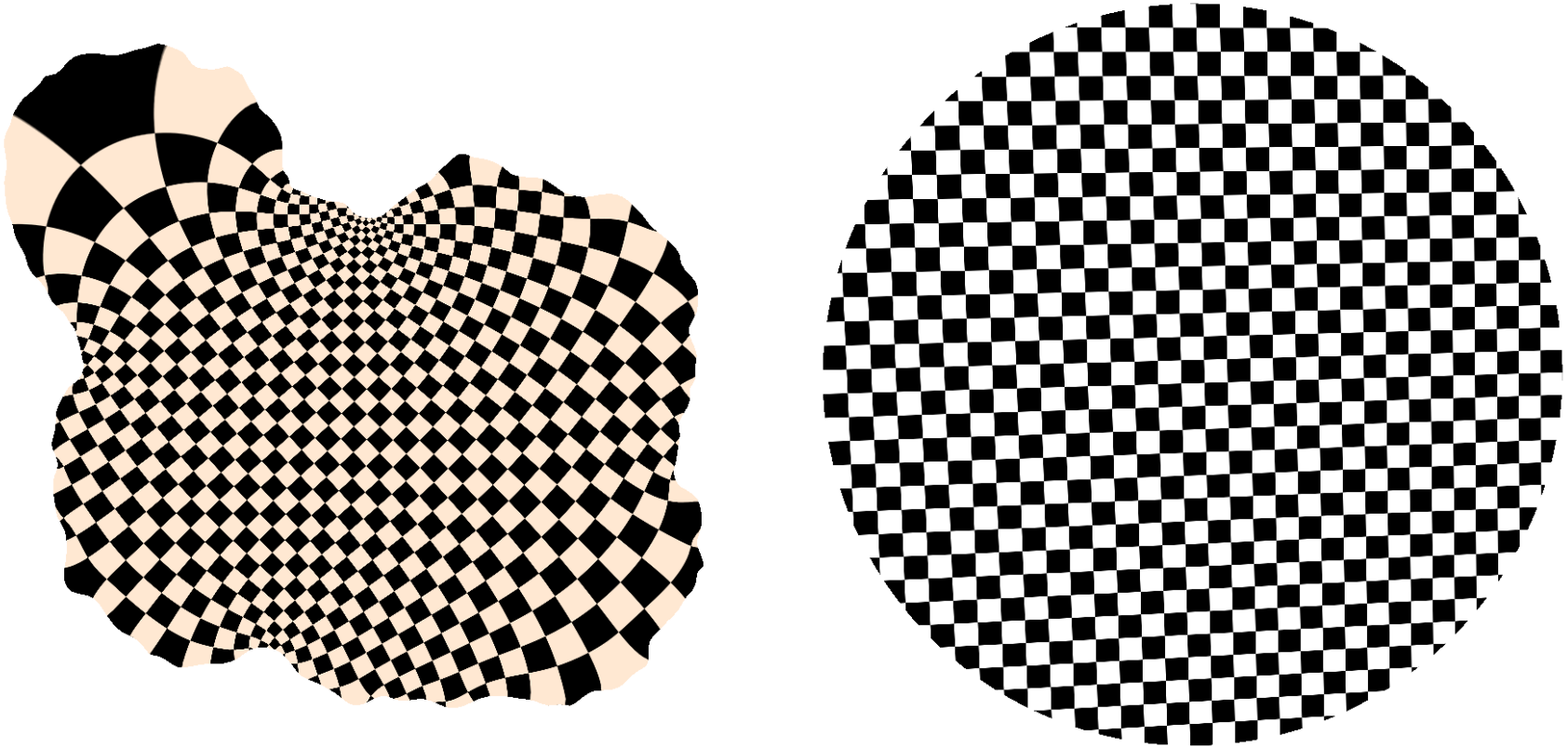


Figure: Riemann Mapping

Riemann Mapping

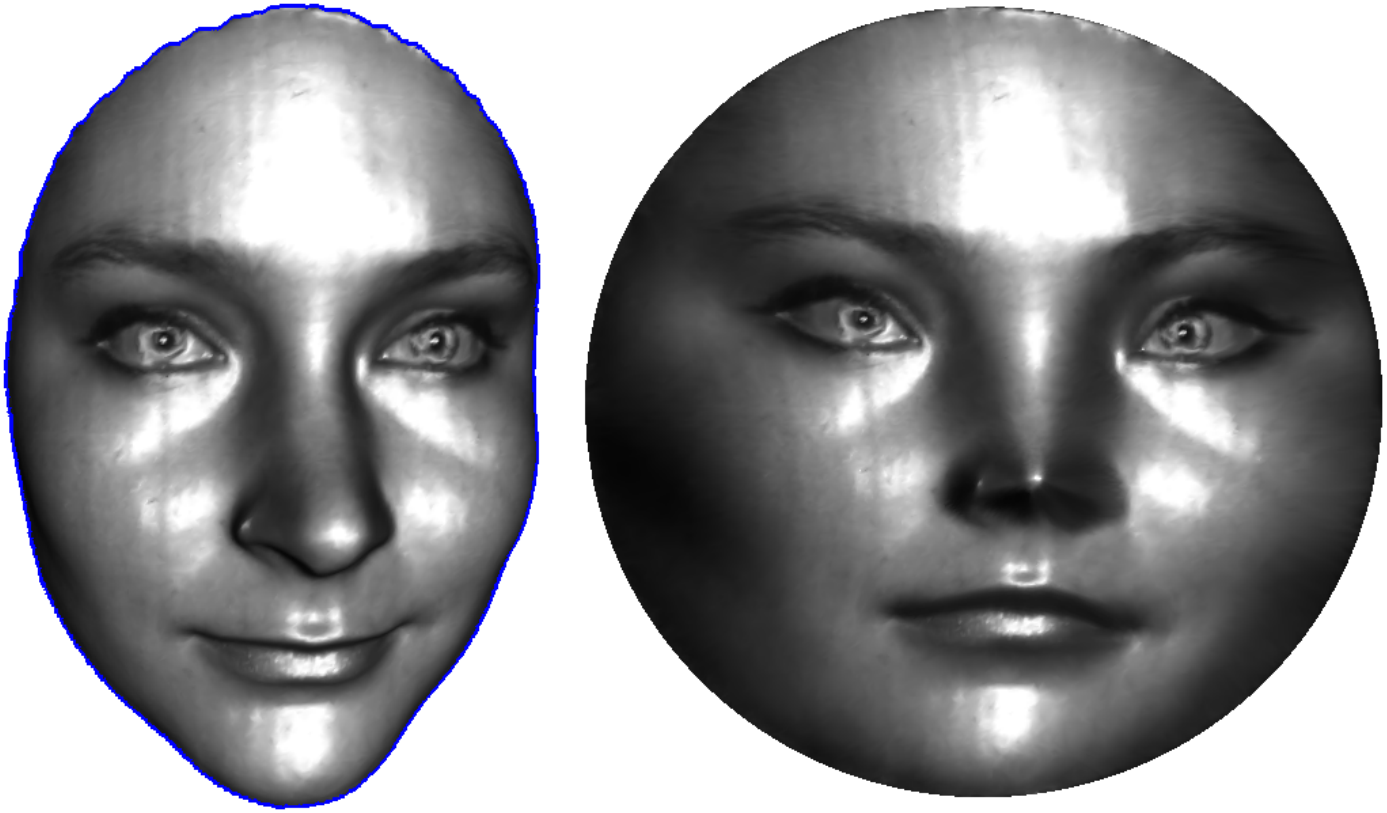


Figure: Riemann Mapping

Schwartz Lemma

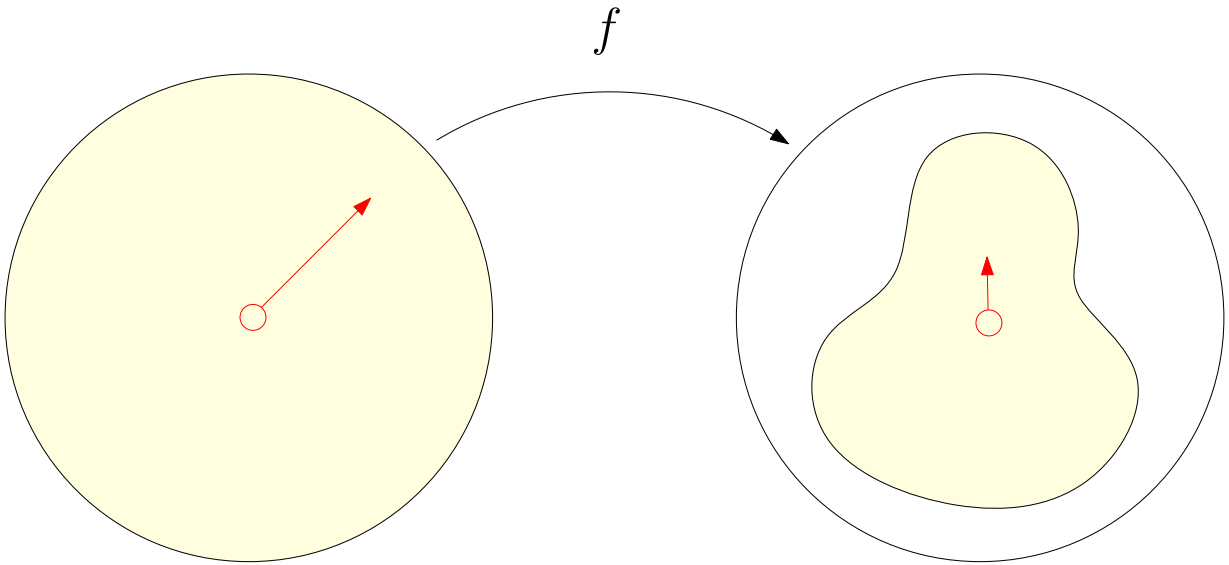


Figure: Schwartz lemma.

Conformal Mapping for Annulus

(a)Topological annulus

(b) Conformal module

Figure: Canonical conformal mapping for topological annulus.

Conformal Module for Topological Annulus

Theorem (Conformal Module for Annulus)

Suppose Ω is a doubly connected domain on $\mathbb{C}$, then Ω is conformally equivalent to a canonical annulus.

Slit Map

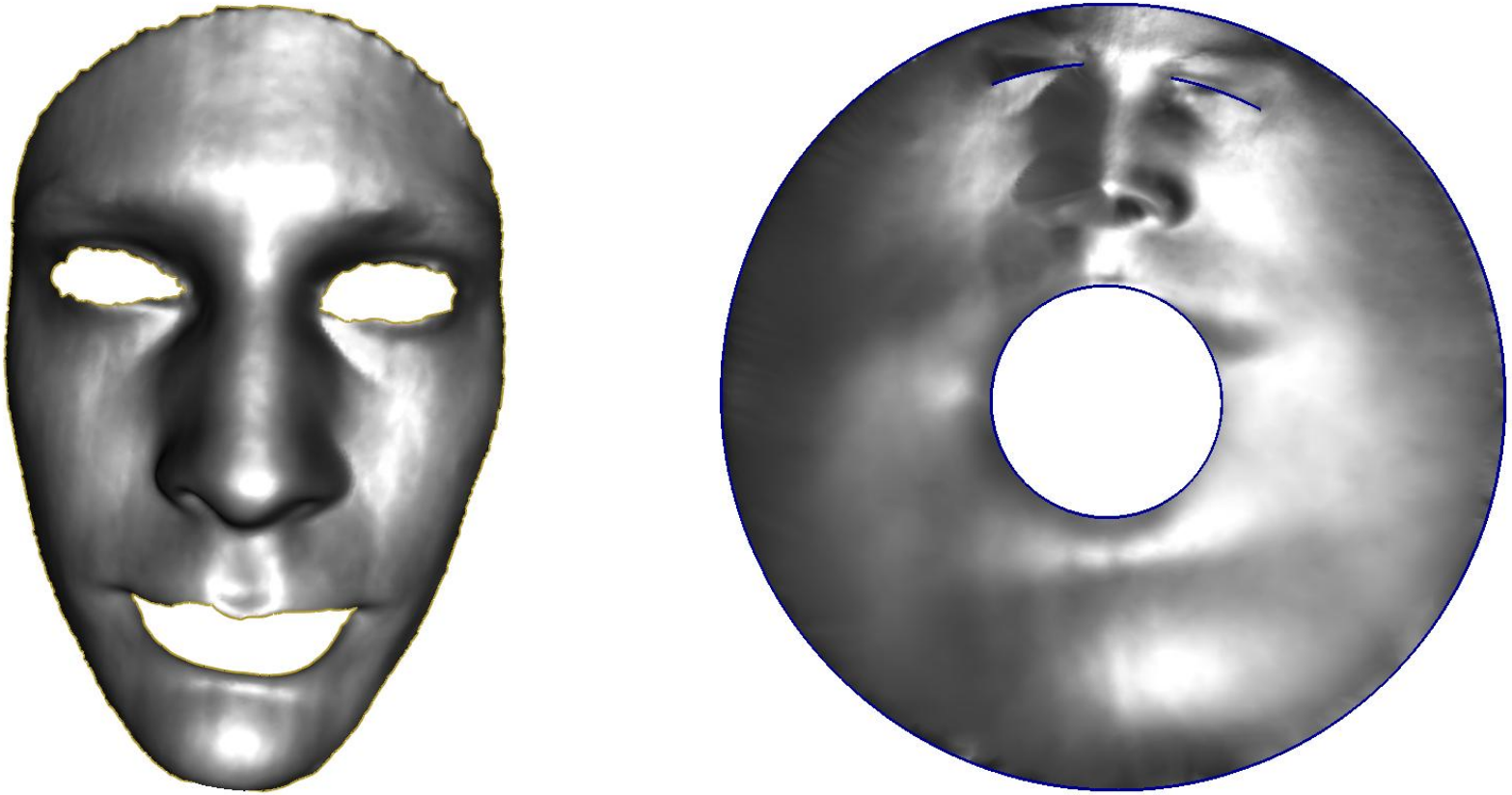


Figure: Slit map.

Slit Map

Definition (slit domain)

A connected open set (domain) $\Omega \subset \mathbb{C}$ is called a slit domain, if every connected component of its boundary $\partial\Omega$ is either a point or a horizontal closed interval.

Theorem (Hilbert)

Given any domain $\Omega \subset \mathbb{C}$, its boundary has finite number of connected components, then Ω is conformal equivalent to a slit domain.

Conformal Module for Poly-annulus

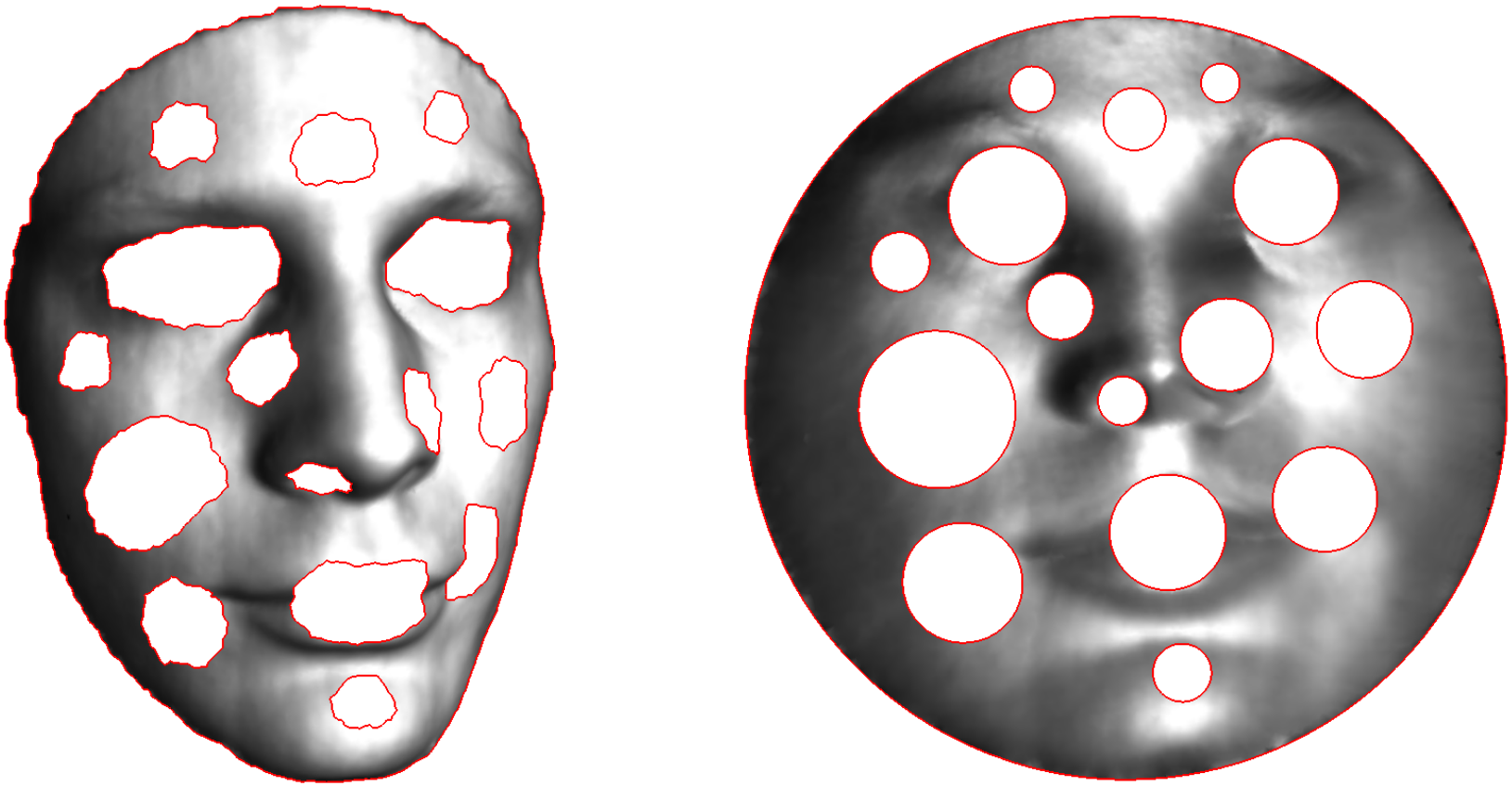


Figure: Conformal mapping from a poly-annulus to a circle domain.

Circle Domain

Definition (Circle Domain)

Suppose $\Omega \subset \hat{\mathbb{C}}$ is a planar domain, if $\partial\Omega$ has finite number of connected components, each of them is either a circle or a point, then Ω is called a circle domain.

Theorem (Koebe)

Suppose S is of genus zero, ∂S has finite number of connected components, then S is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.

Circle Domain Mapping - Uniqueness

Theorem (Uniqueness)

Given two circle domains $C_1, C_2 \subset \hat{\mathbb{C}}$, $f : C_1 \to C_2$ is a univalent holomorphic function, then f is a linear rational, namely a Möbus transformation.

Proof.

Assume both C_1 and C_2 include ∞, and $f(\infty) = \infty$. Since f is holomorhic, it maps the boundary circles of C_1 to those of C_2. By Schwartz reflection principle, f can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. **??**, the multiple reflected domains cover the whole $\hat{\mathbb{C}}$, hence f can be extended to the whole $\hat{\mathbb{C}}$, since $f(\infty) = \infty$, f is a linear function. If $f(\infty) \neq \infty$, we can use a Möbius map to transform $f(\infty)$ to ∞. $\square$

Circle Domain Mapping - Existence

Theorem (Existence)

On the z-plane, every n-connected domain Ω can be mapped to a circle domain on the ζ-plane by a univalent holomorphic function. Choose a point $a \in \Omega$, there is a unique map which maps a to $\zeta = \infty$, and in a neighborhood of $z = a$, the map has the power series

$$\frac{1}{z-a} + a_1(z-a) + \cdots \text{ if } a \neq \infty$$

$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$

Koebe's Iteration - Error Estimate

We proved the convergence rate of Koebe's iteration.

Theorem (Convergence Rate of Koebe's Iteration)

In the Koebe's iteration, when $k > mn$,

$$|g_k(w) - w| \leq \frac{1}{4\delta}\left(\frac{4}{\pi}\pi\rho^2 + \frac{\pi}{2\log\mu^{-1}}16\pi\rho^2\right)\mu^{4m}.$$

This shows μ controls the convergence rate.

Liuville Theorem

Theorem (Liuville)

Suppose a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is bounded, $|f(z)| < C$, for all $z \in \mathbb{C}$, then $f(z) = const$.

Proof.

According to Cauchy's formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^{n+1}} dz,$$

here Γ is a circle centered at a with radius r,

$$|f'(a)| = \left| \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z-a)^2} dz \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{C}{r} d\theta = \frac{C}{r},$$

let $r \to \infty$, the derivative goes to 0. Hence the holomorhic function $f(z)$ is constant. □

Liuville Theorem

The unit sphere $\mathbb{S}^2$ is conformal equivalent to the augmented complex plane $\hat{\mathbb{C}}$. Complex plane $\mathbb{C}$ and the unit open disk $\mathbb{D}$ are open sets, therefore they are not homeomorphic to the compact set $\mathbb{S}^2$. Liuville theorem shows $\mathbb{C}$ and $\mathbb{D}$ are not conformally equivalent to each other.

Corollary

The complex plane $\mathbb{C}$ and the unit disk $\mathbb{D}$ are not conformally equivalent.

Proof.

Suppose they are equivalent, there is a biholomorphic function $f : \mathbb{C} \to \mathbb{D}$, according to Liuville, $f(z)$ is constant. Contradiction to biholomorphic function. $\square$

Crescent and Full-Moon Theorem

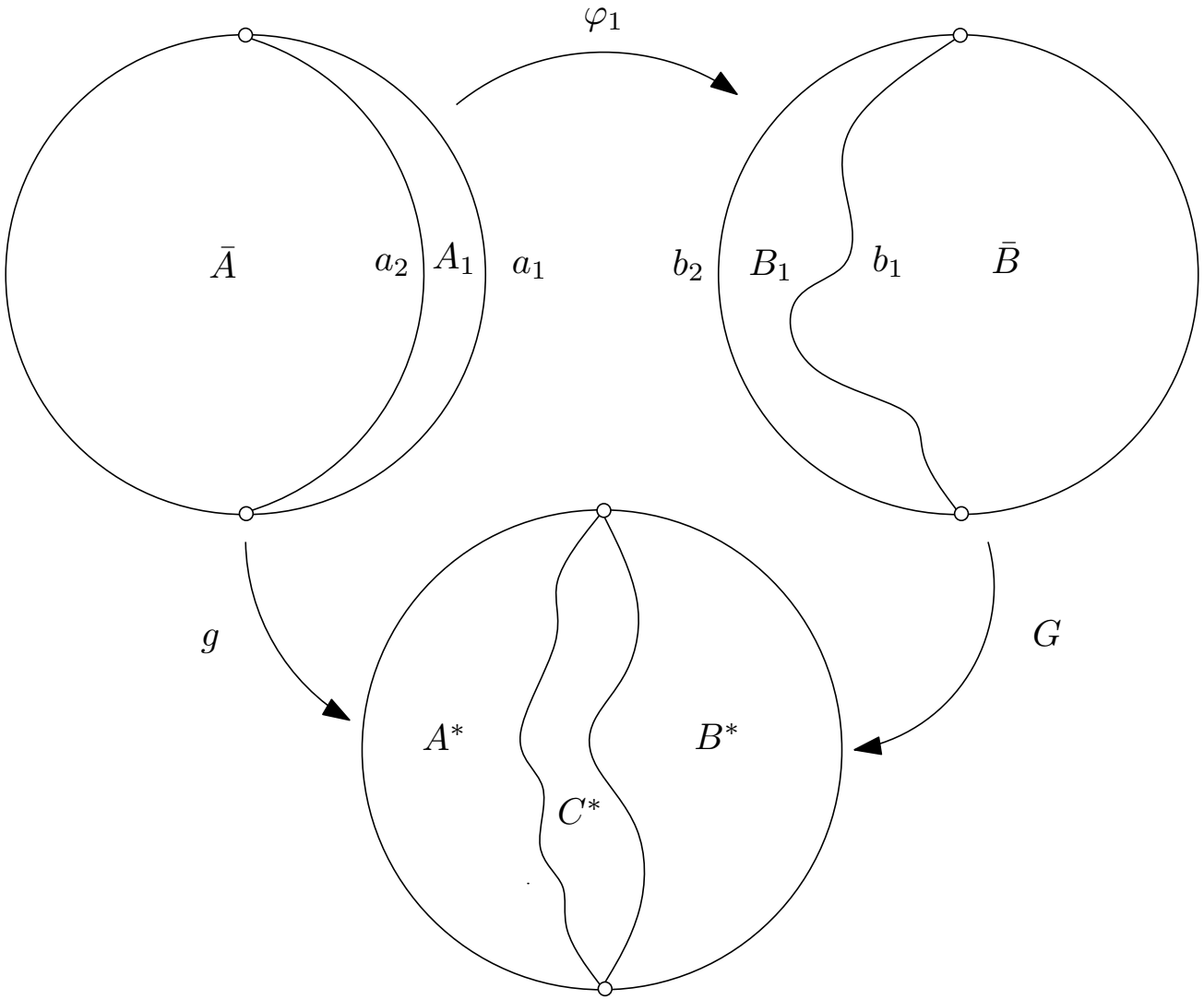


Figure: Analytic extension result.

Crescent and Full Moon

Lemma (Crescent and Full Moon)

As shown in Fig. **??**, *the boundaries of the crescent domain* A_1 *are circular arcs* a_1 *and* a_2, *they have intersection angle* $\pi/2^m$, $m \in \mathbb{Z}^+$. *A conformal map* $\varphi_1 : A_1 \to B_1$ *is defined on the crescent* A_1, $\varphi_1(a_k) = b_k$, $k = 1, 2$, b_2 *is a circular arc. Then there exist analytic functions,* $g, G : \mathbb{D} \to \mathbb{D}$, *as shown Fig. 36, satisfying*

1. $A^* = g(\bar{A})$, $C^* = g(A_1)$;
2. $B^* = G(\bar{B})$, $C^* = G(B_1)$;
3. $g|_{A_1} = G \circ \varphi_1|_{A_1}$;

and the restriction on a_k*'s and* b_k*'s, the mappings* g *and* G *are homeomorphisms.*

Uniformization

Theorem (Open Riemann Surface Uniformization)

Simply connected open Riemann surface is conformal equivalent to the whole complex plane $\mathbb{C}$ or the unit open disk $\mathbb{D}$.

Proof.

Construct a sequence of holomorphic functions

$$\varphi_{1,n}(s) = \varphi_n \circ \varphi_1^{-1},$$

univalent on R_1, and normalized at $s = 0$, $\varphi_{1,n}(0) = 0$, $\varphi'_{1,n}(0) = 1$. Then $\{\varphi_{1,n}\}$ is a normal family. We choose subsequence $\Gamma_1 \subset \{\varphi_{1,n}\}$, which converges to univalent function in the interior of R_1, denoted as

$$\Gamma_1 : \varphi_1^1(p), \varphi_2^1(p), \varphi_3^1(p), \cdots$$

converges to a univalent function $\varphi_0(p)$ in E_1. □

Uniformization

Theorem (Compact Riemann Surface Uniformization)

Compact simply connected Riemann surface is conformal equivalent to the unit sphere.

Proof.

Suppose $\tilde{M}$ has a triangulation $\mathcal{T}$, which includes a finite number of faces,

$$\mathcal{T}_n = \Delta_1 + \Delta_2 + \cdots + \Delta_n,$$

the last triangle Δ_n has three common eges with T_{n-1}. Choose an interior point $q \in \Delta_n$, remove this point, we obtain an open Riemann surface,

$$\tilde{M}_0 = \tilde{M} \setminus \{q\},$$

according to open Riemann surface uniformization theorem, there is a conformal mapping, $\varphi : \tilde{M}_0 \to \mathbb{C}$, $s = \varphi(p)$, which maps the open Riemann surface either to a unit disk or the whole complex plane. □

Surface Uniformization

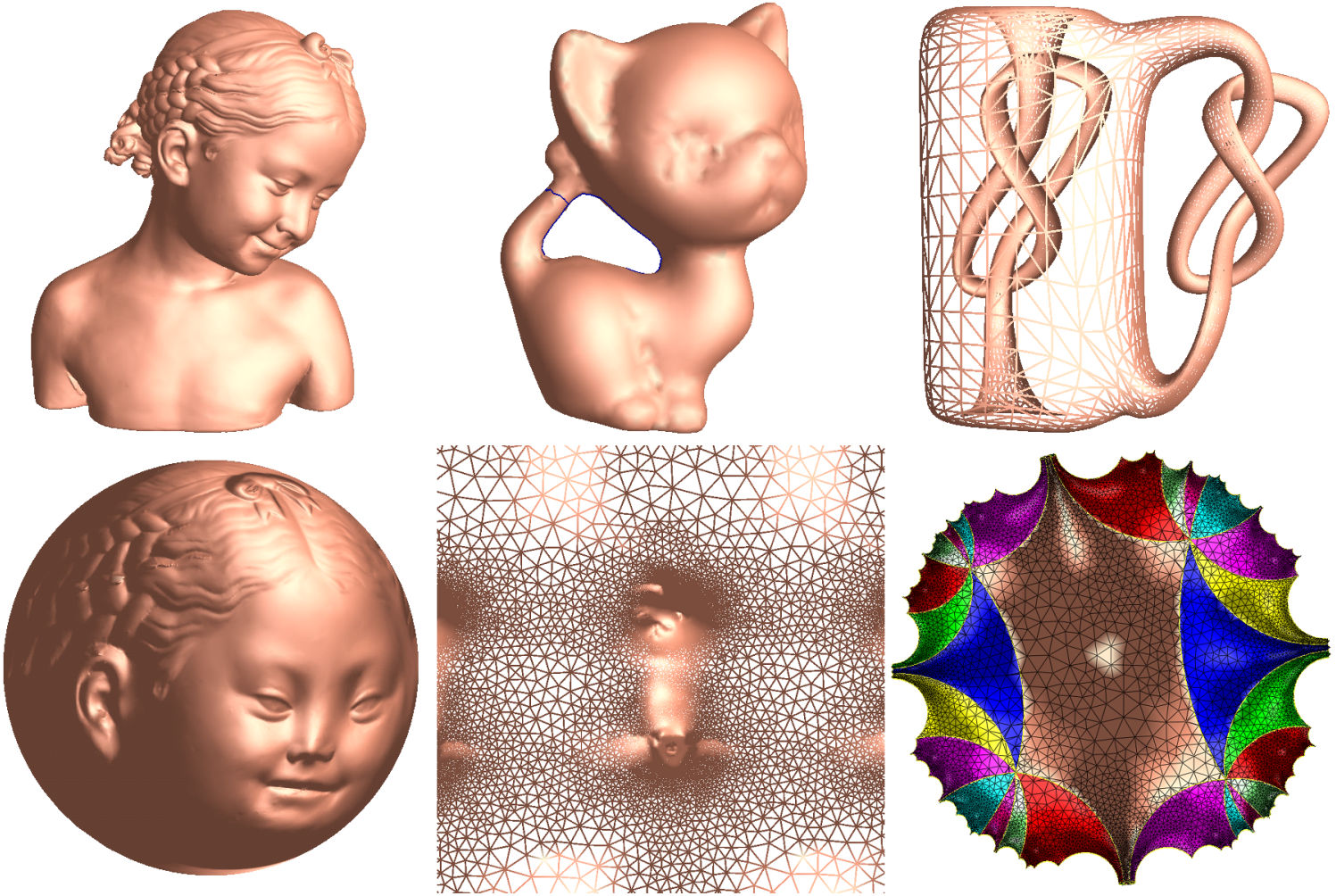


Figure: Closed surface uniformization.

Surface Uniformization

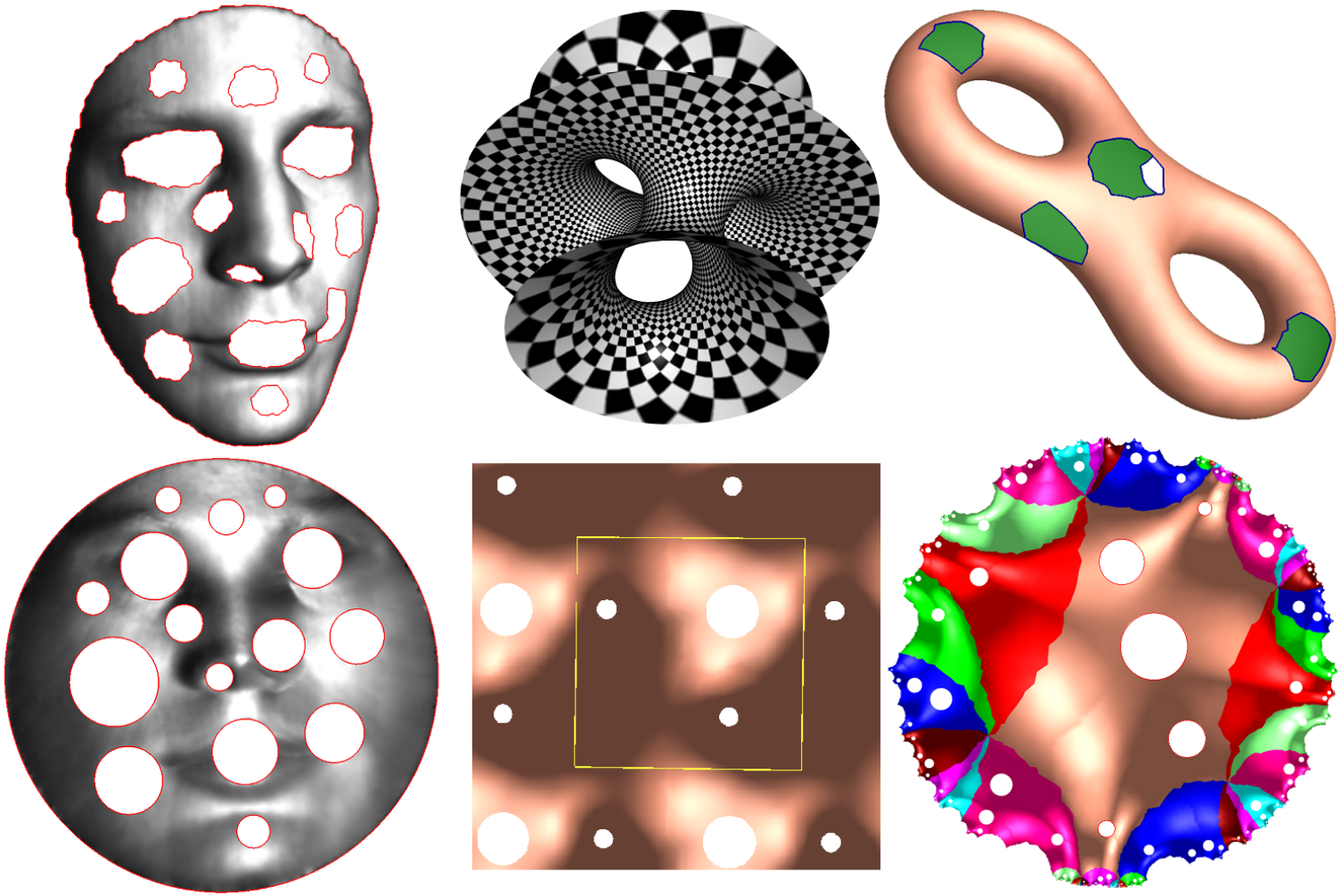


Figure: Open surface uniformization.

Slit Map Algorithm

Input: A genus zero mesh with $n+1$ boundary components M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;
Output: A slit map : $M \to D$, D is a circular slit domain.

1. Compute exact harmonic 1-forms $\omega_1, \omega_2, \ldots, \omega_n$;
2. Compute closed, non-exact harmonic 1-forms $h_1, h_2, \ldots, h_n$;
3. Compute conjugate harmonic 1-forms ${}^*\omega_1, {}^*\omega_2, \ldots, {}^*\omega_n$;
4. Find special holomorphic 1-form φ

$$\Im \int_{\gamma_0} \varphi = 2\pi, \Im \int_{\gamma_1} \varphi = -2\pi, \Im \int_{\gamma_k} \varphi = 0, k = 2, 3, \ldots, n.$$

5. Slit map $f : M \to D$, choose a fixed based point $p \in M$,

$$f(q) := \exp \int_p^q \varphi$$

the integration path can be chosen arbitrarily.

Slit Map Algorithm - Fill Holes

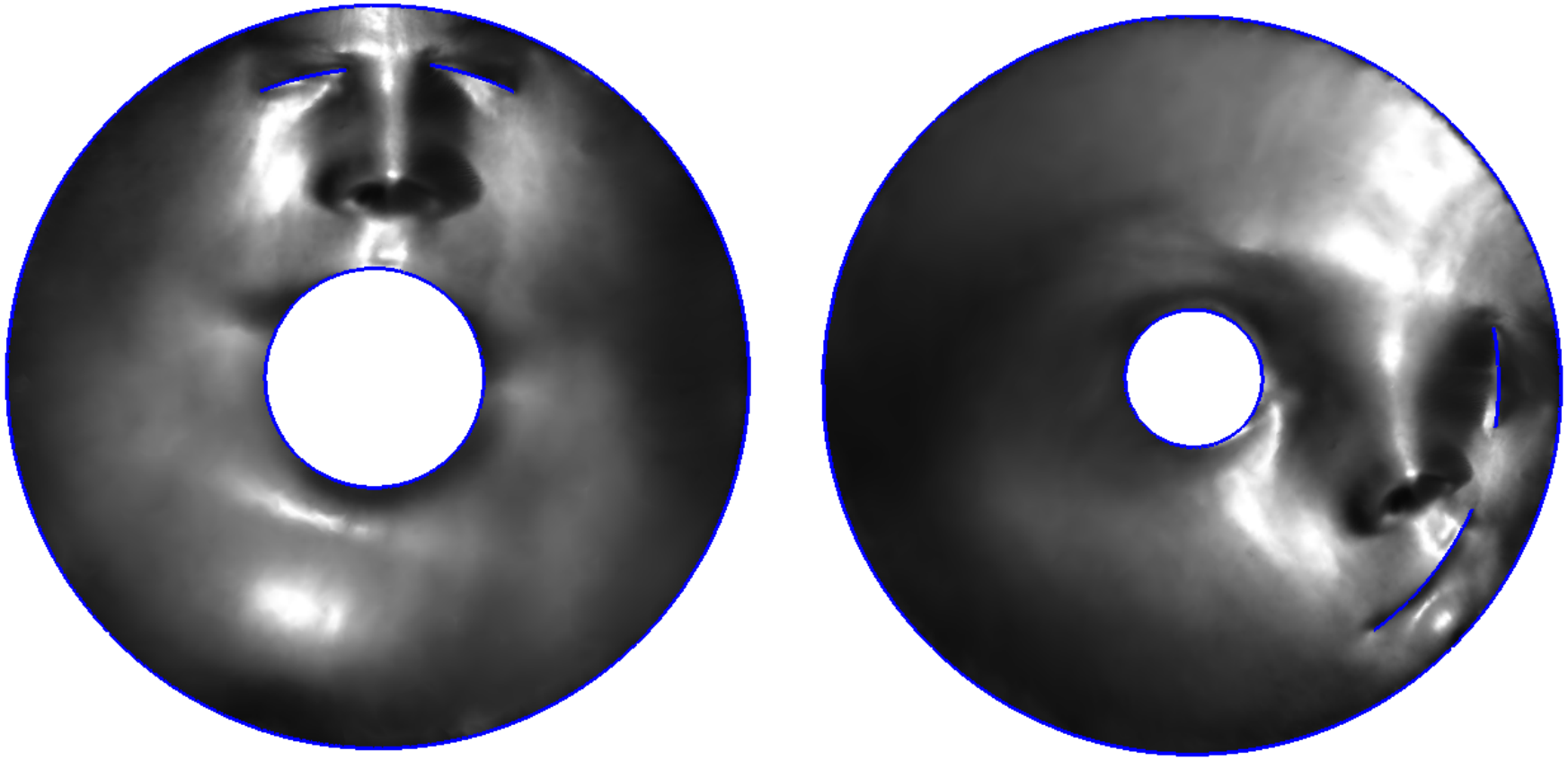


Figure: Slit maps.

Slit Map Algorithm

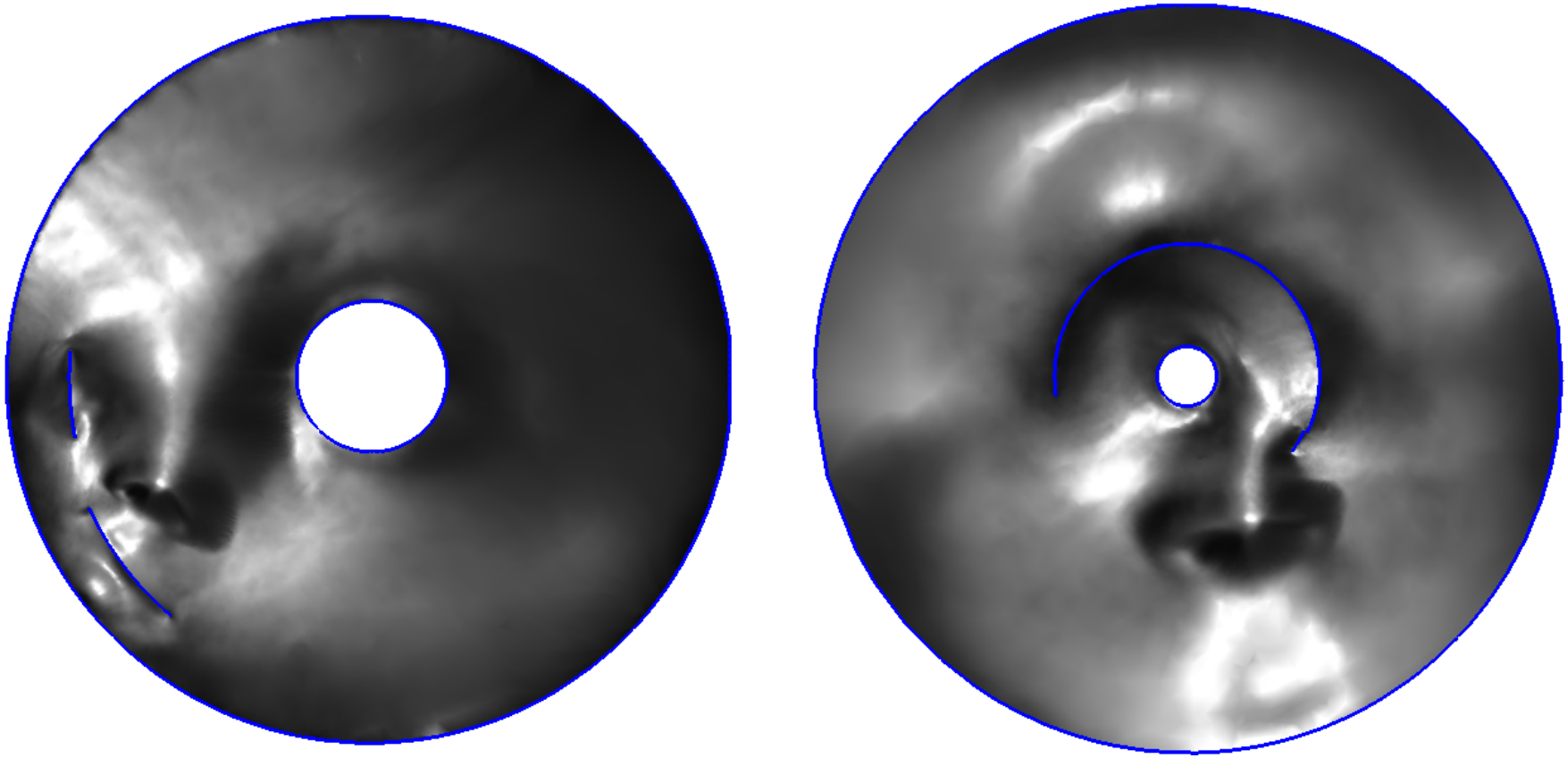


Figure: Slit maps.

Slit Map Algorithm

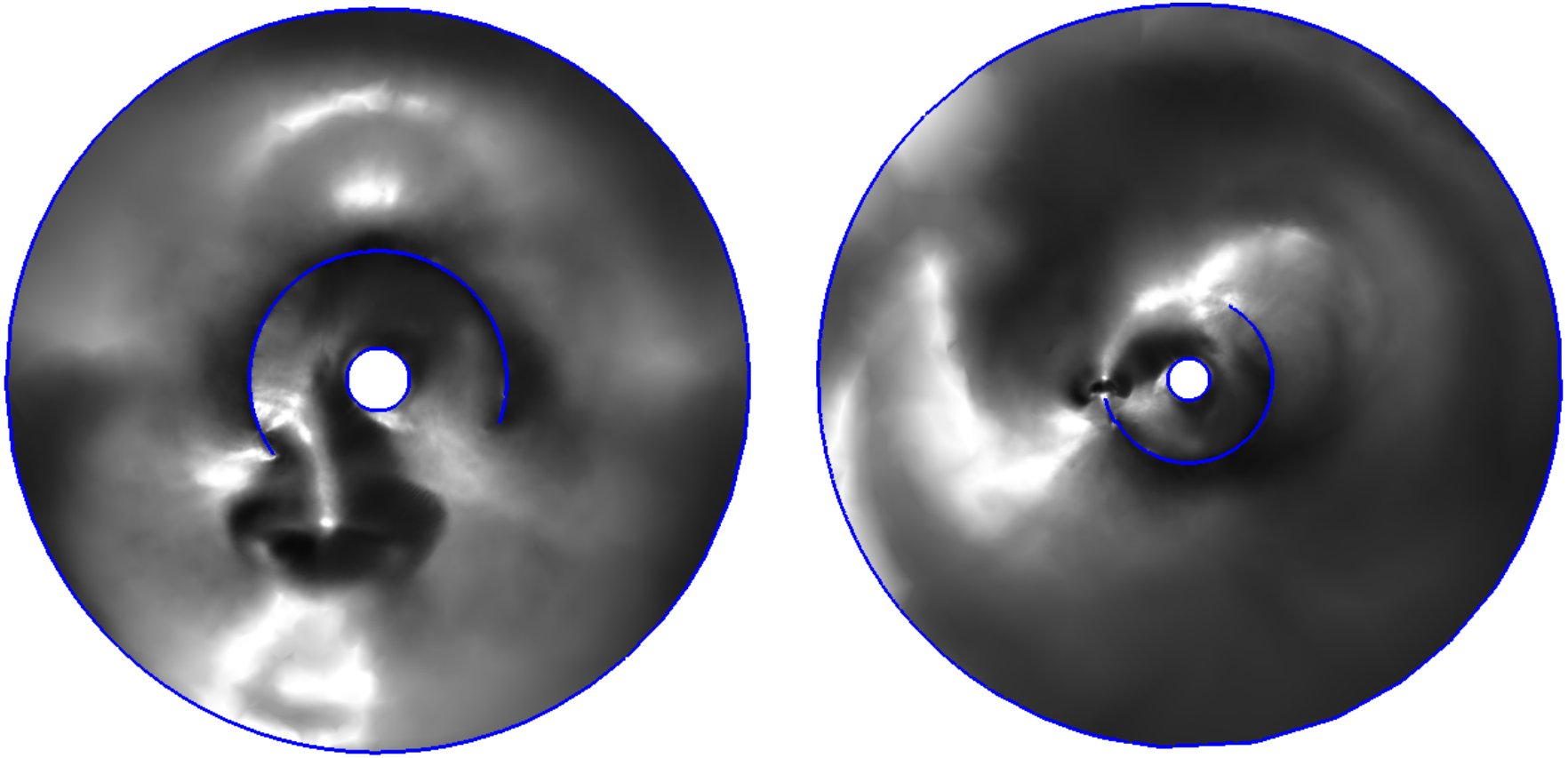


Figure: Slit maps.

Koebe Iteration Algorithm

Input: Poly annulus M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;
Output:Conformal map $\varphi : M \to \mathbb{D}$, where $\mathbb{D}$ is a circle domain.

1. Compute a slit map, map the surface to the circular slit domain $f : M \to \mathbb{C}$, γ_0 and γ_k are mapped to the exetior and interior circular boundary of $\mathbb{C}$;
2. Fill the inner circle using Delaunay refinement mesh generation;
3. Repeat step 1 and 2, fill all the holes step by step;

Koebe Iteration Method

Figure: Slit map.

Koebe Iteration Method

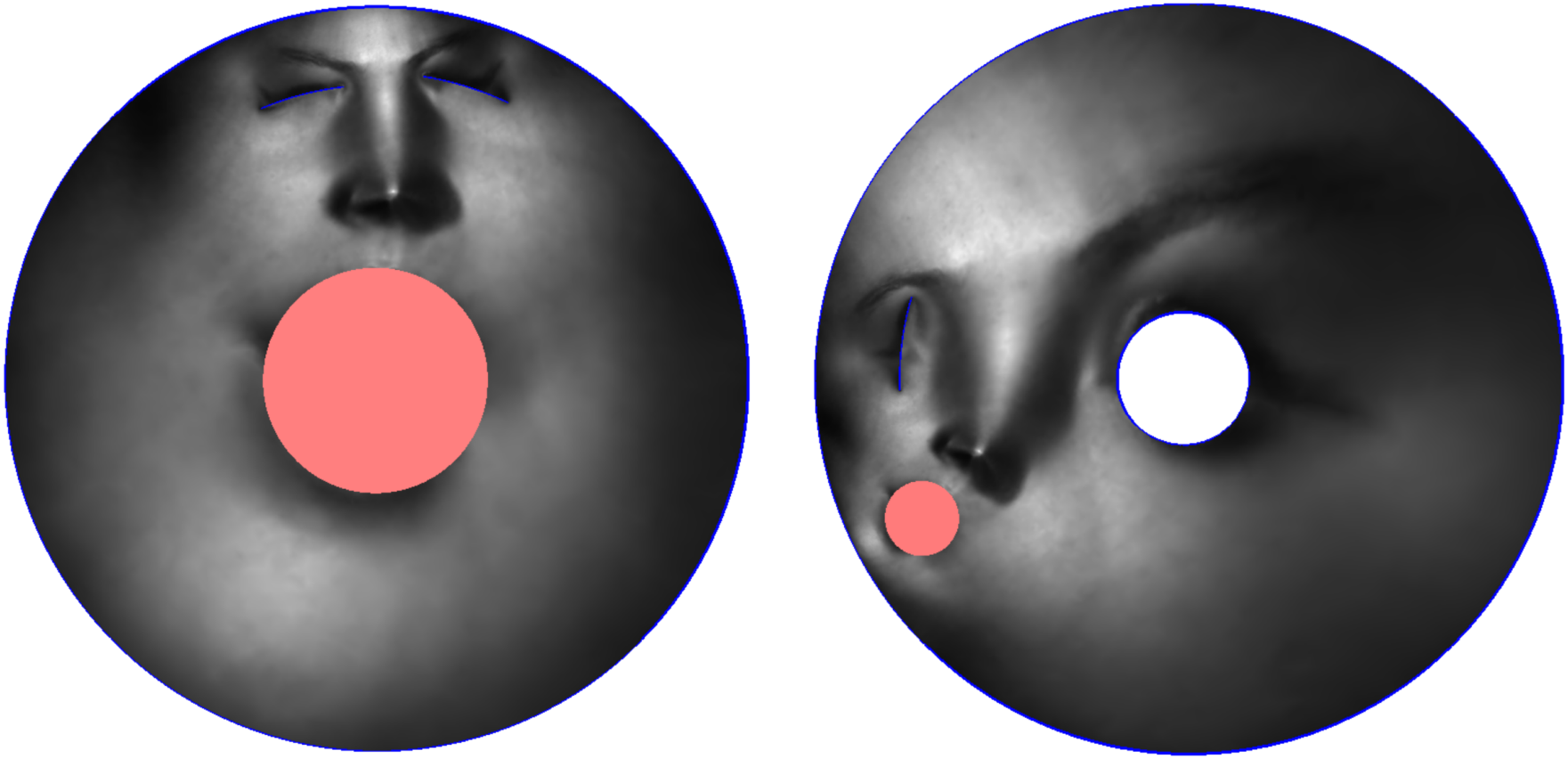


Figure: Hole filling and slit map.

Koebe Iteration Method

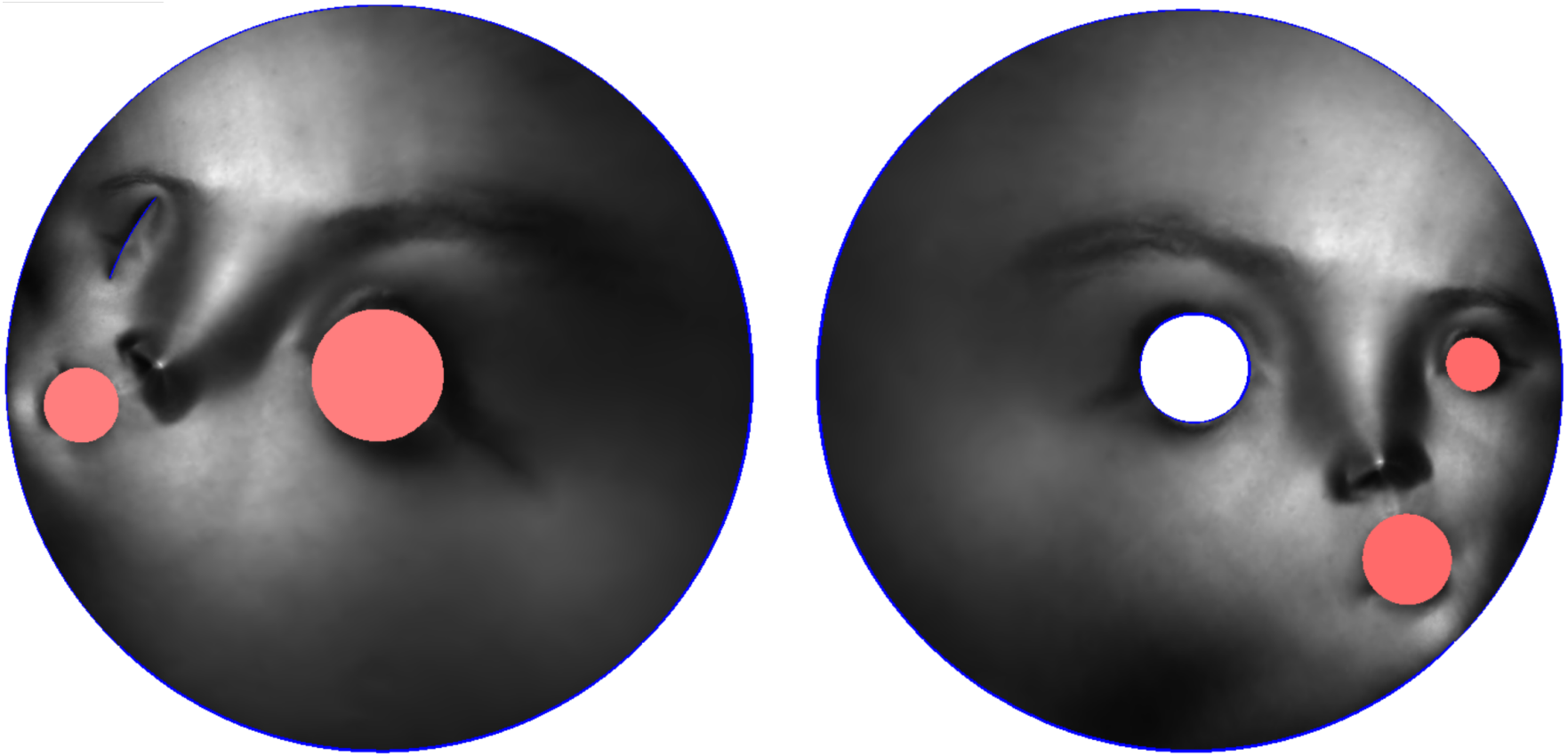


Figure: Hole filling and slit map.

Koebe Iteration Method

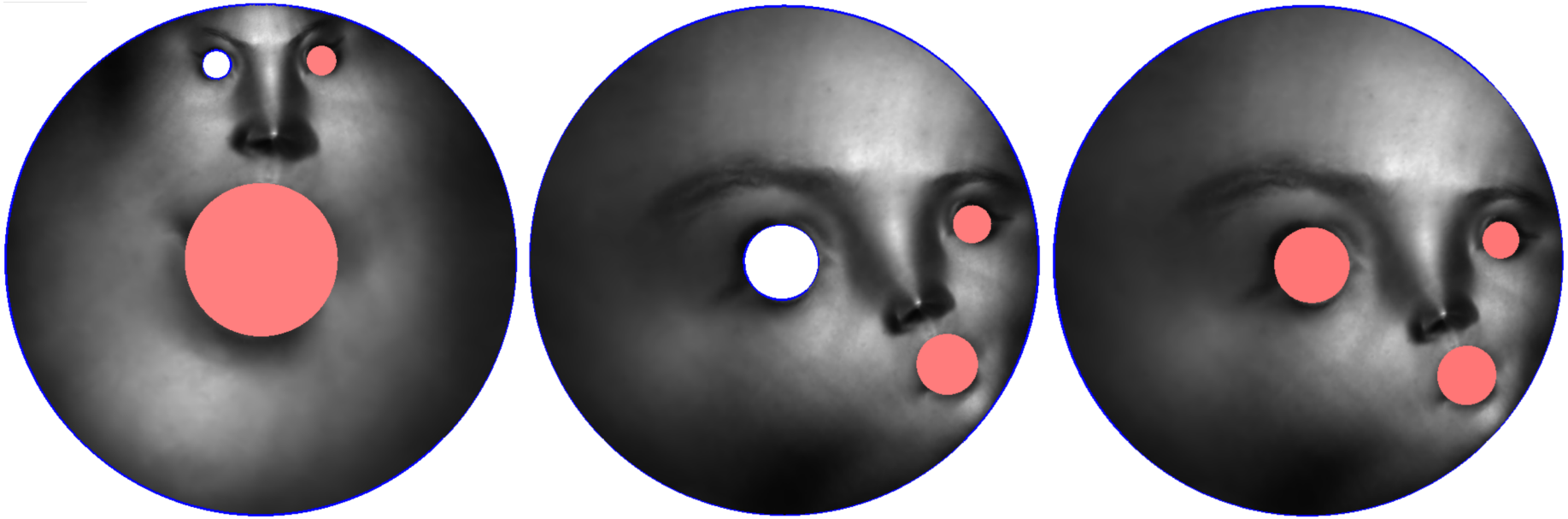
Figure: All holes are filled.

Koebe Iteration Algorithm - Iteration

4. Punch a hole at the k-th inner boundary;
5. Compute a conformal map, to map the surface onto a canonical planar annulus;
6. Fill the inner circular hole;
7. Repeat step 4 through 6, each time punch a different hole, until the process convergences.

Koebe Iteration Method

Koebe Iteration Method



Gram–Schmidt Orthonormalization

Input: Poly annulus M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;
Output: $n-1$ orthonormal non-exact harmonic 1-forms.

1. **for** $i = 0$ to n **do**
2. **while** true **do**
3. generate a random non-exact harmonic 1-form ω_i;
4. **for** $j = 0$ to $i-1$ **do**
5. $w \leftarrow \langle \omega_i, \omega_j \rangle = \int_M \omega_i \wedge^* \omega_j$;
6. $\omega_i \leftarrow \omega_i - w * \omega_j$;
7. **endfor**
8. $w \leftarrow \langle \omega_i, \omega_i \rangle$;
9. **if** $w > 0.5$ **then** break;
10. **endwhile**
11. $\omega_i \leftarrow \omega_i / \sqrt{w}$
12. **endfor**

Hole Filling Algorithm

Input: Poly annulus M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;
Output: A topologgical disk $\bar{M}$, such that all holes are filled.

1. $M_0 \leftarrow M$;
2. **for** $k = 0$ **to** n
3. Compute a circular slit map, map the surface to the circular slit domain $f_k : M_k \to \mathbb{C}$, γ_0 and γ_k are mapped to the exterior and interior circular boundary of $\mathbb{C}$;
4. Generate a mesh D_k using the inner boundary of $f_k(M_k)$ using Delaunay refinement mesh generation;
5. Fill the inner circle of $f_k(M_k)$ to obtain M_{k+1};

$$M_{k+1} \leftarrow f_k(M_k) \cup D_k.$$

6. **endfor**
7. $\bar{M} \leftarrow M_{n+1}$, return $\bar{M}$.

Assignement Six - Slit Map

Figure: Circular slit map.

Assignment Six - Koebe Iteration Method

Surface Ricci Flow

Surface Ricci Flow

Definition (Hamilton's Surface Ricci Flow)

A closed surface with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

The normalized surface Ricci flow,

$$\frac{dg_{ij}}{dt} = \frac{2\pi\chi(S)}{A(0)} - 2Kg_{ij},$$

where $A(0)$ is the initial surface area.

The normalized surface Ricci flow is area-preserving, the Ricci flow will converge to a metric such that the Gaussian curvature is constant $\frac{2\pi\chi(S)}{A(0)}$ every where.

Ricci Flow

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

- Ricci flow

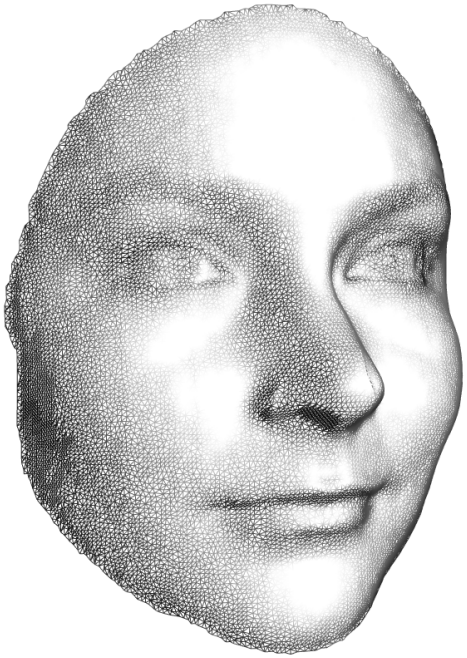
$$\frac{du}{dt} = \bar{K} - K.$$

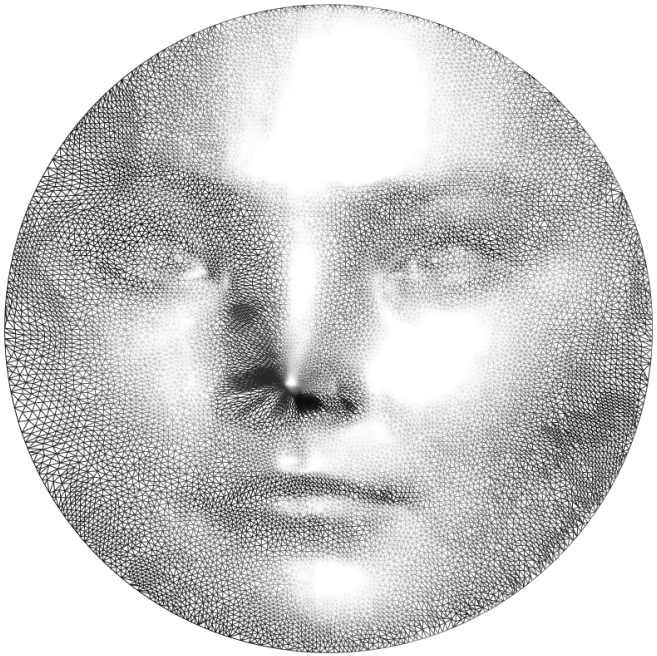
Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

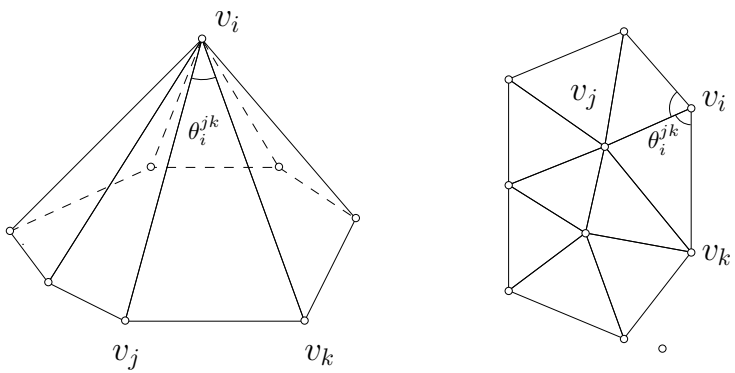
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

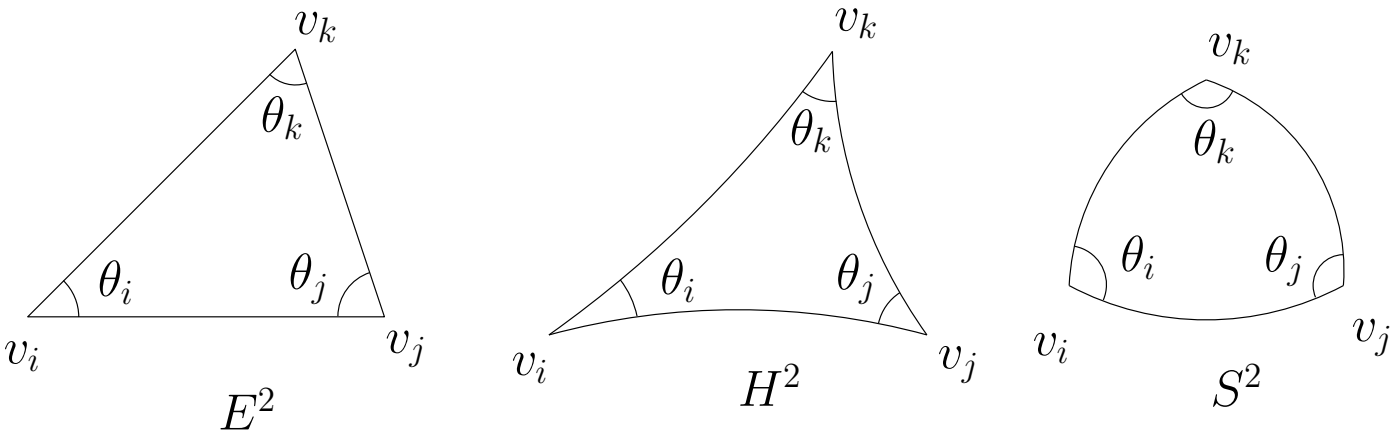
$$K(v_i) = 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M; K(v_i) = \pi - \sum_{jk} \theta_{jk}, v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



cosine laws

$$\begin{aligned}
\cos l_i &= \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} && \mathbb{S}^2 \\
\cosh l_i &= \frac{\cosh\theta_i + \cosh\theta_j \cosh\theta_k}{\sinh\theta_j \sinh\theta_k} && \mathbb{H}^2 \\
1 &= \frac{\cos\theta_i + \cos\theta_j \cos\theta_k}{\sin\theta_j \sin\theta_k} && \mathbb{E}^2
\end{aligned}$$

Conformal Factor

Definition (Discrete Conformal Factor)

The discrete conformal factor is defined as $u : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{E}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Edge Length

Definition (Edge Length)

The edge lengths are given by

$$\begin{cases} l_{ij}^2 &= 2\eta_{ij}e^{u_i+u_j} + \varepsilon_i e^{2u_i} + \varepsilon_j e^{2u_j} \quad \mathbb{E}^2 \\ \cosh l_{ij} &= \frac{4\eta_{ij}e^{u_i+u_j}+(1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})}{(1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})} \quad \mathbb{H}^2 \\ \cos l_{ij} &= \frac{-4\eta_{ij}e^{u_i+u_j}+(1-\varepsilon_i e^{2u_i})(1-\varepsilon_j e^{2u_j})}{(1+\varepsilon_i e^{2u_i})(1+\varepsilon_j e^{2u_j})} \quad \mathbb{S}^2 \end{cases}$$

Edge Length

Scheme	ε_i	ε_j	η_{ij}
Tangential Circle Packing	$+1$	$+1$	$+1$
Thurston's Circle Packing	$+1$	$+1$	$[0,1]$
Inversive Distance Circle Packing	$+1$	$+1$	$(0,\infty)$
Yamabe Flow	0	0	$(0,\infty)$
Virtual Distance Circle Packing	-1	-1	$(0,\infty)$
Mixed Type	$\{-1,0,+1\}$	$\{-1,0,+1\}$	$(0,\infty)$

Table: Parameters for schemes.

Entropy Energy

Definition (Entroy on a mesh)

A discrete surface with $\mathbb{S}^2, \mathbb{E}^2, \mathbb{H}^2$ background geometry, and a circle packing metric $(\Sigma, \gamma, \eta, \varepsilon)$. The discrete entropy energy for the whole mesh is defined as

$$E_= \int^{(u_1, u_2, \cdots, u_n)} \sum_{i=1}^{n} (\bar{K}_i - K_i) du_i.$$

The mesh entropy can be represented as the face energies

$$E_\sigma = \sum_{i=1}^{n} (\bar{K}_i - 2\pi) u_i + \sum_{f \in F} E_f.$$

Discrete Conformal Equivalence

Definition (Discrete Conformal Equivalence)

Two polyhedral metrics $\mathbf{d}$ and $\mathbf{d}'$ on a marked surface (S, V) are discrete conformal equivalent, if there is a series polyhedral metrics on (S, V),

$$\mathbf{d} = \mathbf{d}_1, \mathbf{d}_2, \cdots, \mathbf{d}_m = \mathbf{d}'$$

and a series of triangulations $\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m$, such that

1. every triangulation $\mathcal{T}_k$ is Delaunay on the metric $\mathbf{d}_k$;
2. if $\mathcal{T}_i = \mathcal{T}_{i+1}$, then there is a conformal factor $\mathbf{u} : V \to \mathbb{R}$, such that $\mathbf{d}_{i+1} = \mathbf{u} * \mathbf{d}_i$, namely the two polyhedral metrics differ by a vertex scaling operation;
3. if $\mathcal{T}_i \neq \mathcal{T}_{i+1}$, then there is an isometric transformation $h : (S, V, \mathbf{d}_i) \to (S, V, \mathbf{d}_{i+1})$, this transformation is homotopic to the identity map of (S, V), preserving the vertices.

Main Theorem

Existence and Uniqueness of the Solution to the Discrete Surface Ricci Flow:

Theorem (Existence and Uniqueness)

Suppose $(S, V, \mathbf{d})$ *is a closed polyhedral surface, the for any* $K^* : V \to (-\infty, 2\pi)$, *satisfying the Gauss-Bonnet condition* $\sum_{v \in V} K^*(v) = 2\pi\chi(S)$, *there exists a polyhedral metric* $\mathbf{d}^*$

1. $\mathbf{d}^*$ *is discrete conformal equivalent to the metric* $\mathbf{d}$;
2. $\mathbf{d}^*$ *induces the discrete Gaussian curvature* K^*.

All such kind of polyhedral metrics differ by a global scaling. Furthermore, $\mathbf{d}^*$ *can be obtained by discrete surface Ricci flow.*

Convergence Theorem

Theorem (Convergence of Discrete Curvature Flow)

Given a curved triangle with a Riemannian metric $(S, \mathbf{g})$*, three corner angles are* $\pi/3$*. Given a* (δ, c) *geodesic subdivision sequence* $(\mathcal{T}_n, L_n)$*, for any edge* $e \in E(\mathcal{T}_n)$*,* $L_n(e)$ *is geodesic length under the metric* $\mathbf{g}$*. Then there exists discrete conformal factor* $w_n \in \mathbb{R}^{V(\mathcal{T}_n)}$*, for* n *big enough,* $C_n = (S, \mathcal{T}_n, w_n * L_n)$*, such that*

a. C_n *is isometric to the planar equilateral triangle* $\triangle$*, and* C_n *is* $\delta_\Delta/2$*-triangulation, where the constant* δ_Δ *doesn't depend on the surface;*

b. *Discrete uniformization maps* $\varphi_n : C_n \to \triangle$*, satisfy*

$$\lim_{n\to\infty} \| \varphi_n|_{V(\mathcal{T}_n)} - \varphi|_{V(\mathcal{T}_n)} \|_\infty = 0, \tag{5}$$

uniformly converge to the smooth uniformization map $\varphi : (S, \mathbf{g}) \to (\triangle, dzd\bar{z})$*.*

Assignment 8: Canonical Fundamental Group Generator

Step 1

Compute a set of canonical fundamental group generators of S,

$$\pi_1(S,p) = \langle a_1, b_1, \cdots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

Based on Assignment 7 to compute handle loops and tunnel loops.

Assignment 8:Canonical Fundamental Group Generator

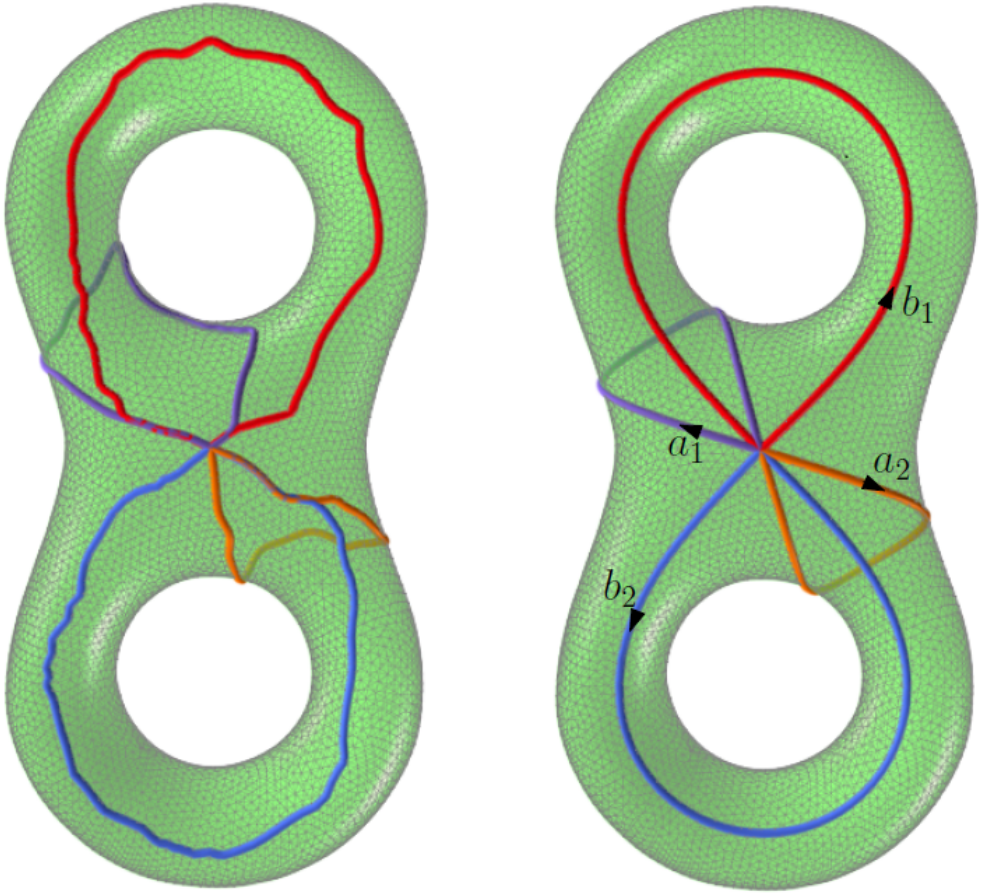


Figure: A set of canonical fundamental group generators.

Assignment 8: Algorithm - Hyperbolic Ricci Flow

Step 2

Use hyperbolic Ricci flow to compute the uniformization metric.

1. Set the target curvature $\bar{K}$ to zeros everywhere;
2. set the conforaml factor u to zeros for all vertices;
3. set the edge length
$$l_{ij} \leftarrow e^{\frac{u_i}{2}} y_{ij} e^{\frac{u_j}{2}}$$
4. Use hyperbolic cosine law to compute corner angles θ_{ij}^k
5. Compute the vertex curvature K_i
6. Compute the gradient of the entropy energy $\nabla E = (\bar{K}_i - K_i)$
7. Compute the Hessian matrix of the entropy energy H
8. Solve linear system $H\delta u = \nabla E$
9. $\mathbf{u} \leftarrow \mathbf{u} + \lambda \delta \mathbf{u}$
10. Repeat step 3 through 9, until $\|\nabla E\| < \varepsilon$.

Assignment 8: Alogrithm- Hyperbolic Isometric Embedding

Step 3

1. Slice the mesh along the canonical fundamental group generators to get a fundamentable domain $\bar{S}$;
2. isometrically embed a face f_0 onto $\mathbb{H}^2$
3. enqueue the face f_0 to the queue Q, set f_0 as processed
4. **while** queue is not empty
5. $f_0 \leftarrow$ Pop Q
6. **for** each face f adjacent to f_0 and unprocessed, embed it on $\mathbb{H}^2$
7. suppose $f = [v_0, v_1, v_2]$, v_0 and v_1 has been embedded, $\varphi(v_2)$ is the intersection of two hyperbolic circles, $c(v_0, l_{02})$ and $c(v_1, l_{12})$, and the orientation is counter-clockwise
8. enqueue f to the queue Q
9. **endfor**
10. **while**

Assignment 8: Algorithm - Funchsian Group Generator

Step 4

1. Locate the segments of $\varphi(\bar{S})$ to $a_k, b_k, a_k^{-1}, b_k^{-1}$, $k = 1, 2, \cdots, g$;
2. choose a pair of segments, γ, and γ^{-1},
3. Find a Möbius transformation α_k, such that

$$\alpha_k(b_k(0)) = b_k^{-1}(1), \quad \alpha_k(b_k(1)) = b_k^{-1}(0).$$

and

$$\beta_k(b_k^{-1}(0)) = b_k(1), \quad \beta_k(b_k^{-1}(1)) = b_k(0).$$

4. Output the Fuchsian group genrators $\{\alpha_k, \beta_k\}_{k=1}^{g}$.

Assignment 8: Algorithm - Fuchsian Group Generator

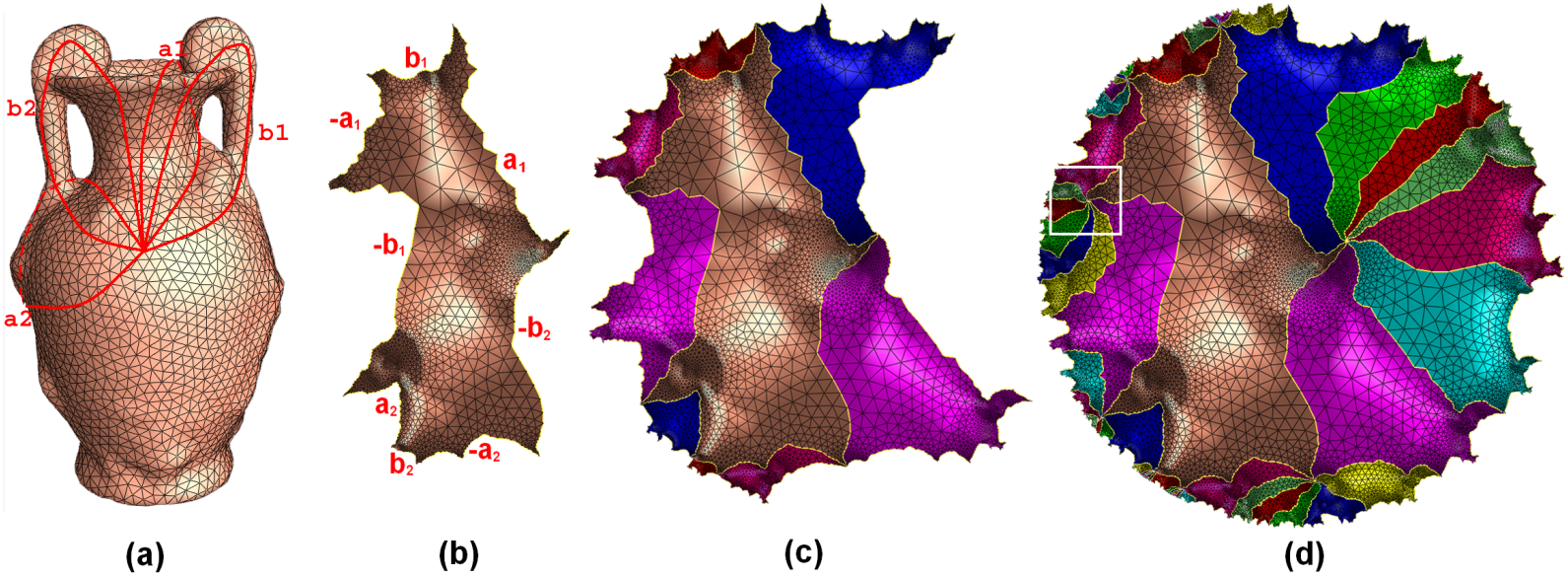


Figure: Fuchsian group generators.

Assignment 8: Algorithm - Hyperbolic Geodesic Boundary

Step 5

1. Use the Fuchsian group generators to compute the Fuchsian transformations
2. Transform the embedding image of the fundamental domain to tessellate the hyperbolic disk
3. Replace each boundary segment of the fundamental domain by the unique hyperbolic geodesic
4. Recompute the fundamental domain and its embedding
5. Generator a finite portion of the universal covering space

Hyperbolic Geodesic Boundary

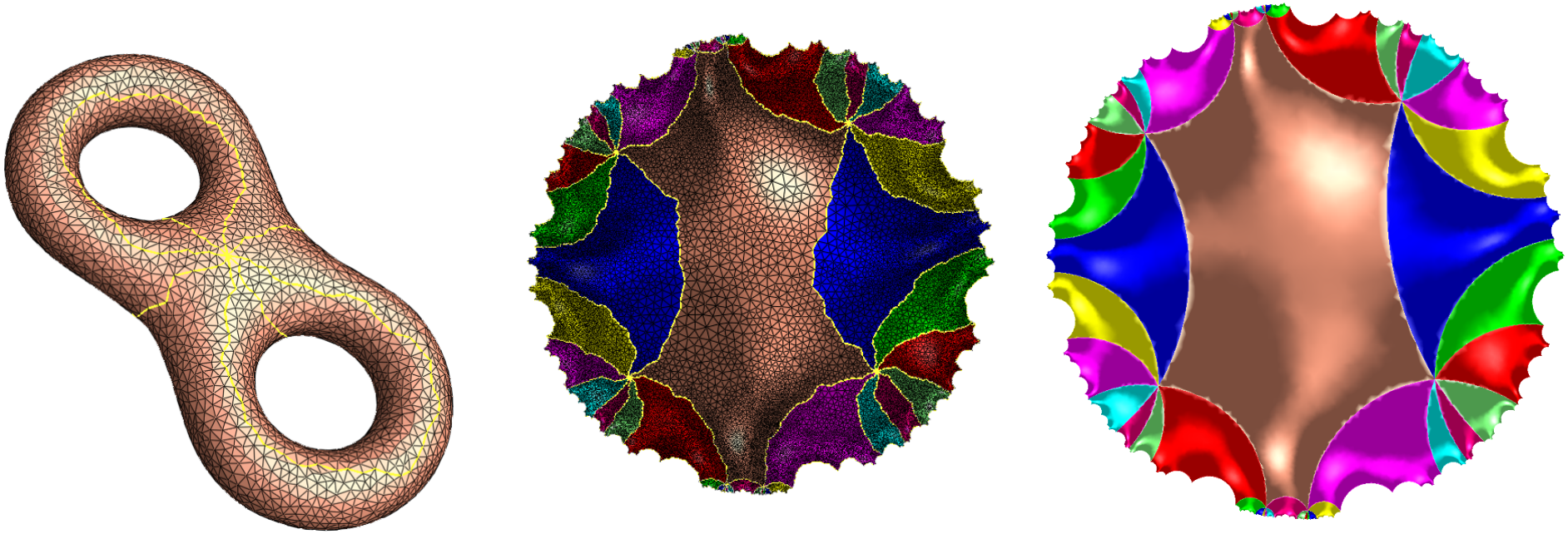


Figure: Fuchsian group generators.