CSE581 CARDINALITIES OF SETS

BASIC DEFINITIONS AND FACTS

Cardinality definition Sets A and B have the same cardinality iff $\exists f(\ f:\ \ A \overset{1-1,onto}{\longrightarrow} B)$.

Cardinality notations If sets A and B have the same cardinality we denote it as: $|A| = |B|$ or $cardA = cardB$, or $A \sim B$. We also say that A and B are *equipotent*.

Cardinality We put the above notations together in one definition:
$|A| = |B|$ or $cardA = cardB$, or $A \sim B$ iff $\exists f(\ f:\ \ A \overset{1-1,onto}{\longrightarrow} B)$.

Finite A set A is finite iff $\exists n \in N\ \ \exists f\ (f:\ \ \{0, 1, 2, ..., n-1\} \overset{1-1,onto}{\longrightarrow} A)$, i.e. we say: a set A is finite iff $\exists n \in N(|A| = n)$.

Infinite A set A is infinite iff A is NOT finite.

Aleph zero $\aleph_0$ (Aleph zero) is a cardinality of N (Natural numbers).

For any set, set A has a cardinality $\aleph_0$ ($|A| = \aleph_0$) iff $A \sim N$, (or $|A| = |N|$, or $cardA = cardN$).

Countable A set A is countable iff A is finite or $|A| = \aleph_0$.

Infinitely countable A set A is infinitely countable iff $|A| = \aleph_0$.

Uncountable A set A is uncountable iff A is NOT countable.

Observe that it means that
A set A is uncountable iff A is infinite and $|A| \neq \aleph_0$.

Continuum $\mathcal{C}$ (Continuum) is a cardinality of Real numbers, i.e. $\mathcal{C} = |\mathcal{R}|$.

We sat that a set A has a cardinality $\mathcal{C}$ ($|A| = \mathcal{C}$) iff $|A| = |R|$.

Cardinality $A \leq$ **cardinality** B We define $|A| \leq |B|$ iff $A \sim C$ and $C \subseteq B$.

Simple Fact If $A \subseteq B$ then $|A| \leq |B|$.

For any cardinal numbers $\mathcal{N}, \mathcal{M}$, we say that

$\mathcal{N} \leq \mathcal{M}$ iff for any sets A, B, such that $|A| = \mathcal{N}$ and $|B| = \mathcal{M}$ we have $|A| \leq |B|$.

Cardinality $A <$ **cardinality** B $|A| < |B|$ iff $|A| \leq |B|$ and $|A| \neq |B|$.

For any cardinal numbers $\mathcal{N}, \mathcal{M}$ we say that

$\mathcal{N} < \mathcal{M}$ iff for any sets A, B, such that $|A| = \mathcal{N}$ and $|B| = \mathcal{M}$ we have $|A| < |B|$.

Cantor Theorem For any set A, $|A| < |\mathcal{P}(A)|$.

Cantor-Berstein Theorem For any sets A, B,
If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

For any cardinal numbers $\mathcal{N}, \mathcal{M}$, we have that

If $\mathcal{N} \leq \mathcal{M}$ and $\mathcal{M} \leq \mathcal{N}$, then $\mathcal{N} = \mathcal{M}$.

ARITHMETIC OF CARDINAL NUMBERS

Sum ($\mathcal{N} + \mathcal{M}$) We define:
$\mathcal{N} + \mathcal{M} = |A \cup B|$, where A, B are such that $|A| = \mathcal{N}$, $|B| = \mathcal{M}$ and $A \cap B = \emptyset$.

Multiplication ($\mathcal{N} \cdot \mathcal{M}$) We define:
$\mathcal{N} \cdot \mathcal{M} = |A \times B|$, where A, B are such that $|A| = \mathcal{N}$, $|B| = \mathcal{M}$.

Power ($\mathcal{M}^{\mathcal{N}}$) $\mathcal{M}^{\mathcal{N}} = card\{f : \ f : A \longrightarrow B\}$, where A, B are such that $|A| = \mathcal{N}$, $|B| = \mathcal{M}$.

Observe that the definition says that $\mathcal{M}^{\mathcal{N}}$ is the cardinality of all functions that map a set A (of cardinality $\mathcal{N}$) into a set B (of cardinality $\mathcal{M}$).

Power $2^{\mathcal{N}}$ We define:

$$2^{\mathcal{N}} = card\{f : \ f : A \longrightarrow \{0, 1\} \}, \text{ where } |A| = \mathcal{N}.$$

$2^{\mathcal{N}}$ Theorems We prove the following.

1. $2^{\mathcal{N}} = card\mathcal{P}(A)$, where $|A| = \mathcal{N}$.

2. $2^{\aleph_0} = \mathcal{C}$.

Power Properties $\mathcal{N}^{\mathcal{P}+\mathcal{T}} = \mathcal{N}^{\mathcal{P}} \cdot \mathcal{N}^{\mathcal{T}}$. $(\mathcal{N}^{\mathcal{P}})^{\mathcal{T}} = \mathcal{N}^{\mathcal{P} \cdot \mathcal{T}}$.

ARITHMETIC OF n, $\aleph_0$, $\mathcal{C}$

Union 1 $\aleph_0 + \aleph_0 = \aleph_0$.
Union of two infinitely countable sets is an infinitely countable set.

Union 2 $\aleph_0 + n = \aleph_0$.
Union of a finite (cardinality n) and infinitely countable set is an infinitely countable set.

Union 3 $\aleph_0 + \mathcal{C} = \mathcal{C}$.
Union of an infinitely countable set and a set of the same cardinality as Real numbers has the same cardinality as the set of Real numbers.

Union 4 $\mathcal{C} + \mathcal{C} = \mathcal{C}$.
Union of two sets of cardinality the same as Real numbers has the same cardinality as the set of Real numbers.

Cartesian Product 1 $\aleph_0 \cdot \aleph_0 = \aleph_0$.
Cartesian Product of two infinitely countable sets is an infinitely countable set.

Cartesian Product 2 $n \cdot \aleph_0 = \aleph_0$.
Cartesian Product of a finite set and an infinitely countable set is an infinitely countable set.

Cartesian Product 3 $\aleph_0 \cdot \mathcal{C} = \mathcal{C}$.
Cartesian Product of an infinitely countable set and a set of the same cardinality as Real numbers has the same cardinality as the set of Real numbers.

Cartesian Product 4 $\mathcal{C} \cdot \mathcal{C} = \mathcal{C}$.
Cartesian Product of two sets of cardinality the same as Real numbers has the same cardinality as the set of Real numbers.

Power 1 $2^{\aleph_0} = \mathcal{C}$.
The set of all subsets of Natural numbers (or any set equipotent with natural numbers) has the same cardinality as the set of Real numbers.

Power 2 $\aleph_0^{\aleph_0} = \mathcal{C}$.

There are $\mathcal{C}$of all functions that map N into N.

There are $\mathcal{C}$ sequences (all sequences) that can be form out of an infinitely countable set.

$\aleph_0^{\aleph_0} = \{f : \quad f : N \longrightarrow N\} = \mathcal{C}$.

Power 3 $\mathcal{C}^{\mathcal{C}} = 2^{\mathcal{C}}$.

There are $2^{\mathcal{C}}$ of all functions that map R into R.

The set of all real functions of one variable has the same cardinality as the set of all subsets of Real numbers.

Inequalities $n < \aleph_0 < \mathcal{C}$.

Theorem If A is a finite set, A^* is the set of all finite sequences formed out of A, then A^* has $\aleph_0$ elements.

Shortly: If $|A| = n$, then $|A^*| = \aleph_0$.