ORDER RELATIONS, LATTICES, BOOLEAN ALGEBRAS

Order Relation  \( R \subset A \times A \) is an order on \( A \) iff \( R \) is 1. Reflexive, 2. Antisymmetric, 3. Transitive, i.e.
1. \( \forall a \in A \ (a, a) \in R \)
2. \( \forall a, b \in A \ ((a, b) \in R \cap (b, a) \in R \Rightarrow a = b) \)
3. \( \forall a, b, c \in A \ ((a, b) \in R \cap (b, c) \in R \Rightarrow (a, c) \in R) \)

Total Order  \( R \subset (A \times A) \) is a total order on \( A \) iff \( R \) is an order and any two elements of \( A \) are comparable, i.e.
\( \forall a, b \in A \ ((a, b) \in R \cup (b, a) \in R) \).

Historical names  Order is also called partial order and total order is also called a linear order.

Notations  Order relations are usually denoted by \( \leq \). We use, in our lecture notes the notation \( \preceq \) as a symbol for order relation.
Remember, that even if we use \( \leq \) as the order relation symbol, it is a SYMBOL for ANY order relation and not only a symbol for a natural order \( \leq \) in number sets.

Poset  A set \( A \neq \emptyset \) ordered by a relation \( R \) is called a poset. We write it as a tuple: \((A, R), (A, \leq), (A, \preceq)\). Name poset stands for "partially ordered set".

Diagram  Diagram or Hasse Diagram of order relation is a graphical representation of a poset. It is a simplified graph constructed as follows.
1. As the relation is REFLEXIVE, i.e. \((a, a) \in R\) for all \( a \in A \), we draw a point \( a \) instead of a point \( a \) with the loop.
2. As the relation is antisymmetric we draw a point \( b \) above point \( a \) (connected, but without the arrow) to indicate that \((a, b) \in R\).
3. As the relation in transitive, we connect points \( a, b, c \) without arrows.

Special elements  in a poset \((A, \preceq)\) are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least)  \( a_0 \in A \) is a smallest (least) element in the poset \((A, \preceq)\) iff \( \forall a \in A \ (a_0 \preceq a) \).

Greatest (largest)  \( a_0 \in A \) is a greatest (largest) element in the poset \((A, \preceq)\) iff \( \forall a \in A \ (a \preceq a_0) \).

Maximal (formal)  \( a_0 \in A \) is a maximal element in the poset \((A, \preceq)\) iff \( \neg \exists a \in A \ (a_0 \preceq a \land a_0 \neq a) \).

Maximal (informal)  \( a_0 \in A \) is a maximal element in the poset \((A, \preceq)\) iff on the diagram of \((A, \preceq)\) there is no element placed above \( a_0 \).

Minimal  \( a_0 \in A \) is a minimal element in the poset \((A, \preceq)\) iff \( \neg \exists a \in A \ (a \preceq a_0 \land a_0 \neq a) \).

Minimal (informal)  \( a_0 \in A \) is a minimal element in the poset \((A, \preceq)\) iff on the diagram of \((A, \preceq)\) there is no element placed below \( a_0 \).

Lower Bound  Let \( B \subseteq A \) and \((A, \preceq)\) is a poset. \( a_0 \in A \) is a lower bound of a set \( B \) iff \( \forall b \in B \ (a_0 \leq b) \).
Lattice Unit Axioms

Let \(0\) (lattice zero) and \(1\) (lattice unit) exist in a lattice, we will write \(A\) as a lattice with unit and zero.

The greatest element in a lattice (if exists) is denoted by \(1\) and is called a lattice special elements.

Conditions \(l1- l5\) from above are called distributive lattice axioms.

Distributive lattice

A lattice \(A\) is called a lattice iff for all \(a, b, c \in A\) the following conditions hold:

\[
\begin{align*}
  l1 & \quad a \lor b = b \lor a \
  l2 & \quad (a \lor b) \lor c = a \lor (b \lor c) \quad \text{and} \quad (a \land b) \land c = a \land (b \land c) \
  l3 & \quad a \land (a \lor b) = a \quad \text{and} \quad a \lor (a \land b) = a.
\end{align*}
\]

Lattice axioms

The conditions \(l1- l3\) from above definition are called lattice axioms.

Lattice orderings

Let \((A, \leq)\) be a poset. The relations:

\[
\begin{align*}
  a \leq b \quad \text{iff} \quad a \lor b = b, \quad a \leq b \quad \text{iff} \quad a \land b = a
\end{align*}
\]

are order relations in \(A\) and are called a lattice orderings.

Distributive lattice

A lattice \((A, \lor, \land)\) is called a distributive lattice iff for all \(a, b, c \in A\) the following conditions hold:

\[
\begin{align*}
  l4 & \quad a \lor (b \land c) = (a \lor b) \land (a \lor c) \
  l5 & \quad a \land (b \lor c) = (a \land b) \lor (a \land c).
\end{align*}
\]

Distributive lattice axioms

Conditions \(l1- l5\) from above are called a distributive lattice axioms.

Lattice special elements

The greatest element in a lattice (if exists) is denoted by \(1\) and is called a lattice unit. The least (smallest) element in \(A\) (if exists) is denoted by \(0\) and called a lattice zero.

Lattice with unit and zero

If \(0\) (lattice zero) and \(1\) (lattice unit) exist in a lattice, we will write the lattice as: \((A, \lor, \land, 0, 1)\) and call is a lattice with zero and unit.

Lattice Unit Axioms

Let \((A, \lor, \land)\) be a lattice. An element \(x \in A\) is called a lattice unit iff for any \(a \in A\) \(x \land a = a\) and \(x \lor a = x\).

If such element \(x\) exists we denote it by \(1\) and we write the unit axioms as follows.

\[
\begin{align*}
  l6 & \quad 1 \land a = a
\end{align*}
\]
Lattice Zero Axioms Let \((A, \cup, \cap)\) be a lattice. An element \(x \in A\) is called a lattice zero iff for any \(a \in A\) \(x \cap a = x\) and \(x \cup a = a\).

We denote the lattice zero by 0 and write the zero axioms as follows.

\[
\begin{align*}
\text{l8} & \quad 0 \cap a = 0 \\
\text{l9} & \quad 0 \cup a = a.
\end{align*}
\]

Complement Let \((A, \cup, \cap, 1, 0)\) be a lattice with unit and zero. An element \(x \in A\) is called a complement of an element \(a \in A\) iff \(a \cup x = 1\) and \(a \cap x = 0\).

Complement axioms Let \((A, \cup, \cap, 1, 0)\) be a lattice with unit and zero. The complement of \(a \in A\) is usually denoted by \(-a\) and the above conditions that define the complement above are called complement axioms. The complement axioms are usually written as follows.

\[
\begin{align*}
\text{c1} & \quad a \cup -a = 1 \\
\text{c2} & \quad a \cap -a = 0.
\end{align*}
\]

Boolean Algebra A distributive lattice with zero and unit such that each element has a complement is called a Boolean Algebra.

Boolean Algebra Axioms A lattice \((A, \cup, \cap, 1, 0)\) is called a Boolean Algebra iff the operations \(\cap, \cup\) satisfy axioms \(l1 - l5, 0 \in A\) and \(1 \in A\) satisfy axioms \(l6 - l9\) and each element \(a \in A\) has a complement \(-a \in A\), i.e.

\[
\begin{align*}
\text{l1o} & \quad \forall a \in A \exists -a \in A \left( (a \cup -a = 1) \cap (a \cap -a = 0) \right).
\end{align*}
\]

SOME BASIC FACTS

Uniqueness In any poset \((A, \preceq)\), if a greatest and a least elements exist, then they are unique.

Finite Posets If \((A, \preceq)\) is a finite poset (i.e. \(A\) is a finite set), then a unique maximal (if exists) is the largest element and a unique minimal (if exists) is the least element.

Infinite Posets It is possible to order an infinite set \(A\) in such a way that the poset \((A, \preceq)\) has a unique maximal element (minimal element) and no largest element (least element).

Any poset In any poset, the largest element is a unique maximal element and the least element is the unique minimal element.

Lower, upper bounds A set \(B \subseteq A\) of a poset \((A, \preceq)\) can have none, finite or infinite number of lower or upper bounds, depending of ordering.

Finite lattice If \((A, \cup, \cap)\) is a finite lattice (i.e. \(A\) is a finite set), then 1 and 0 always exist.

Infinite lattice If \((A, \cup, \cap)\) is an infinite lattice (i.e. the set \(A\) is infinite ), then 1 or 0 might or might not exist.

For example:

\((N \leq)\) is a lattice with 0 (the number 0) and no 1.

\((Z \leq)\) is a lattice without 0 and without 1.

Finite Boolean Algebra Non- generate Finite Boolean Algebras always have \(2^n\) elements \((n \geq 1)\).